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Matrix-injection-based transformation method for discrete-time systems with time-varying delay

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Dear editor,

In order to identify the influence of time-varying delays on the stability of control systems represented in a discrete-time mode, developing stability criteria of discrete-time delayed systems has received serious attention in the past few decades (see [1–3] and their references).

Consider the following linear discrete-time system with a time-varying delay:

$$\begin{cases} x(k+1) = Ax(k) + A_d x(k-d(k)), \ k \geqslant 0, \\ x(k) = \phi(k), \quad k \in [-h_2, 0], \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ and $\phi(k)$ denote the system state and the initial condition, respectively; A and A_d are the system matrices; and $d(k)$ is the time-varying delay satisfying

$$0 = h_0 < 1 \leqslant h_1 \leqslant d(k) \leqslant h_2 \tag{2}$$

with h_1 and h_2 being constant. If no confusion arises, let $d = d(k)$, $h_{12} = h_2 - h_1$, $h_{1d} = d - h_1 + 1$, $h_{2d} = h_2 - d + 1$.

Among the methodologies for developing stability criteria, the Lyapunov function method is popular since it can be used to easily develop delay-dependent criteria, for delay margin calculations, by introducing the following summation term:

$$V_r(x_k) = \sum_{l=1}^{2}(h_l - h_{l-1}) \sum_{i=-h_l}^{-h_{l-1}-1} \sum_{j=k+i}^{k-1} \eta^{\mathrm{T}}(j) R_l \eta(j), \tag{3}$$

where $R_i \geqslant 0, i = 1, 2$, and $\eta(k) = x(k+1) - x(k)$. The following terms appear in the forward difference:

$$\mathcal{S}_1 = h_1 \sum_{i=k-h_1}^{k-1} \eta^{\mathrm{T}}(i) R_1 \eta(i), \tag{4}$$

$$\mathcal{S}_2 = h_{12} \sum_{i=k-d}^{k-h_1-1} \eta^{\mathrm{T}}(i) R_2 \eta(i) + h_{12} \sum_{i=k-h_2}^{k-d-1} \eta^{\mathrm{T}}(i) R_2 \eta(i). \tag{5}$$

Owing to the influence on the conservatism of criteria, many efforts have been devoted to estimating these terms, such as Wirtinger-like inequalities (WLIs) [4], auxiliary function-based inequality (AFBI) [5], and polynomial-based inequality [6]. However, to date, these improved inequalities have not been used to estimate $\mathcal{S}_2$ for avoiding the introduction of a non-convex cubic function with respect to the delay. Although the vector extension idea developed by [7] can achieve this target, too many decision variables are required. Therefore, to make better use of the aforementioned improved inequalities, it is vital to directly determine the negativity of the said cubic functions.

In this study, a matrix-injection-based transformation method is developed to solve the above problem. This method gives an equivalent condition to ensure the negativity of cubic functions and it is tractable due to its convexity of the delay. By constructing an augmented Lyapunov function and using the proposed method combined with AFBI and an extended reciprocally convex matrix inequality to estimate the forward difference of the function, a stability criterion of system (1) is established and its merit is demonstrated via a numerical example. The notations used, if not given in this study, are all listed in Appendix A.

Main results. The following delay-dependent summation vectors appear in the lower bound of $\mathcal{S}_2$ when $\mathcal{S}_2$ is estimated via the AFBI:

$$v_1(k) = \frac{\sum_{i=k-d}^{k-h_1} \sum_{j=i}^{k-h_1} x(j)}{h_{1d}(h_{1d}+1)}, \ v_2(k) = \frac{\sum_{i=k-h_2}^{k-d} \sum_{j=i}^{k-d} x(j)}{h_{2d}(h_{2d}+1)}. \tag{6}$$

Similar to the discussion in [8], the Lyapunov function candidate should contain at least one cross term with vectors, $v_1(k)$ and $v_2(k)$, to fully benefit from the AFBI. As an effective method, the following augmented term is helpful:

$$V_1(x_k) = \xi^{\mathrm{T}}(k) P \xi(k), \quad P = [P_{ij}]_{5\times 5} > 0, \tag{7}$$

the forward difference of which can be expressed as

$$\Delta V_1(x_k) = \zeta^{\mathrm{T}}(k) \Psi_p(d) \zeta(k), \tag{8}$$

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where $\Psi_p(d)$ is a cubic function of the delay and thus is non-convex with respect to the delay. In developing the criteria, it is important to find a tractable condition of the cubic function. For this purpose, a matrix-injection-based transformation method is developed as the following lemma.

Lemma 1. Assume that $y \in \mathbb{R}$ is a time-varying parameter, $\varsigma \in \mathbb{R}^m$ is a vector, and $\Xi(y) \in \mathbb{S}^m$ is a time-varying matrix, expressed as $\Xi(y) = y^3\Gamma_1^{\mathrm{T}}\Xi_1\Gamma_1 + y^2\mathrm{He}\{\Gamma_1^{\mathrm{T}}\Xi_2\Gamma_2\} + \Xi_3(y)$ with $\Xi_1 \in \mathbb{S}^p$, $\Xi_3(y) \in \mathbb{S}^m$ being convex with respect to y, $\Xi_2 \in \mathbb{R}^{p\times q}$, $\Gamma_1 \in \mathbb{R}^{p\times m}$ and $\Gamma_2 \in \mathbb{R}^{q\times m}$ being obtained under the requirement that Γ_1 satisfies the least number of rows. Then the following holds:

$$y^3\left[\varsigma^{\mathrm{T}}\Gamma_1^{\mathrm{T}}\Xi_1\Gamma_1\varsigma\right] + y^2\left[\varsigma^{\mathrm{T}}\mathrm{He}\{\Gamma_1^{\mathrm{T}}\Xi_2\Gamma_2\}\varsigma\right] + \varsigma^{\mathrm{T}}\Xi_3(y)\varsigma < 0, \quad (9)$$

if there exist $N_1 \in \mathbb{R}^{p\times p}$ and $N_2 \in \mathbb{R}^{m\times p}$ such that

$$\mathcal{G}(y) = \begin{bmatrix} \mathcal{G}_{11}(y) & \mathcal{G}_{12}(y) \\ * & \mathcal{G}_{22}(y) \end{bmatrix} < 0, \quad (10)$$

where $\mathcal{G}_{11}(y) = \Xi_3(y) - \mathrm{He}\{yN_2\Gamma_1\}$, $\mathcal{G}_{12}(y) = N_2 - y(\Gamma_1^{\mathrm{T}}N_1^{\mathrm{T}} - \Gamma_2^{\mathrm{T}}\Xi_2^{\mathrm{T}})$, and $\mathcal{G}_{22}(y) = y\Xi_1 + N_1 + N_1^{\mathrm{T}}$.

Based on Lemma 1, a stability criterion is developed using the AFBI and an extended reciprocally convex matrix inequality to estimate $\mathcal{S}_2$.

Theorem 1. For given h_1 and h_2, system (1) with the delay satisfying (2) is asymptotically stable if there exist matrices $P = [P_{ij}]_{5\times 5} \in \mathbb{S}_+^{5n}$, $\{Q_i, R_i\} \in \mathbb{S}_+^n$, matrices $N_1 \in \mathbb{R}^{2n\times 2n}$, $N_2 \in \mathbb{R}^{10n\times 2n}$, $S_i \in \mathbb{R}^{3n\times 3n}$, $i = 1, 2$, such that the following holds for both $d = h_1$ and $d = h_2$:

$$\Psi(d) = \begin{bmatrix} \Psi_1(d) & \begin{bmatrix} \frac{(d-h_1)E_3^{\mathrm{T}}S_1^{\mathrm{T}} + (h_2-d)E_2^{\mathrm{T}}S_2}{h_{12}} \\ 0_{2n\times 3n} \end{bmatrix} \\ * & -\mathrm{diag}\{R_2, 3R_2, 5R_2\} \end{bmatrix} < 0. \quad (11)$$

Note that the detailed proof and the advantages of Lemma 1 and Theorem 1 are discussed in Appendixes B and C.

A numerical example. Consider system (1) with

$$A = \begin{bmatrix} 1.00 & 0.01 \\ -0.10 & 0.99 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0.003 & 0.001 \\ 0.010 & 0.005 \end{bmatrix}. \quad (12)$$

For different h_1, the allowably maximal h_2 provided by Theorem 1 and those reported in literature are listed in Table 1, where the number of decision variables (NDVs) and the maximal order of linear matrix inequalities (MoLs) indicate the criteria complexity.

Table 1 The allowably maximal h_2 and the complexity indexes[a)]

Criteria	h_1				Complexity indexes	
	20	30	40	50	NDVs	MoLs
Th.5 [4]	90	110	126	136	$10.5n^2+3.5n$	$7n$
Th.1 [5]	97	124	144	156	$29.5n^2+8.5n$	$12n$
Th.3 [6]	101	129	146	158	$64n^2+4n$	$16n$
Th.1 [7]	127	140	150	158	$79.5n^2+4.5n$	$17n$
Th.1	129	141	151	160	$56.5n^2+4.5n$	$15n$

a) Th. denotes Theorem.

The results show the advantages of Lemma 1 from two aspects. (1) Compared with the criteria in [5,6], Theorem 1 more obviously improves the results obtained by the WBI-based criterion in [4]. The reason is that, due to the usage of Lemma 1, both $\mathcal{S}_1$ and $\mathcal{S}_2$ are established by using tighter AFBI to significantly reduce the conservatism. (2) Compared with the criteria in [7], Theorem 1 not only provides less conservative results but also requires lower complexity, which greatly shows the advantage of Lemma 1 in comparison with the treatments for avoiding d^3-dependent terms applied in [7].

Conclusion. This study has investigated the stability of discrete-time systems with a time-varying delay. To handle the negativity condition of the forward difference of Lyapunov function, a matrix-injection-based method has been developed to convert the original negativity condition to an equivalent tractable matrix inequality by injecting a few matrices. By using this method and an augmented Lyapunov function, a stability criterion with less conservatism for the delayed linear discrete-time system has been developed. The case study via a numerical example has shown the merit of the criterion. It is predictable that the proposed method, together with the methods recently reported by [9], can further improve the results.

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Supporting information Appendixes A–C. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

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