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• **Supplementary File** •

Matrix-injection-based transformation method for discrete-time systems with time-varying delay

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Appendix A Preliminaries

Firstly, the standard notations to be used are listed in Table A1.

Table A1 Notations

Notations	Explanations
$\mathbb{R}^n$	the sets of n-dimensional vectors
$\mathbb{R}^{m\times n}$	the sets of $m \times n$-dimensional real matrices
$\mathbb{S}_+^{n\times n}$	the sets of $n \times n$-dimensional symmetric positive definite matrices
$X > 0\ (\geqslant 0)$	symmetric and positive-definite (semi-positive-definite) matrix, X
X^T	the transpose of matrix X
X^{-1}	the inverse of matrix X
$\|\cdot\|$	the Euclidean vector norm
$\mathrm{col}\{y_1, y_2, \ldots, y_n\}$	$\left[y_1^T, y_2^T, \ldots, y_n^T\right]^T$
$\mathrm{He}\{X\}$	$X + X^T$
$\mathrm{diag}\{X, Y\}$	$\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$
$[\varsigma]^T[X][\star]$	$[\varsigma]^T[X][\varsigma]$
$\begin{bmatrix} X & Z \\ * & Y \end{bmatrix}$	$\begin{bmatrix} X & Z \\ Z^T & Y \end{bmatrix}$

Secondly, for the sake of subsequent description, the system concerned is recalled as follows

$$\begin{cases} x(k+1) = Ax(k) + A_d x(k-d(k)),\ k \geqslant 0 \\ x(k) = \phi(k), \quad k \in [-h_2, 0] \end{cases} \tag{A1}$$

where $x(k) \in \mathbb{R}^n$ is the system state, $\phi(k)$ is the initial condition, A and A_d are the system matrices, and the delay $d(k)$ satisfies

$$0 = h_0 < 1 \leqslant h_1 \leqslant d(k) \leqslant h_2 \tag{A2}$$

with h_i, $i = 1, 2$ being constant. Define $d = d(k)$, $h_{12} = h_2 - h_1$, $h_{1d} = d - h_1 + 1$, and $h_{2d} = h_2 - d + 1$.

Thirdly, the lemmas to be used for deriving the main results are given as follows

Lemma 1. [2] For matrix $R > 0$, integers a and b with $b > a$, any sequence of discrete-time variable x well defined, the following inequality holds

$$(b-a)\sum_{i=a}^{b-1} \eta^T(i) R \eta(i) \geqslant \sum_{i=1}^{3} (2i-1)\chi_i^T R \chi_i \tag{A3}$$

where $\chi_1 = x(b) - x(a)$, $\chi_2 = x(b) + x(a) - \sum\limits_{i=a}^{b} \frac{2x(i)}{b-a+1}$, $\chi_3 = x(b) - x(a) + \sum\limits_{i=a}^{b} \frac{6x(i)}{b-a+1} - \sum\limits_{i=a}^{b}\sum\limits_{j=i}^{b} \frac{12x(j)}{(b-a+1)(b-a+2)}$.

Lemma 2. [1] For a scalar $\alpha \in [0, 1]$, a matrices $X > 0$, and matrices S_1 and S_2, the following inequality holds

$$\begin{bmatrix} \frac{1}{\alpha}X & 0 \\ 0 & \frac{1}{1-\alpha}X \end{bmatrix} \geqslant \begin{bmatrix} X + (1-\alpha)(X - S_2 X^{-1} S_2^T) & (1-\alpha)S_1 + \alpha S_2 \\ * & X + \alpha(X - S_1^T X^{-1} S_1) \end{bmatrix} \tag{A4}$$

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Appendix B Proof and discussion of Lemma 1

The part gives the detailed proof of the following lemma (i.e., Lemma 1 in the Letter) and also briefly discusses its advantages.

Lemma 3. Assume that $y \in \mathbb{R}$ is a time-varying parameter, $\varsigma \in \mathbb{R}^m$ is a vector, and $\Xi(y) \in \mathbb{S}^m$ is a time-varying matrix, which can be rewritten as

$$\Xi(y) = y^3\Gamma_1^T\Xi_1\Gamma_1 + y^2\mathrm{He}\{\Gamma_1^T\Xi_2\Gamma_2\} + \Xi_3(y) \tag{B1}$$

where $\Xi_1 \in \mathbb{S}^p$, $\Xi_3(y) \in \mathbb{S}^m$ is convex w.r.t. y, $\Xi_2 \in \mathbb{R}^{p\times q}$, $\Gamma_1 \in \mathbb{R}^{p\times m}$ and $\Gamma_2 \in \mathbb{R}^{q\times m}$ are obtained under the requirement that Γ_1 has least number of rows. Then the following holds

$$\begin{aligned}\mathcal{F}(y) &= \varsigma^T\Xi(y)\varsigma \\ &= y^3\left[\varsigma^T\Gamma_1^T\Xi_1\Gamma_1\varsigma\right] + y^2\left[\varsigma^T\mathrm{He}\{\Gamma_1^T\Xi_2\Gamma_2\}\varsigma\right] + \varsigma^T\Xi_3(y)\varsigma \\ &< 0\end{aligned} \tag{B2}$$

if and only if there exists $N_1 \in \mathbb{R}^{p\times p}$ and $N_2 \in \mathbb{R}^{m\times p}$ such that

$$\mathcal{G}(y) = \begin{bmatrix}\mathcal{G}_{11}(y) & \mathcal{G}_{12}(y)\\ * & \mathcal{G}_{22}(y)\end{bmatrix} < 0 \tag{B3}$$

where $\mathcal{G}_{11}(y) = \Xi_3(y) - \mathrm{He}\{yN_2\Gamma_1\}$, $\mathcal{G}_{12}(y) = N_2 - y(\Gamma_1^TN_1^T + \Gamma_2^T\Xi_2^T)$, and $\mathcal{G}_{22}(y) = y\Xi_1 + N_1 + N_1^T$.

Proof. On the one hand, if there exist matrices N_1 and N_2 satisfying (B3), then, for a vector $\rho = \mathrm{col}\{\varsigma, y\Gamma_1\varsigma\}$, $\Theta_1 = \rho^T\mathcal{G}(y)\rho < 0$ holds. Moreover, for the same N_1 and N_2, the following holds

$$\Theta_2 = 2[(y\Gamma_1\varsigma)^TN_1 + \varsigma^TN_2][(y\Gamma_1\varsigma) - y\Gamma_1\varsigma] = 0 \tag{B4}$$

Then, carrying out simple mathematical computations leads to $\mathcal{F}(y) = \Theta_1 - \Theta_2 < 0$. Therefore, if there exists N_1 and N_2 satisfying (B3), then (B2) holds. On the other hand, based on Schur complement, (B3) is equivalent to

$$\mathcal{G}_{22}(y) = y\Xi_1 + N_1 + N_1^T < 0 \tag{B5}$$

$$\bar{\mathcal{F}}(y) = \mathcal{G}_{11}(y) - \mathcal{G}_{12}(y)[\mathcal{G}_{22}(y)]^{-1}\mathcal{G}_{12}^T(y) < 0 \tag{B6}$$

Letting $N_2 = -y(\Gamma_1^T(N_1 + y\Xi_1) + \Gamma_2^T\Xi_2^T)$ and carrying out mathematical computations lead that $\bar{\mathcal{F}}(y)$ reduces to $\mathcal{F}(y)$. Thus, if $\mathcal{F}(y) < 0$ holds, then matrices $N_i, i = 1,2$ satisfying (B5) and (B6) can be found by setting $y\Xi_1 + N_1 + N_1^T < 0$ and $N_2 = -y(\Gamma_1^T(N_1 + y\Xi_1) + \Gamma_2^T\Xi_2^T)$. Therefore, if (B2) holds, then there must exist N_1 and N_2 satisfying (B3). This completes the proof. ■

Remark 1. In Lemma 3, by injecting free matrices N_1 and N_2, the negativity condition of cubic function given by $\mathcal{F}(y) < 0$ is converted to an equivalent matrix inequality $\mathcal{G}(y) < 0$, which is convex w.r.t. y. Then, the condition that $\mathcal{F}(y) < 0$ holds for a time-varying parameter y within $[y_1, y_2]$ can be guaranteed by requiring $\mathcal{G}(y_i) < 0, i = 1,2$. On the one hand, this matrix-injection-based transformation method does not introduce extra conservatism due to the equivalence of two conditions. On the other hand, the computational burden of transformed condition increases due to the injection of N_1 and N_2. It is found from (B4) that the number of rows of Γ_1 is related to the dimensions of $N_i, i = 1,2$ and in turn affects the computational burden of criteria obtained. That is the reason of minimizing the number of rows of Γ_1 during the expression of (B1).

Appendix C Proof and discussion of Theorem 1

The part gives the detailed proof of the following lemma (i.e., Lemma 1 in the Letter) and also briefly discusses its advantages.

The following stability criterion is developed.

Theorem 1. For given h_1 and h_2, system (A1) with the delay satisfying (A2) is asymptotically stable if there exist matrices $P = [P_{ij}]_{5\times 5} \in \mathbb{S}_+^{5n}$, $\{Q_i, R_i\} \in \mathbb{S}_+^n$, matrices $N_1 \in \mathbb{R}^{2n\times 2n}$, $N_2 \in \mathbb{R}^{10n\times 2n}$, $S_i \in \mathbb{R}^{3n\times 3n}$, $i = 1,2$, such that the following holds for both $d = h_1$ and $d = h_2$

$$\Psi(d) = \begin{bmatrix}\Psi_1(d) & \begin{bmatrix}\frac{(d-h_1)E_3^TS_1^T + (h_2-d)E_2^TS_2}{h_{12}}\\ 0_{2n\times 3n}\end{bmatrix}\\ * & -\mathrm{diag}\{R_2, 3R_2, 5R_2\}\end{bmatrix} < 0 \tag{C1}$$

where

$$\begin{aligned}\Psi_1(d) &= \begin{bmatrix}\Upsilon(d) - dN_2E_A - dE_A^TN_2^T & N_2 - dE_A^TN_1^T + dE_B^TP_B^T\\ * & dP_A + N_1 + N_1^T\end{bmatrix}\\ \Upsilon(d) &= \Upsilon_1(d) + \Upsilon_2 + \Upsilon_3 - \Upsilon_4(d)\\ \Upsilon_1(d) &= \Psi_p(d) - d^3E_A^TP_AE_A - d^2\mathrm{He}\{E_A^TP_BE_B\}\\ \Psi_p(d) &= \mathrm{He}\{E_{p0}^TP(E_{p1} - E_{p2})\} + E_{p1}^TPE_{p1} - E_{p2}^TPE_{p2}\\ E_{p0} &= \mathrm{col}\{e_1, (h_1+1)e_5, e_{p1}, (h_1+1)(h_1+2)e_8, e_{p0}\}\\ e_{p0} &= h_{1d}(h_{1d}+1)e_9 + h_{2d}(h_{2d}+1)e_{10} + (h_{2d}-1)h_{1d}e_6 - h_{2d}e_3\\ e_{p1} &= h_{1d}e_6 + h_{2d}e_7 - e_3\end{aligned}$$

$$
\begin{aligned}
E_{p1} &= \mathrm{col}\{e_s, -e_2, -e_4, -(h_1+1)e_5, -e_{p1}\}\\
E_{p2} &= \mathrm{col}\{e_0, -e_1, -e_2, -(h_1+1)e_1, -(h_{12}+1)e_2\}\\
E_A &= \begin{bmatrix} e_a \\ e_b \end{bmatrix}\\
E_B &= \begin{bmatrix} E_{p1}|_{d=0} - E_{p2} \\ e_b \\ e_c \end{bmatrix}\\
P_A &= \begin{bmatrix} 0_{n\times n} & -P_{55} \\ -P_{55}^T & 0_{n\times n} \end{bmatrix}\\
P_B &= \begin{bmatrix} P_5 & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times 5n} & \frac{P_{55}}{2} - P_{35}^T & -P_{55}^T \end{bmatrix}\\
P_5 &= [P_{51}, P_{52}, P_{53}, P_{54}, P_{55}]\\
e_a &= e_9 + e_{10} - e_6\\
e_b &= e_6 - e_7\\
e_c &= (3-2h_1)e_9 - (3+2h_2)e_{10} + (h_1+h_2-1)e_6 + e_3\\
\Upsilon_2 &= e_1^TQ_1e_1 - e_2^T\left(Q_1 - Q_2\right)e_2 - e_4^TQ_2e_4\\
\Upsilon_3 &= e_s^T\left(h_1^2R_1 + h_{12}^2R_2\right)e_s - E_1^T\hat{R}_1E_1\\
\Upsilon_4(d) &= \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}^T \begin{bmatrix} \frac{2h_2-d-h_1}{h_{12}}\hat{R} & \frac{(d-h_1)S_2+(h_2-d)S_1}{h_{12}} \\ * & \frac{h_2+d-2h_1}{h_{12}}\hat{R} \end{bmatrix} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}\\
E_i &= \mathrm{col}\{e_i - e_{i+1}, e_i + e_{i+1} - 2e_{i+4}, e_i - e_{i+1} + 6e_{i+4} - 12e_{i+7}\}, i = 1,2,3\\
\hat{R}_i &= \mathrm{diag}\{R_i, 3R_i, 5R_i\}, i = 1,2\\
e_i &= [0_{n\times(i-1)n}, I_{n\times n}, 0_{n\times(10-i)n}], i = 1,2,\ldots,10\\
e_s &= (A-I)e_1 + A_de_3\\
e_0 &= 0_{n\times 10n}
\end{aligned}
$$

Proof. *Step 1: construct the functional candidate.* Construct the following Lyapunov functional candidate

$$
V(x_k) = V_1(x_k) + V_2(x_k) + V_r(x_k) \tag{C2}
$$

where

$$
\begin{aligned}
V_1(x_k) &= \xi^T(k)P\xi(k)\\
V_2(x_k) &= \sum_{i=k-h_1}^{k-1} x^T(i)Q_1x(i) + \sum_{i=k-h_2}^{k-h_1-1} x^T(i)Q_2x(i)\\
V_r(x_k) &= \sum_{l=1}^{2}(h_l - h_{l-1}) \sum_{i=-h_l}^{-h_{l-1}-1}\sum_{j=k+i}^{k-1} \eta^T(j)R_l\eta(j) = h_1\sum_{i=-h_1}^{-1}\sum_{j=k+i}^{k-1}\eta^T(j)R_1\eta(j) + h_{12}\sum_{i=-h_2}^{-h_1-1}\sum_{j=k+i}^{k-1}\eta^T(j)R_2\eta(j)
\end{aligned}
$$

where

$$
\xi(k) = col\left\{x(k), \sum_{i=k-h_1}^{k-1}x(i), \sum_{i=k-h_2}^{k-h_1-1}x(i), \omega_1(k), \omega(k)\right\}, \qquad \omega_1(k) = \sum_{i=k-h_1}^{k-1}\sum_{j=i}^{k-1}x(j), \qquad \omega(k) = \sum_{i=k-h_2}^{k-h_1-1}\sum_{j=i}^{k-h_1-1}x(j).
$$

and $P > 0$, $Q_i > 0$, $R_i > 0$, $i = 1, 2$, which guarantees the positiveness of $V(x_k)$.

Step 2: Define necessary notations for simplifying the expression.

Define the following augmented vector

$$
\zeta(k) = col\left\{x(k), x(k-h_1), x(k-d), x(k-h_2), \frac{\sum\limits_{i=k-h_1}^{k}x(i)}{h_1+1}, \frac{\sum\limits_{i=k-d}^{k-h_1}x(i)}{h_{1d}}, \frac{\sum\limits_{i=k-h_2}^{k-d}x(i)}{h_{2d}}, \frac{\sum\limits_{i=k-h_1}^{k}\sum\limits_{j=i}^{k}x(j)}{(h_1+1)(h_1+2)}, v_1(k), v_2(k)\right\}
$$

$$
\begin{aligned}
0_{n\times 1} = e_0\zeta(k), &\qquad e_0 = 0_{n\times 10n} = [0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0]\\
x(k) = e_1\zeta(k), &\qquad e_1 = [I_n, 0_{n\times 9n}] = [I\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0]\\
x(k-h_1) = e_2\zeta(k), &\qquad e_2 = [0_{n\times n}, I_n, 0_{n\times 8n}] = [0\ \ I\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0]
\end{aligned}
$$

$$\begin{aligned}
x(k-d) = e_3\zeta(k), \qquad & e_3 = [0_{n\times 2n}, I_n, 0_{n\times 7n}] = [0\ \ 0\ \ I\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0]\\
x(k-h_2) = e_4\zeta(k), \qquad & e_4 = [0_{n\times 3n}, I_n, 0_{n\times 6n}] = [0\ \ 0\ \ 0\ \ I\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0]\\
\frac{\sum_{i=k-h_1}^{k} x(i)}{h_1+1} = e_5\zeta(k), \qquad & e_5 = [0_{n\times 4n}, I_n, 0_{n\times 5n}] = [0\ \ 0\ \ 0\ \ 0\ \ I\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0]\\
\frac{\sum_{i=k-d}^{k-h_1} x(i)}{h_{1d}} = e_6\zeta(k), \qquad & e_6 = [0_{n\times 5n}, I_n, 0_{n\times 4n}] = [0\ \ 0\ \ 0\ \ 0\ \ 0\ \ I\ \ 0\ \ 0\ \ 0\ \ 0]\\
\frac{\sum_{i=k-h_2}^{k-d} x(i)}{h_{2d}} = e_7\zeta(k), \qquad & e_7 = [0_{n\times 6n}, I_n, 0_{n\times 3n}] = [0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ I\ \ 0\ \ 0\ \ 0]\\
\frac{\sum_{i=k-h_1}^{k}\sum_{j=i}^{k} x(j)}{(h_1+1)(h_1+2)} = e_8\zeta(k), \qquad & e_8 = [0_{n\times 7n}, I_n, 0_{n\times 2n}] = [0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ I\ \ 0\ \ 0]\\
v_1(k) = \frac{\sum_{i=k-d}^{k-h_1}\sum_{j=i}^{k-h_1} x(j)}{h_{1d}(h_{1d}+1)} = e_9\zeta(k), \qquad & e_9 = [0_{n\times 8n}, I_n, 0_{n\times n}] = [0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ I\ \ 0]\\
v_2(k) = \frac{\sum_{i=k-h_2}^{k-d}\sum_{j=i}^{k-d} x(j)}{h_{2d}(h_{2d}+1)} = e_{10}\zeta(k), \qquad & e_{10} = [0_{n\times 9n}, I_n] = [0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ I]\\
\eta(k) = x(k+1) - x(k) = e_s\zeta(k), \qquad & e_s = (A-I)e_1 + A_d e_3\\
& e_i = [0_{n\times(i-1)n}, I_n, 0_{n\times(10-i)n}], i = 1, 2, 3, \ldots, 10
\end{aligned}$$

Then

$$\begin{aligned}
\omega_1(k) &= \sum_{i=k-h_1}^{k-1}\sum_{j=i}^{k-1} x(j) = \sum_{i=k-h_1}^{k-1}\left(\sum_{j=i}^{k} x(j) - x(k)\right) = \sum_{i=k-h_1}^{k}\sum_{j=i}^{k} x(j) - (h_1+1)x(k)\\
&= [(h_1+1)(h_1+2)e_8 - (h_1+1)e_1]\zeta(k) \qquad \text{(C3)}\\
\omega_1(k+1) &= \sum_{i=k-h_1+1}^{k}\sum_{j=i}^{k} x(j) = \sum_{i=k-h_1}^{k}\sum_{j=i}^{k} x(j) - \sum_{j=k-h_1}^{k} x(j)\\
&= [(h_1+1)(h_1+2)e_8 - (h_1+1)e_5]\zeta(k) \qquad \text{(C4)}\\
\omega(k) &= \sum_{i=k-h_2}^{k-h_1-1}\sum_{j=i}^{k-h_1-1} x(j) = \sum_{i=k-h_2}^{k-h_1-1}\left(\sum_{j=i}^{k-h_1} x(j) - x(k-h_1)\right) = \sum_{i=k-h_2}^{k-h_1-1}\sum_{j=i}^{k-h_1} x(j) - h_{12}x(k-h_1)\\
&= \left\{\sum_{i=k-h_2}^{k-h_1}\sum_{j=i}^{k-h_1} x(j)\right\} - (h_{12}+1)x(k-h_1)\\
&= \left\{\sum_{i=k-d}^{k-h_1}\sum_{j=i}^{k-h_1} x(j) + \sum_{i=k-h_2}^{k-d}\sum_{j=i}^{k-d} x(j) + (h_2-d)\sum_{j=k-d}^{k-h_1} x(j) - h_{2d}x(k-d)\right\} - (h_{12}+1)x(k-h_1)\\
&= \Big\{h_{1d}(h_{1d}+1)e_9 + h_{2d}(h_{2d}+1)e_{10} + (h_{2d}-1)h_{1d}e_6 - h_{2d}e_3\Big\}\zeta(k) - (h_{12}+1)e_2\zeta(k) \qquad \text{(C5)}\\
&= [e_{p0} - (h_{12}+1)e_2]\zeta(k) \qquad \text{(C6)}\\
\omega(k+1) &= \sum_{i=k-h_2+1}^{k-h_1}\sum_{j=i}^{k-h_1} x(j) = \sum_{i=k-h_2}^{k-h_1}\sum_{j=i}^{k-h_1} x(j) - \sum_{j=k-h_2}^{k-h_1} x(j)\\
&= \left\{\sum_{i=k-h_2}^{k-h_1}\sum_{j=i}^{k-h_1} x(j)\right\} - \left\{\sum_{j=k-h_2}^{k-d} x(j) + \sum_{j=k-d}^{k-h_1} x(j) - x(k-d)\right\}\\
&= \Big\{h_{1d}(h_{1d}+1)e_9 + h_{2d}(h_{2d}+1)e_{10} + (h_{2d}-1)h_{1d}e_6 - h_{2d}e_3\Big\}\zeta(k) - \left\{\sum_{j=k-d}^{k-h_1} x(j) + \sum_{j=k-h_2}^{k-d} x(j) - x(k-d)\right\}\\
&= \Big\{h_{1d}(h_{1d}+1)e_9 + h_{2d}(h_{2d}+1)e_{10} + (h_{2d}-1)h_{1d}e_6 - h_{2d}e_3\Big\}\zeta(k) - \Big\{h_{1d}e_6 + h_{2d}e_7 - e_3\Big\}\zeta(k) \qquad \text{(C7)}\\
&= (e_{p0} - e_{p1})\zeta(k) \qquad \text{(C8)}
\end{aligned}$$

with $e_{p0} = h_{1d}(h_{1d}+1)e_9 + h_{2d}(h_{2d}+1)e_{10} + (h_{2d}-1)h_{1d}e_6 - h_{2d}e_3$ and $e_{p1} = h_{1d}e_6 + h_{2d}e_7 - e_3$.

Step 3: Calculate the forward difference of $V(x_k)$*.*

Calculating the forward difference of $V_1(x_k)$ yields:

$$\begin{aligned}
\Delta V_1(x_k) &= \xi^T(k+1)P\xi(k+1) - \xi^T(k)P\xi(k)\\
&= \begin{bmatrix} x(k+1)\\ \sum_{i=k-h_1+1}^{k} x(i)\\ \sum_{i=k-h_2+1}^{k-h_1} x(i)\\ \omega_1(k+1)\\ \omega(k+1)\end{bmatrix}^T P \begin{bmatrix} x(k+1)\\ \sum_{i=k-h_1+1}^{k} x(i)\\ \sum_{i=k-h_2+1}^{k-h_1} x(i)\\ \omega_1(k+1)\\ \omega(k+1)\end{bmatrix} - \begin{bmatrix} x(k)\\ \sum_{i=k-h_1}^{k-1} x(i)\\ \sum_{i=k-h_2}^{k-h_1-1} x(i)\\ \omega_1(k)\\ \omega(k)\end{bmatrix}^T P \begin{bmatrix} x(k)\\ \sum_{i=k-h_1}^{k-1} x(i)\\ \sum_{i=k-h_2}^{k-h_1-1} x(i)\\ \omega_1(k)\\ \omega(k)\end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= \begin{bmatrix} x(k)+\eta(k) \\ \sum_{i=k-h_1}^{k} x(i) - x(k-h_1) \\ \sum_{i=k-h_2}^{k-d} x(i) + \sum_{i=k-d}^{k-h_1} x(i) - x(k-d) - x(k-h_2) \\ \omega_1(k+1) \\ \omega(k+1) \end{bmatrix}^T P \begin{bmatrix} x(k)+\eta(k) \\ \sum_{i=k-h_1}^{k} x(i) - x(k-h_1) \\ \sum_{i=k-h_2}^{k-d} x(i) + \sum_{i=k-d}^{k-h_1} x(i) - x(k-d) - x(k-h_2) \\ \omega_1(k+1) \\ \omega(k+1) \end{bmatrix}^T \\
&- \begin{bmatrix} x(k) \\ \sum_{i=k-h_1}^{k} x(i) - x(k) \\ \sum_{i=k-h_2}^{k-d} x(i) + \sum_{i=k-d}^{k-h_1} x(i) - x(k-h_1) - x(k-d) \\ \omega_1(k) \\ \omega(k) \end{bmatrix}^T P \begin{bmatrix} x(k) \\ \sum_{i=k-h_1}^{k} x(i) - x(k) \\ \sum_{i=k-h_2}^{k-d} x(i) + \sum_{i=k-d}^{k-h_1} x(i) - x(k-h_1) - x(k-d) \\ \omega_1(k) \\ \omega(k) \end{bmatrix} \\
&= \begin{bmatrix} (e_s+e_1)\zeta(k) \\ [(h_1+1)e_5 - e_2]\zeta(k) \\ [h_{1d}e_6 + h_{2d}e_7 - e_3 - e_4]\zeta(k) \\ ((h_1+1)(h_1+2)e_8 - (h_1+1)e_5)\zeta(k) \\ (e_{p0} - e_{p1})\zeta(k) \end{bmatrix}^T P \begin{bmatrix} (e_s+e_1)\zeta(k) \\ [(h_1+1)e_5 - e_2]\zeta(k) \\ [h_{1d}e_6 + h_{2d}e_7 - e_3 - e_4]\zeta(k) \\ ((h_1+1)(h_1+2)e_8 - (h_1+1)e_5)\zeta(k) \\ (e_{p0} - e_{p1})\zeta(k) \end{bmatrix} \\
&- \begin{bmatrix} e_1\zeta(k) \\ [(h_1+1)e_5 - e_1]\zeta(k) \\ [h_{1d}e_6 + h_{2d}e_7 - e_2 - e_3]\zeta(k) \\ ((h_1+1)(h_1+2)e_8 - (h_1+1)e_1)\zeta(k) \\ (e_{p0} - (h_{12}+1)e_2)\zeta(k) \end{bmatrix}^T P \begin{bmatrix} e_1\zeta(k) \\ [(h_1+1)e_5 - e_1]\zeta(k) \\ [h_{1d}e_6 + h_{2d}e_7 - e_2 - e_3]\zeta(k) \\ ((h_1+1)(h_1+2)e_8 - (h_1+1)e_1)\zeta(k) \\ (e_{p0} - (h_{12}+1)e_2)\zeta(k) \end{bmatrix} \\
&= \zeta^T(k) \left\{ \left(\begin{bmatrix} e_1 \\ (h_1+1)e_5 \\ e_{p1} \\ (h_1+1)(h_1+2)e_8 \\ e_{p0} \end{bmatrix} + \begin{bmatrix} e_s \\ -e_2 \\ -e_4 \\ -(h_1+1)e_5 \\ -e_{p1} \end{bmatrix} \right)^T P \left(\begin{bmatrix} e_1 \\ (h_1+1)e_5 \\ e_{p1} \\ (h_1+1)(h_1+2)e_8 \\ e_{p0} \end{bmatrix} + \begin{bmatrix} e_s \\ -e_2 \\ -e_4 \\ -(h_1+1)e_5 \\ -e_{p1} \end{bmatrix} \right) \right\} \zeta(k) \\
&-\zeta^T(k) \left\{ \left(\begin{bmatrix} e_1 \\ (h_1+1)e_5 \\ e_{p1} \\ (h_1+1)(h_1+2)e_8 \\ e_{p0} \end{bmatrix} + \begin{bmatrix} e_0 \\ -e_1 \\ -e_2 \\ -(h_1+1)e_1 \\ -(h_{12}+1)e_2 \end{bmatrix} \right)^T P \left(\begin{bmatrix} e_1 \\ (h_1+1)e_5 \\ e_{p1} \\ (h_1+1)(h_1+2)e_8 \\ e_{p0} \end{bmatrix} + \begin{bmatrix} e_0 \\ -e_1 \\ -e_2 \\ -(h_1+1)e_1 \\ -(h_{12}+1)e_2 \end{bmatrix} \right) \right\} \zeta(k) \\
&= \zeta^T(k) \left\{ \left(E_{p0} + E_{p1}\right)^T P \left[E_{p0} + E_{p1}\right] \right\} \zeta(k) - \zeta^T(k) \left\{ \left[E_{p0} + E_{p2}\right]^T P \left[E_{p0} + E_{p2}\right] \right\} \zeta(k) \\
&= \zeta^T(k) \Big\{ E_{p1}^T P E_{p1} + E_{p0}^T P (E_{p1} - E_{p2}) + (E_{p1} - E_{p2})^T P E_{p0} - E_{p2}^T P E_{p2} \Big\} \zeta(k) \\
&= \zeta^T(k) \Psi_p(d) \zeta(k) \qquad \text{(C9)}
\end{aligned}
$$

where

$$
\begin{aligned}
\Psi_p(d) &= He\left\{E_{p0}^T P (E_{p1} - E_{p2})\right\} + E_{p1}^T P E_{p1} - E_{p2}^T P E_{p2} && \text{(C10)} \\
E_{p0} &= \mathrm{col}\{e_1, (h_1+1)e_5, e_{p1}, (h_1+1)(h_1+2)e_8, e_{p0}\} && \text{(C11)} \\
E_{p1} &= \mathrm{col}\{e_s, -e_2, -e_4, -(h_1+1)e_5, -e_{p1}\} && \text{(C12)} \\
E_{p2} &= \mathrm{col}\{e_0, -e_1, -e_2, -(h_1+1)e_1, -(h_{12}+1)e_2\} && \text{(C13)}
\end{aligned}
$$

Calculating the forward difference of $V_2(x_k)$ yields:

$$
\begin{aligned}
\Delta V_2(x_k) &= x^T(k)Q_1x(k) - x^T(k-h_1)Q_1x(k-h_1) + x^T(k-h_1)Q_2x(k-h_1) - x^T(k-h_2)Q_2x(k-h_2) \\
&= \zeta(k)^T (e_1^T Q_1 e_1 + e_2^T (Q_2 - Q_1) e_2 - e_4^T Q_2 e_4) \zeta(k) \\
&= \zeta(k)^T \Upsilon_2 \zeta(k) \qquad \text{(C14)}
\end{aligned}
$$

where $\Upsilon_2 = e_1^T Q_1 e_1 + e_2^T (Q_2 - Q_1) e_2 - e_4^T Q_2 e_4$.

Calculating the forward difference of $V_r(x_k)$ yields:

$$
\begin{aligned}
\Delta V_r(x_k) &= h_1^2 \eta^T(k) R_1 \eta(k) - h_1 \sum_{i=k-h_1}^{k-1} \eta^T(i) R_1 \eta(i) + h_{12}^2 \eta^T(k) R_2 \eta(k) - h_{12} \sum_{i=k-h_2}^{k-h_1-1} \eta^T(i) R_2 \eta(i) \\
&= \eta^T(k) \Big[h_1^2 R_1 + h_{12}^2 R_2\Big] \eta(k) - \mathcal{S}_1 - \mathcal{S}_2 \qquad \text{(C15)}
\end{aligned}
$$

where

$$\mathcal{S}_1 = h_1 \sum_{i=k-h_1}^{k-1} \eta^T(i) R_1 \eta(i) \qquad \mathcal{S}_2 = h_{12} \sum_{i=k-h_2}^{k-h_1-1} \eta^T(i) R_2 \eta(i) = h_{12} \sum_{i=k-d}^{k-h_1-1} \eta^T(i) R_2 \eta(i) + h_{12} \sum_{i=k-h_2}^{k-d-1} \eta^T(i) R_2 \eta(i)$$

Combining (C2), (C9), (C14), and (C15) yields

$$\begin{aligned}\Delta V(x_k) &= \Delta V_1(x_k) + \Delta V_2(x_k) + \Delta V_r(x_k) \\ &= \zeta^T(k)\Psi_p(d)\zeta(k) + \zeta^T(k)\Upsilon_2\zeta(k) + \eta^T(k)\left[h_1^2 R_1 + h_{12}^2 R_2\right]\eta(k) - \mathcal{S}_1 - \mathcal{S}_2 \\ &= \zeta^T(k)\Big(\Psi_p(d) + \Upsilon_2 + e_s^T\left(h_1^2 R_1 + h_{12}^2 R_2\right)e_s\Big)\zeta(k) - \mathcal{S}_1 - \mathcal{S}_2 \end{aligned} \tag{C16}$$

Step 4: Handle the related terms in $\Delta V(x_k)$.

Firstly, notation $\Psi_p(d)$ in $\Delta V_1(x_k)$ of (C16) is transformed in to a new form. It is found that E_{p0} and E_{p1}, respectively defined in (C11) and (C12), can be rewritten as follows:

$$\begin{aligned} E_{p0} &= \begin{bmatrix} e_1 \\ (h_1+1)e_5 \\ e_{p1} \\ (h_1+1)(h_1+2)e_8 \\ e_{p0} \end{bmatrix} = \begin{bmatrix} e_1 \\ (h_1+1)e_5 \\ h_{1d}e_6 + h_{2d}e_7 - e_3 \\ (h_1+1)(h_1+2)e_8 \\ h_{1d}(h_{1d}+1)e_9 + h_{2d}(h_{2d}+1)e_{10} + (h_{2d}-1)h_{1d}e_6 - h_{2d}e_3 \end{bmatrix} \\ &= d^2 \begin{bmatrix} e_0 \\ e_0 \\ e_0 \\ e_0 \\ e_{10}+e_9-e_6 \end{bmatrix} + d \begin{bmatrix} e_0 \\ e_0 \\ e_6 - e_7 \\ e_0 \\ (3-2h_1)e_9 - (3+2h_2)e_{10} + (h_2+h_1-1)e_6 + e_3 \end{bmatrix} + \begin{bmatrix} e_1 \\ (h_1+1)e_5 \\ e_{p1}|_{d=0} \\ (h_1+1)(h_1+2)e_8 \\ e_{p0}|_{d=0} \end{bmatrix} \\ &= d^2 \begin{bmatrix} e_0 \\ e_0 \\ e_0 \\ e_0 \\ e_a \end{bmatrix} + d \begin{bmatrix} e_0 \\ e_0 \\ e_b \\ e_0 \\ e_c \end{bmatrix} + \begin{bmatrix} e_1 \\ (h_1+1)e_5 \\ e_{p1}|_{d=0} \\ (h_1+1)(h_1+2)e_8 \\ e_{p0}|_{d=0} \end{bmatrix} \qquad \textit{with} \left\{ \begin{aligned} e_a &= e_{10}+e_9-e_6 \\ e_b &= e_6 - e_7 \\ e_c &= (3-2h_1)e_9 - (3+2h_2)e_{10} + (h_2+h_1-1)e_6 + e_3 \end{aligned} \right\} \\ &= d^2 E_a + dE_b + E_c \qquad \textit{with} \left\{ E_a = \begin{bmatrix} e_0 \\ e_0 \\ e_0 \\ e_0 \\ e_a \end{bmatrix}, \quad E_b = \begin{bmatrix} e_0 \\ e_0 \\ e_b \\ e_0 \\ e_c \end{bmatrix}, \quad E_c = \begin{bmatrix} e_1 \\ (h_1+1)e_5 \\ e_{p1}|_{d=0} \\ (h_1+1)(h_1+2)e_8 \\ e_{p0}|_{d=0} \end{bmatrix} = E_{p0}|_{d=0} \right\} \end{aligned}$$

$$\begin{aligned} E_{p1} &= \begin{bmatrix} e_s \\ -e_2 \\ -e_4 \\ -(h_1+1)e_5 \\ -e_{p1} \end{bmatrix} = \begin{bmatrix} e_s \\ -e_2 \\ -e_4 \\ -(h_1+1)e_5 \\ -h_{1d}e_6 - h_{2d}e_7 + e_3 \end{bmatrix} = \begin{bmatrix} e_s \\ -e_2 \\ -e_4 \\ -(h_1+1)e_5 \\ -d(e_6-e_7) - e_{p1}|_{d=0} \end{bmatrix} \\ &= \begin{bmatrix} e_0 \\ e_0 \\ e_0 \\ e_0 \\ -e_b \end{bmatrix} + \begin{bmatrix} e_s \\ -e_2 \\ -e_4 \\ -(h_1+1)e_5 \\ -e_{p1}|_{d=0} \end{bmatrix} = dE_d + E_f \qquad \textit{with} \left\{ E_d = \begin{bmatrix} e_0 \\ e_0 \\ e_0 \\ e_0 \\ -e_b \end{bmatrix}, \quad E_f = \begin{bmatrix} e_s \\ -e_2 \\ -e_4 \\ -(h_1+1)e_5 \\ -e_{p1}|_{d=0} \end{bmatrix} = E_{p1}|_{d=0} \right\} \end{aligned}$$

$$E_{p2} = \begin{bmatrix} e_0 \\ -e_1 \\ -e_2 \\ -(h_1+1)e_1 \\ -(h_{12}+1)e_2 \end{bmatrix} = E_{p1}|_{d=0} - E_e = E_f - E_e \qquad \textit{with} \left\{ E_e = E_{p1}|_{d=0} - E_{p2} = \begin{bmatrix} e_s \\ e_1 - e_2 \\ e_2 - e_4 \\ (h_1+1)e_1 - (h_1+1)e_5 \\ (h_{12}+1)e_2 - e_{p1}|_{d=0} \end{bmatrix} \right\}$$

Therefore,

$$\begin{aligned} \Psi_p(d) &= He\left\{E_{p0}^T P(E_{p1} - E_{p2})\right\} + E_{p1}^T P E_{p1} - E_{p2}^T P E_{p2} \\ &= He\left\{(d^2E_a + dE_b + E_c)^T P(dE_d + E_f - (E_f - E_e))\right\} + (dE_d + E_f)^T P(dE_d + E_f) - (E_f - E_e)^T P(E_f - E_e) \end{aligned}$$

$$
\begin{aligned}
&= He\left\{(d^2E_a+dE_b+E_c)^TP(dE_d+E_e)\right\}+(dE_d+E_f)^TP(dE_d+E_f)-(E_f-E_e)^TP(E_f-E_e)\\
&= d^3He\left\{E_a^TPE_d\right\}+d^2\left[He\left\{E_a^TPE_e+E_b^TPE_d\right\}+E_d^TPE_d\right]\\
&\quad +He\left\{dE_b^TPE_e+dE_c^TPE_d+E_c^TPE_e\right\}+dE_d^TPE_f+dE_f^TPE_d+E_f^TPE_f-(E_f-E_e)^TP(E_f-E_e)\\
&= d^3He\left\{E_a^TPE_d\right\}+d^2\left[He\left\{E_a^TPE_e+E_b^TPE_d\right\}+E_d^TPE_d\right]+\Upsilon_1(d) \qquad \text{(C17)}
\end{aligned}
$$

It is found that $\Upsilon_1(d)$ is linear function of d, and d^3- and d^2-dependent terms appear in the following parts:

$$
\mathrm{He}\{E_{p0}^TP(E_{p1}-E_{p2})\}+E_{p1}^TPE_{p1}=\mathrm{He}\{(d^2E_a+dE_b+E_c)^TP(dE_d+E_e)\}+(dE_d+E_f)^TP(dE_d+E_f) \qquad \text{(C18)}
$$

More specifically, extracting the d^3- and d^2-dependent terms from (C17), respectively denoted as $\Gamma_3(d^3)$ and $\Gamma_2(d^2)$, and deleting the related zero-value parts yield

$$
\Gamma_3(d^3) = d^3\mathrm{He}\{E_a^TPE_d\}=d^3\mathrm{He}\{e_a^TP_{55}(-e_b)\}=d^3\begin{bmatrix}e_a\\ e_b\end{bmatrix}^T\begin{bmatrix}0_{n\times n} & -P_{55}\\ -P_{55} & 0_{n\times n}\end{bmatrix}\begin{bmatrix}e_a\\ e_b\end{bmatrix}=d^3E_A^TP_AE_A \qquad \text{(C19)}
$$

$$
\begin{aligned}
\Gamma_2(d^2) &= d^2\left[\mathrm{He}\{E_a^TPE_e+E_b^TPE_d\}+E_d^TPE_d\right] \qquad \text{(C20)}\\
&= d^2\left[\mathrm{He}\{E_a^TPE_e+E_d^TP^TE_b\}+E_d^TPE_d\right] \qquad \text{(C21)}\\
&= d^2\left[\mathrm{He}\left\{e_a^TP_5E_e-e_b^T\begin{bmatrix}P_{35}\\ P_{55}\end{bmatrix}^T\begin{bmatrix}e_b\\ e_c\end{bmatrix}\right\}+e_b^TP_{55}e_b\right]=d^2\mathrm{He}\left\{\begin{bmatrix}e_a\\ e_b\end{bmatrix}^T\begin{bmatrix}P_5 & 0_{n\times n} & 0_{n\times n}\\ 0_{n\times 5n} & \frac{P_{55}}{2}-P_{35}^T & -P_{55}^T\end{bmatrix}\begin{bmatrix}E_e\\ e_b\\ e_c\end{bmatrix}\right\}\\
&= d^2\mathrm{He}\{E_A^TP_BE_B\} \qquad \text{(C22)}
\end{aligned}
$$

where

$$
E_A=\begin{bmatrix}e_a\\ e_b\end{bmatrix},\ E_B=\begin{bmatrix}E_e\\ e_b\\ e_c\end{bmatrix}=\begin{bmatrix}E_{p1}|_{d=0}-E_{p2}\\ e_b\\ e_c\end{bmatrix},\ P_A=\begin{bmatrix}0_{n\times n} & -P_{55}\\ -P_{55}^T & 0_{n\times n}\end{bmatrix},\ P_B=\begin{bmatrix}P_5 & 0_{n\times n} & 0_{n\times n}\\ 0_{n\times 5n} & \frac{P_{55}}{2}-P_{35}^T & -P_{55}^T\end{bmatrix},\ P_5=[P_{51},P_{52},P_{53},P_{54},P_{55}]
$$

Combining (C17), (C19), and (C22) yields

$$
\begin{aligned}
\Psi_p(d) &= He\left\{E_{p0}^TP(E_{p1}-E_{p2})\right\}+E_{p1}^TPE_{p1}-E_{p2}^TPE_{p2}\\
&= d^3He\left\{E_a^TPE_d\right\}+d^2\left[He\left\{E_a^TPE_e+E_b^TPE_d\right\}+E_d^TPE_d\right]+\Upsilon_1(d)\\
&= d^3E_A^TP_AE_A+d^2\mathrm{He}\{E_A^TP_BE_B\}+\Upsilon_1(d) \qquad \text{(C23)}
\end{aligned}
$$

Therefore, the transformation from (15) to (16) in the manuscript is completed.

Secondly, $\mathcal{S}_1$ and $\mathcal{S}_2$ in (C16) are estimated by using Lemmas 1 and 2. Based on $R_1>0$ in (C2), using (A3) to estimate the R_1-dependent summation term, $\mathcal{S}_1$, in (C16) yields:

$$
\begin{aligned}
\mathcal{S}_1 &\geqslant \zeta^T(k)\begin{bmatrix}e_1-e_2\\ e_1+e_2-2e_5\\ e_1-e_2+6e_5-12e_8\end{bmatrix}^T\begin{bmatrix}R_1 & 0 & 0\\ 0 & 3R_1 & 0\\ 0 & 0 & 5R_1\end{bmatrix}\begin{bmatrix}e_1-e_2\\ e_1+e_2-2e_5\\ e_1-e_2+6e_5-12e_8\end{bmatrix}\zeta(k)\\
&= \zeta^T(k)E_1^T\hat{R}_1E_1\zeta(k) \qquad \text{(C24)}
\end{aligned}
$$

where $\hat{R}_1=\begin{bmatrix}R_1 & 0 & 0\\ 0 & 3R_1 & 0\\ 0 & 0 & 5R_1\end{bmatrix}$, $E_1=\begin{bmatrix}e_1-e_2\\ e_1+e_2-2e_5\\ e_1-e_2+6e_5-12e_8\end{bmatrix}$.

Based on $R_2>0$, using (A3) to estimate R_2-dependent summation term, $\mathcal{S}_2$, in (C16) yields

$$
\begin{aligned}
\mathcal{S}_2 &= h_{12}\sum_{i=k-d}^{k-h_1-1}\eta^T(i)R_2\eta(i)+h_{12}\sum_{i=k-h_2}^{k-d-1}\eta^T(i)R_2\eta(i)\\
&\geqslant \frac{h_{12}}{d-h_1}\begin{bmatrix}x(k-h_1)-x(k-d)\\ x(k-h_1)+x(k-d)-2\sum\limits_{i=k-d}^{k-h_1}\frac{x(i)}{h_{1d}}\\ x(k-h_1)-x(k-d)+6\sum\limits_{i=k-d}^{k-h_1}\frac{x(i)}{h_{1d}}-12\sum\limits_{i=k-d}^{k-h_1}\sum\limits_{j=i}^{k-h_1}\frac{x(j)}{h_{1d}(h_{1d}+1)}\end{bmatrix}^T\begin{bmatrix}R_2 & 0 & 0\\ 0 & 3R_2 & 0\\ 0 & 0 & 5R_2\end{bmatrix}\begin{bmatrix}\star\end{bmatrix}\\
&\quad +\frac{h_{12}}{h_2-d}\begin{bmatrix}x(k-d)-x(k-h_2)\\ x(k-d)+x(k-h_2)-2\sum\limits_{i=k-h_2}^{k-d}\frac{x(i)}{h_{2d}}\\ x(k-d)-x(k-h_2)+6\sum\limits_{i=k-h_2}^{k-d}\frac{x(i)}{h_{2d}}-12\sum\limits_{i=k-h_2}^{k-d}\sum\limits_{j=i}^{k-d}\frac{x(j)}{h_{2d}(h_{2d}+1)}\end{bmatrix}^T\begin{bmatrix}R_2 & 0 & 0\\ 0 & 3R_2 & 0\\ 0 & 0 & 5R_2\end{bmatrix}\begin{bmatrix}\star\end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{h_{12}}{d-h_1}\begin{bmatrix}(e_2-e_3)\zeta(k)\\(e_2+e_3-2e_6)\zeta(k)\\(e_2-e_3-6e_6+12e_9)\zeta(k)\end{bmatrix}^T\begin{bmatrix}R_2&0&0\\0&3R_2&0\\0&0&5R_2\end{bmatrix}\begin{bmatrix}(e_2-e_3)\zeta(k)\\(e_2+e_3-2e_6)\zeta(k)\\(e_2-e_3-6e_6+12e_9)\end{bmatrix}\\
&\quad+\frac{h_{12}}{h_2-d}\begin{bmatrix}(e_3-e_4)\zeta(k)\\(e_3+e_4-2e_7)\zeta(k)\\(e_3-e_4+6e_7-12e_{10}\zeta(k)\end{bmatrix}^T\begin{bmatrix}R_2&0&0\\0&3R_2&0\\0&0&5R_2\end{bmatrix}\begin{bmatrix}(e_3-e_4)\zeta(k)\\(e_3+e_4-2e_7)\zeta(k)\\(e_3-e_4+6e_7-12e_{10}\zeta(k)\end{bmatrix}\\
&= \frac{h_{12}}{d-h_1}\zeta^T(k)E_2^T\hat{R}_2E_2\zeta(k)+\frac{h_{12}}{h_2-d}\zeta^T(k)E_3^T\hat{R}_2E_3\zeta(k)
\end{aligned}\tag{C25}
$$

where

$$
\hat{R}_2=\begin{bmatrix}R_2&0&0\\0&3R_2&0\\0&0&5R_2\end{bmatrix},\quad E_2=\begin{bmatrix}e_2-e_3\\e_2+e_3-2e_6\\e_2-e_3-6e_6+12e_9\end{bmatrix},\quad E_3=\begin{bmatrix}e_3-e_4\\e_3+e_4-2e_7\\e_3-e_4+6e_7-12e_{10}\end{bmatrix}
$$

Furthermore, for any matrices S_1 and S_2, using (A4) to estimate (C25) yields

$$
\begin{aligned}
\mathcal{S}_2 &\geqslant \frac{h_{12}}{d-h_1}\zeta^T(k)E_2^T\hat{R}_2E_2\zeta(k)+\frac{h_{12}}{h_2-d}\zeta^T(k)E_3^T\hat{R}_2E_3\zeta(k)\\
&= \zeta^T(k)\begin{bmatrix}E_2\\E_3\end{bmatrix}^T\begin{bmatrix}\frac{h_{12}}{d-h_1}\hat{R}_2&0\\0&\frac{h_{12}}{h_2-d}\hat{R}_2\end{bmatrix}\begin{bmatrix}E_2\\E_3\end{bmatrix}\zeta(k)\\
&\geqslant \zeta^T(k)\begin{bmatrix}E_2\\E_3\end{bmatrix}^T\begin{bmatrix}\hat{R}_2+\frac{h_2-d}{h_{12}}(\hat{R}_2-S_2\hat{R}_2^{-1}S_2^T)&\frac{h_2-d}{h_{12}}S_1+\frac{d-h_1}{h_{12}}S_2\\\frac{h_2-d}{h_{12}}S_1^T+\frac{d-h_1}{h_{12}}S_2^T&\hat{R}_2+\frac{d-h_1}{h_{12}}(\hat{R}_2-S_1^T\hat{R}_2^{-1}S_1)\end{bmatrix}\begin{bmatrix}E_2\\E_3\end{bmatrix}\zeta(k)\\
&= \zeta^T(k)\begin{bmatrix}E_2\\E_3\end{bmatrix}^T\left(\begin{bmatrix}\hat{R}_2+\frac{h_2-d}{h_{12}}\hat{R}_2&\frac{h_2-d}{h_{12}}S_1+\frac{d-h_1}{h_{12}}S_2\\\frac{h_2-d}{h_{12}}S_1^T+\frac{d-h_1}{h_{12}}S_2^T&\hat{R}_2+\frac{d-h_1}{h_{12}}\hat{R}_2\end{bmatrix}+\begin{bmatrix}-\frac{h_2-d}{h_{12}}S_2\hat{R}_2^{-1}S_2^T&0\\0&-\frac{d-h_1}{h_{12}}S_1^T\hat{R}_2^{-1}S_1\end{bmatrix}\right)\begin{bmatrix}E_2\\E_3\end{bmatrix}\zeta(k)\\
&= \zeta^T(k)\left(\Upsilon_4(d)-\bar{\Upsilon}_4(d)\right)\zeta(k)
\end{aligned}\tag{C26}
$$

where

$$
\Upsilon_4(d)=\begin{bmatrix}E_2\\E_3\end{bmatrix}^T\begin{bmatrix}\frac{2h_2-d-d_1}{h_{12}}\hat{R}_2&\frac{h_2-d}{h_{12}}S_1+\frac{d-h_1}{h_{12}}S_2\\ *&\frac{h_2+d-2h_1}{h_{12}}\hat{R}_2\end{bmatrix}\begin{bmatrix}E_2\\E_3\end{bmatrix},\quad \bar{\Upsilon}_4(d)=\begin{bmatrix}E_2\\E_3\end{bmatrix}^T\begin{bmatrix}\frac{h_2-d}{h_{12}}S_2\hat{R}_2^{-1}S_2^T&0\\0&\frac{d-h_1}{h_{12}}S_1^T\hat{R}_2^{-1}S_1\end{bmatrix}\begin{bmatrix}E_2\\E_3\end{bmatrix}
$$

Step 5: give the upper bound of $\Delta V(x_k)$ by combining the above discussions.

Based on (C16), (C23), (C24), and (C26), the forward difference of Lyapunov function, $\Delta V(x_k)$, is estimated as

$$
\begin{aligned}
\Delta V(x_k) &\leqslant \zeta^T(k)\left[\Psi(d^3,d^2,d)+\Upsilon_2+\Upsilon_3-\Upsilon_4(d)+\bar{\Upsilon}_4(d)\right]\zeta(k)\\
&= \zeta^T(k)\left[d^3E_A^TP_AE_A+d^2He\left\{E_A^TP_BE_B\right\}+\Upsilon_1(d)+\Upsilon_2+\Upsilon_3-\Upsilon_4(d)+\bar{\Upsilon}_4(d)\right]\zeta(k)\\
&= \zeta^T(k)\left[d^3E_A^TP_AE_A+d^2He\left\{E_A^TP_BE_B\right\}+\Upsilon(d)+\bar{\Upsilon}_4(d)\right]\zeta(k)\qquad \textit{with}\ \Upsilon(d)=\Upsilon_1(d)+\Upsilon_2+\Upsilon_3-\Upsilon_4(d)\\
&= d^3\left[\zeta^T(k)E_A^TP_AE_A\zeta(k)\right]+d^2\left[\zeta^T(k)\mathrm{He}\{E_A^TP_BE_B\}\zeta(k)\right]+\zeta^T(k)\Phi(d)\zeta(k)\quad \textit{with}\ \ \Phi(d)=\Upsilon(d)+\bar{\Upsilon}_4(d)
\end{aligned}\tag{C27}
$$

Step 6: deal with the time-varying delay d in (C27) *by using the proposed Lemma 3, convex combination technique, and Schur completement.*

For any matrices, N_1 and N_2, with appropriate dimensions, it follows from Lemma 3 that

$$
\begin{aligned}
\bar{\Psi}(d) &= \Psi_{11}(d)+\mathrm{diag}\{\bar{\Upsilon}_4(d),0_{2n\times 2n}\}\\
&= \begin{bmatrix}\Upsilon(d)+\bar{\Upsilon}_4(d)-dN_2E_A-dE_A^TN_2^T&N_2-dE_A^TN_1^T+dE_B^TP_B^T\\ *&dP_A+N_1+N_1^T\end{bmatrix}\\
&= \begin{bmatrix}\Phi(d)-dN_2E_A-dE_A^TN_2^T&N_2-dE_A^TN_1^T+dE_B^TP_B^T\\ *&dP_A+N_1+N_1^T\end{bmatrix}<0,\quad \forall d\in[h_1,h_2]
\end{aligned}\tag{C28}
$$

$$\Downarrow$$

$$
d^3\left[\zeta^T(k)E_A^TP_AE_A\zeta(k)\right]+d^2\left[\zeta^T(k)\mathrm{He}\{E_A^TP_BE_B\}\zeta(k)\right]+\zeta^T(k)\Phi(d)\zeta(k)<0,\qquad \forall d\in[h_1,h_2]\tag{C29}
$$

On the other side, it follows from the convexity of $\bar{\Psi}(d)$ with respect to d that

$$
\bar{\Psi}(h_i)<0, i=1,2\tag{C30}
$$

$$\Downarrow$$

$$
\bar{\Psi}(d)<0,\forall d\in[h_1,h_2]\tag{C31}
$$

and it follows from Schur complement that

$$\Psi(h_i) =< 0, i = 1, 2 \tag{C32}$$
$$\Downarrow$$
$$\bar{\Psi}(h_i) < 0, i = 1, 2 \tag{C33}$$

More specifically, the derivation from (C32) to (C33) is given as follows

$$\underbrace{\Psi(h_1) = \begin{bmatrix} \Psi_{11}(d) & \Psi_{12}(d) \\ \Psi_{12}^T(d) & -\hat{R}_2 \end{bmatrix}\Big|_{d=h_1} = \begin{bmatrix} \begin{bmatrix} \Upsilon(h_1) - h_1N_2E_A - h_1E_A^TN_2^T & N_2 - h_1E_A^TN_1^T + h_1E_B^TP_B^T \\ * & h_1P_A + N_1 + N_1^T \end{bmatrix} & \begin{bmatrix} E_2^TS_2 \\ 0_{2n\times 3n} \end{bmatrix} \\ \begin{bmatrix} S_2^TE_2 & 0_{3n\times 2n} \end{bmatrix} & -\hat{R}_2 \end{bmatrix} < 0}$$
$$\Downarrow$$
$$\overbrace{\bar{\Psi}(h_1) = \begin{bmatrix} \Upsilon(h_1) + E_2^TS_2\hat{R}_2^{-1}S_2^TE_2 - h_1N_2E_A - h_1E_A^TN_2^T & N_2 - h_1E_A^TN_1^T + h_1E_B^TP_B^T \\ * & h_1P_A + N_1 + N_1^T \end{bmatrix} < 0}$$

and

$$\underbrace{\Psi(h_2) = \begin{bmatrix} \Psi_{11}(d) & \Psi_{12}(d) \\ \Psi_{12}^T(d) & -\hat{R}_2 \end{bmatrix}\Big|_{d=h_2} = \begin{bmatrix} \begin{bmatrix} \Upsilon(h_2) - h_2N_2E_A - h_2E_A^TN_2^T & N_2 - h_2E_A^TN_1^T + h_2E_B^TP_B^T \\ * & h_2P_A + N_1 + N_1^T \end{bmatrix} & \begin{bmatrix} E_3^TS_1^T \\ 0_{2n\times 3n} \end{bmatrix} \\ \begin{bmatrix} S_1E_3 & 0_{3n\times 2n} \end{bmatrix} & -\hat{R}_2 \end{bmatrix} < 0}$$
$$\Downarrow$$
$$\overbrace{\bar{\Psi}(h_2) = \begin{bmatrix} \Upsilon(h_2) + E_3^TS_1^T\hat{R}_2^{-1}S_1E_3 - h_2N_2E_A - h_2E_A^TN_2^T & N_2 - h_2E_A^TN_1^T + h_2E_B^TP_B^T \\ * & h_2P_A + N_1 + N_1^T \end{bmatrix} < 0}$$

Step 7: Summarize the conditions for ensuring the negative-definiteness of forward difference of functional.
Based on the above discussion, the following relationship is summarized

$$\left.\begin{matrix}\left.\begin{matrix}\left.\begin{matrix} Construct\ \text{(C2)} \\ Calculations \end{matrix}\right\} \Longrightarrow \left\{\begin{matrix} \text{(C9)} \\ \text{(C14)} \\ \text{(C15)} \end{matrix}\right\} \Longrightarrow \text{(C16)} \\ \left.\begin{matrix} \left.\begin{matrix} Construct\ V_1(k) \\ Calculations \end{matrix}\right\} \Longrightarrow \text{(C9) } i.e., \text{ (C10)} \\ Calculations\ from\ \text{(C17)}\ to\ \text{(C23)} \end{matrix}\right\} \Longrightarrow \text{(C23)} \\ \left.\begin{matrix} Lemma\ 1 \\ \textbf{Theorem 1}: \mathbf{R_1} > \mathbf{0} \end{matrix}\right\} \Longrightarrow \text{(C24)} \\ \left.\begin{matrix} \left.\begin{matrix} Lemma\ 1 \\ \textbf{Theorem 1}: \mathbf{R_2} > \mathbf{0} \end{matrix}\right\} \Longrightarrow (C25) \\ Lemma\ 2 \end{matrix}\right\} \Longrightarrow \text{(C26)} \end{matrix}\right\} \Longrightarrow \text{(C27)} \\ \left.\begin{matrix} Lemma\ 3 \\ \left.\begin{matrix} Convex\ combination\ technique \\ \left.\begin{matrix} Schur\ completement \\ \textbf{Theorem 1}: \text{(C1) } i.e., \text{ (C32)} \end{matrix}\right\} \Longrightarrow \text{(C33) } i.e., \text{ (C30)} \end{matrix}\right\} \Longrightarrow \text{(C31) } i.e., \text{ (C28)} \end{matrix}\right\} \Longrightarrow \text{(C29)} \end{matrix}\right\} \Rightarrow \Delta V(x_k) \leqslant -\varepsilon||x(k)||^2 \tag{C34}$$

It is found that conditions for ensuring the negative-definiteness of forward difference of functional is $R_i > 0$, $i = 1, 2$ and (C1) holds for both $d = h_1$ and $d = h_2$.

Step 8: Give a conclusion.

Finally, by considering the previous discussion, system (A1) is stable if $P > 0$, $Q_i > 0$, $R_i > 0$, $i = 1, 2$ and (C1) holds for both $d = h_1$ and $d = h_2$. This completes the proof. ■

Remark 2. The negativity condition of $\Delta V(x_k)$ shown in (C29) can be rewritten as $\begin{bmatrix} \zeta(k) \\ dE_A\zeta(k) \end{bmatrix}^T \bar{\Psi}_1(d) \begin{bmatrix} \zeta(k) \\ dE_A\zeta(k) \end{bmatrix} < 0$ with $\bar{\Psi}_1(d) = \begin{bmatrix} \Phi(d) & dE_B^TP_B^T \\ * & dP_A \end{bmatrix}$. This condition cannot be directly guaranteed by requiring $\bar{\Psi}_1(d) < 0$ due to the constraint of zero-value diagonal elements in P_A. On the contrary, due to the usage of Lemma 3, such constraint is relaxed by injecting free matrix N_1 shown in (C28). The dimension of the injected matrix is related to computational burden of Theorem 1, thus, as discussed in Remark 1, it is necessary to minimize the number of rows of E_A in (C29) during the re-expression of (C27). As shown in $\Gamma_2(d^2)$ and $\Gamma_3(d^3)$, two treatments are used for this purpose, including replacing the terms dependent on high-dimension matrices (E_a, E_b, and E_d) in (C18) by the ones with lower-dimension matrices (e_a, e_b, and e_c), and carrying out the transformation from (C20) to (C21) (E_A will become $E_A = \mathrm{col}\{e_a, e_b, e_c\}$ if no such transformation). Moreover, those treatments do not lead to extra conservatism since they are all based on equivalent calculation.

Remark 3. In [2–5], the usage of auxiliary function-based, polynomial-based, and Bessel-based inequalities achieves the improvement of the WLI-based criterion [6]. However, in order to avoid introducing d^3-dependent terms, those inequalities are not used to estimate $\mathcal{S}_2$ [3, 5] and $w(k)$, which is necessary as discussion in [7], is not included in the Lyapunov functions [2–5], both of which affect the benefit from the increase of estimation accurate caused by tighter inequalities. During the development of Theorem 1, these drawbacks are overcome by using the matrix-injection-based transformation method shown in Lemma 3.

Remark 4. In [8] and [9], wherein $w(k)$ is included and $\mathcal{S}_2$ is estimated by improved inequalities, another method is used to avoid the information of high-order function of d and its key idea is to replace $\zeta(k)$ used in (C27) by $\bar{\zeta}(k) = \mathrm{col}\{\zeta(k), \sum_{i=k-d}^{k-h_1} x(i)$, $\sum_{i=k-h_2}^{k-d} x(i)$, $h_{1d}v_1(k), h_{2d}v_2(k)$. Then the negativity condition of $\Delta V(x_k)$ can be rewritten as the form of $\bar{\zeta}^T(k)\bar{\Psi}_2(d)\bar{\zeta}(k) < 0$, in which $\bar{\Psi}_2(d)$ is convex w.r.t. d. However, this condition cannot be directly guaranteed by requiring $\bar{\Psi}_2(d) < 0$ due to the existence of zero-value diagonal elements therein caused by the absence of quadratic terms related to $\sum_{i=k-d}^{k-h_1} x(i)$, $\sum_{i=k-h_2}^{k-d} x(i)$, $h_{1d}v_1(k)$, and $h_{2d}v_2(k)$. In order to introduce necessary quadratic terms, the summation terms of Lyapunov function are augmented in [8] and several zero-value terms based on the linear relationships among the vectors in $\bar{\zeta}(k)$ are added into $\Delta V(x_k)$ in [9]. On the contrary, the usage of Lemma 3 can be considered as replacing $\zeta(k)$ used in (C27) by $\hat{\zeta}(k) = \mathrm{col}\{\zeta(k), dE_A\zeta(k)\}$ and injecting two matrices N_1 and N_2. Since the number of rows in $\hat{\zeta}(k)$ (i.e., $12n$) is smaller than that of $\bar{\zeta}(k)$ (i.e., $14n$), the injected matrices have less decision variables and the LMI-based condition (C1) has smaller order, which in turn implies lower complexity (See Table **??** for details).

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