SCIENCE CHINA
Information Sciences

• RESEARCH PAPER •

May 2023, Vol. 66 152204:1–152204:14
https://doi.org/10.1007/s11432-022-3633-9

Fixed-time stabilization of discontinuous spatiotemporal neural networks with time-varying coefficients via aperiodically switching control

Xiaofang HU[1,2,3], Leimin WANG[1,2,3*], Chuan-Ke ZHANG[1,2,3], Xiongbo WAN[1,2,3] & Yong HE[1,2,3]

[1] *School of Automation, China University of Geosciences, Wuhan* 430074*, China;*
[2] *Hubei Key Laboratory of Advanced Control and Intelligent Automation for Complex Systems, Wuhan* 430074*, China;*
[3] *Engineering Research Center of Intelligent Technology for Geo-Exploration, Ministry of Education, Wuhan* 430074*, China*

Received 23 July 2022/Revised 21 September 2022/Accepted 23 October 2022/Published online 7 April 2023

Abstract This paper focuses on the challenge of fixed-time control for spatiotemporal neural networks (SNNs) with discontinuous activations and time-varying coefficients. A novel fixed-time convergence lemma is proposed, which facilitates the handling of time-varying coefficients of SNNs and relaxes the restriction on the non-positive definiteness of the derivative of the Lyapunov function. Besides, a more flexible and economical aperiodically switching control technique is presented to stabilize SNNs within a fixed time, effectively reducing the amount of information transmission and control costs. Under the newly established fixed-time convergence lemma and aperiodically switching controller, many more general algebraic conditions are deduced to ensure the fixed-time stabilization of SNNs. Numerical examples are provided to manifest the validity of the results.

Keywords spatiotemporal neural networks, discontinuous activations, time-varying coefficients, fixed-time stabilization, aperiodically switching control

Citation Hu X F, Wang L M, Zhang C-K, et al. Fixed-time stabilization of discontinuous spatiotemporal neural networks with time-varying coefficients via aperiodically switching control. Sci China Inf Sci, 2023, 66(5): 152204, https://doi.org/10.1007/s11432-022-3633-9

1 Introduction

It is well established that the phenomenon of spatial diffusion is ubiquitous, such as chemical reactions [1], disease transmission [2], and biological populations [3]. In fact, whether it is the transmission of information between cells in biological neural networks (NNs), or the movement of electrons in non-uniform electromagnetic fields in the implementation of circuits in artificial NNs, the phenomenon of spatial diffusion also exists [4, 5]. To quantitatively study the phenomenon of spatial diffusion, NNs with reaction-diffusion terms, also known as spatiotemporal neural networks (SNNs), were suggested in [6], showing that the evolution of the states of NNs depends on time variables and spatial variables. Nowadays, SNNs have slowly become a hot research topic, and have been widely applied to engineering fields, for example, the shortest path solving, pattern recognition, and image processing [7,8]. Hence, the study of SNNs has essential theoretical significance and application value.

Noteworthy, most existing research on SNNs focuses on systems with constant parameters and continuous activations [9–11]. Nevertheless, environmental mutations and external disturbances affect the information transmission between neurons [12], that is, the diffusion coefficients and connection weights of SNNs usually alter over time. Thence, it is important to establish a spatiotemporal system with time-varying coefficients. Additionally, NNs with discontinuous activations frequently appear in practice applications because of the switching of the electronic circuit and dry friction [13]. The dynamic behavior

* Corresponding author (email: wangleimin@cug.edu.cn)

of NNs with discontinuous activations was studied for the first time in [14], by virtue of the differential inclusion theory. After that, many researchers examined NNs with discontinuous activations. Compared with NNs with continuous activations, NNs with discontinuous activations are more advantageous in handling linear and nonlinear programming problems [15, 16]. Based on the above two aspects, time-varying coefficients and discontinuous activations are incorporated into this paper to build more general and practical SNNs.

Recently, the fixed-time stabilization (FXS) of SNNs has drawn much attention since it retains the advantages of strong robustness and anti-interference ability of finite-time stabilization, and the convergence time is independent of the initial value [17–20]. To stabilize SNNs within a fixed time, many excellent results have been reported [21–23]. A nonlinear controller was shown in [21] to guarantee the FXS of the error states of SNNs without delays. In [22], an integral type controller was constructed to tackle the challenge of fixed-time control for delayed SNNs. Different from the conventional feedback control schemes in [21,22], an adaptive control algorithm was developed for SNNs in [23], which can automatically adjust the feedback gains of the controller to adapt to dynamic alterations of the controlled system and external environment.

Note that to achieve FXS of SNNs, the control schemes designed in [21–23] are all continuous. In contrast to the continuous control method, the intermittent control strategy, as a discontinuous control method, can effectively decrease the amount of information transmission and control costs. Presently, the research related to the stabilization of SNNs via intermittent control methods mainly focuses on the exponential and finite-time stabilization [24,25]. What they have in common is that the established Lyapunov function $\mathscr{V}(t)$ satisfies $\dot{\mathscr{V}}(t) \leqslant \lambda \mathscr{V}(t)$ when the system is in the non-control phase, which indicates that the state trajectory of the system may increase exponentially. However, existing analytical methods cannot handle the increment of the Lyapunov function in the non-control phase to achieve stabilization in a fixed time. That is, it is challenging to guarantee the FXS of SNNs via strictly intermittent control. This motivates us to discuss the FXS of SNNs by utilizing a semi-intermittent control strategy.

Semi-intermittent control is actually a switching control, which is divided into a strong control phase and a weak control phase. In [26], a periodically semi-intermittent control technique was established to realize FXS. In [27], the requirement of a fixed control period and width was eliminated, and an aperiodically semi-intermittent control scheme was suggested to attain FXS. However, the theoretical results based on semi-intermittent control acquired in [26, 27] are only appropriate for systems with constant coefficients, and they all restrict the derivative of the Lyapunov function to be non-positive. This fact inspires two major issues to be considered in this paper. One is how to explore a novel fixed-time convergence method based on semi-intermittent control to deal with the time-varying parameters of SNNs. The other is whether the constraint on the derivative of the Lyapunov function can be further relaxed so that it can be non-negative.

Driven by the above discussions, this paper focuses on the FXS of SNNs with discontinuous activations and time-varying coefficients via aperiodically switching control. The innovations are summarized below.

- A novel fixed-time convergence lemma is demonstrated. It has two major advantages: one is that it facilitates the treatment of the time-varying coefficients of SNNs. The other is that the derivative of the Lyapunov function can be non-negative at some moments. Moreover, the newly proposed lemma contains some existing results [26–30] as special cases.

- A more general aperiodically switching control approach is proposed. The approach only needs to stabilize SNNs within a fixed time during the strong control phase. When the controller is in the weak control phase, SNNs are only required to achieve Lyapunov stability. Compared to the control strategies shown in [22, 31–35], the designed controller is more flexible and economical.

- Time-varying parameters and discontinuous activations are integrated into the SNNs, which is a generalization of existing SNNs. Several new algebraic criteria are deduced to ensure the FXS of SNNs. The obtained results can be generalized and applied to [36–39].

Notations. $\mathbb{R}_+$ and $\mathbb{Z}_+$ represent the sets of non-negative real numbers and integers, respectively. $\mathbb{R}^n$ stands for n-dimensional Euclidean space. Let $\mathscr{F}_{[\varphi_\iota]}(\varpi_\iota) = \bigcap_{\wp>0}\bigcap_{\mu(\Upsilon)>0}\overline{\mathrm{co}}[\varphi_\iota(B(\varpi_\iota,\wp)\backslash\Upsilon)]$, where $B(\varpi_\iota,\wp) = \{y \in \mathbb{R} : |y-\varpi_\iota| \leqslant \wp\}$, $\overline{\mathrm{co}}[\Omega]$ denotes the closure of the convex hull of Ω, and $\mu(\Upsilon)$ is the Lebesgue measure of Υ. Besides, $\|\varpi(t,\cdot)\| = \left(\int_\Xi \sum_{i=1}^n |\varpi_i(t,\upsilon)|^\varrho \mathrm{d}\upsilon\right)^{\frac{1}{\varrho}}$, where $\Xi = \{(\upsilon_1,\upsilon_2,\ldots,\upsilon_m)^{\mathrm{T}} \| \upsilon_j| \leqslant \nu_j, j=1,2,\ldots,m\}$ is the bounded compact set with smooth boundary $\partial\Xi$ and measure $\mathrm{mes}\Xi > 0$.

2 Problem formulation and preliminaries

Consider SNNs with discontinuous activations and time-varying coefficients as follows:

$$\begin{aligned}\frac{\partial \varpi_i(t,\upsilon)}{\partial t} =& \sum_{j=1}^{m}\frac{\partial}{\partial \upsilon_j}\left(a_{ij}(t)\frac{\partial \varpi_i(t,\upsilon)}{\partial \upsilon_j}\right) - b_i(t)\varpi_i(t,\upsilon)\\ &+\sum_{\iota=1}^{n} c_{i\iota}(t)\varphi_\iota(\varpi_\iota(t,\upsilon)) + \sum_{\iota=1}^{n} d_{i\iota}(t)\varphi_\iota(\varpi_\iota(t-\tau_\iota(t),\upsilon)),\end{aligned} \tag{1}$$

where $i=1,2,\ldots,n$. $\varpi_i(t,\upsilon)$ denotes the state of SNNs at t and $\upsilon\in\Xi$. $a_{ij}(t)$ and $b_i(t)$ are the diffusion and self-feedback coefficients, respectively. $c_{i\iota}(t)$ and $d_{i\iota}(t)$ represent the connection weights. $\tau_\iota(t)$ is the bounded delay and meets $\tau_\iota(t)\in[0,\tau]$. $\varphi_\iota(\cdot)$ stands for the activation function with $\varphi_\iota(0)=0$, and meets the assumptions as follows.

(A1) The function $\varphi_\iota(\cdot)$ is continuous apart from a countable set of isolate points $\{\rho_r^\iota\}$, where the right and left limits $\varphi_\iota^+(\rho_r^\iota)$ and $\varphi_\iota^-(\rho_r^\iota)$ exist. Besides, in every compact subinterval of $\mathbb{R}$, $\varphi_\iota(\cdot)$ has finite discontinuous points.

(A2) For any ϖ_ι, there exist constants ζ_ι, $\xi_\iota>0$, such that $\sup|\phi_\iota|\leqslant\zeta_\iota|\varpi_\iota|+\xi_\iota$, where $\phi_\iota\in\mathscr{F}_{[\varphi_\iota]}(\varpi_\iota)$.

Because of the discontinuity of the activation function $\varphi_\iota(\cdot)$, the solution of system (1) should be defined in the Filippov sense [40]. By virtue of the differential inclusion theory and measurable selection theorem [41], there exists $\phi_\iota(\varpi_\iota)\in\mathscr{F}_{[\varphi_\iota]}(\varpi_\iota)$ satisfying

$$\begin{aligned}\frac{\partial \varpi_i(t,\upsilon)}{\partial t} =& \sum_{j=1}^{m}\frac{\partial}{\partial \upsilon_j}\left(a_{ij}(t)\frac{\partial \varpi_i(t,\upsilon)}{\partial \upsilon_j}\right) - b_i(t)\varpi_i(t,\upsilon)\\ &+\sum_{\iota=1}^{n} c_{i\iota}(t)\phi_\iota(\varpi_\iota(t,\upsilon)) + \sum_{\iota=1}^{n} d_{i\iota}(t)\phi_\iota(\varpi_\iota(t-\tau_\iota(t),\upsilon)).\end{aligned} \tag{2}$$

The initial and Dirichlet boundary values of SNNs are $\varpi_i(\theta,\upsilon)=\Im_i(\theta,\upsilon)$, $(\theta,\upsilon)\in[-\tau,0)\times\Xi$ and $\varpi_i(t,\upsilon)=0$, $(t,\upsilon)\in[-\tau,+\infty)\times\partial\Xi$, respectively.

Remark 1. For general NNs, the dynamic behavior depends only on the time variable. Strictly speaking, the diffusion phenomenon is inevitable when electrons move in a non-uniform electromagnetic field. Thus, considering NNs with reaction-diffusion terms can show the influence of the diffusion phenomenon on the dynamic behavior of NNs. Moreover, in contrast to the NNs with reaction-diffusion terms studied in [35–39], the proposed model has two major improvements. One is that the activation functions are discontinuous. In fact, neuron activations at high gains approximate discontinuous functions, and NNs with discontinuous activations have essential applications in engineering practice [13, 16]. The other is that the coefficients of SNNs are time-varying. Since different interference factors exist in practice, the coefficients of the system are not constant. Consequently, it is more general and practical to incorporate discontinuous activations and time-varying parameters to construct SNNs. It is worth highlighting that because of the discontinuity of the activation functions, the classical theory of differential equations does not apply, and the Lipschitz condition, in the usual sense, no longer holds. Thus, the mathematical complexity induced by the simultaneous presence of discontinuous activations and the Laplace operator in the system brings certain challenges to the theoretical analysis of this study.

Then, many definitions and lemmas are presented.

Definition 1. System (1) is fixed-time stabilizable if for an appropriate controller, there exists a settling time (ST) T independent of the initial value $\Im_i(\theta,\upsilon)$ such that $\lim_{t\to T}|\varpi_i(t,\upsilon)|=0$ and $\varpi_i(t,\upsilon)=0$, $\forall t>T$.

Definition 2. A continuous function $\mathcal{K}:\mathbb{R}_+\to\mathbb{R}_+$ belongs to $\mathscr{K}_\infty$, if $\mathcal{K}$ is strictly monotonically increasing with $\mathcal{K}(0)=0$ and $\lim_{\psi\to+\infty}\mathcal{K}(\psi)=+\infty$.

Lemma 1 ([42])**.** Let $\mathscr{H}(x)$ be a continuous function defined on a bounded compact set $\Omega=\{x=(x_1,x_2,\ldots,x_m)^{\mathrm{T}}\big||x_j|\leqslant\nu_j, j=1,2,\ldots,m\}$ with smooth boundary $\partial\Omega$ and mes$\Omega>0$ and satisfying $\mathscr{H}(x)\big|_{\partial\Omega}=0$. Then for $\varrho\geqslant2$,

$$\int_\Omega|\mathscr{H}(x)|^\varrho\mathrm{d}x\leqslant\frac{\varrho^2\nu_j^2}{4}\int_\Omega|\mathscr{H}(x)|^{\varrho-2}\left|\frac{\partial\mathscr{H}}{\partial x_j}\right|^2\mathrm{d}x. \tag{3}$$

Lemma 2. Let $\mathscr{V}(t,\psi(t)):\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}_+$ be a non-negative continuous function, and there exist $\mathcal{K}_1$, $\mathcal{K}_2\in\mathscr{K}_\infty$, such that $\mathcal{K}_1(\|\psi\|)\leqslant\mathscr{V}(t,\psi(t))\leqslant\mathcal{K}_2(\|\psi\|)$. For convenience, the abbreviation $\mathscr{V}(t)$ denotes $\mathscr{V}(t,\psi(t))$. Besides, the derivative of $\mathscr{V}(t)$ along the trajectories of system (2) satisfies

$$\dot{\mathscr{V}}(t)\leqslant\begin{cases}\mathcal{I}(t)\mathscr{V}(t)-\mathcal{N}(t)\mathscr{V}^p(t)-\mathcal{W}(t)\mathscr{V}^q(t), & t_s\leqslant t<\sigma_s,\\ 0, & \sigma_s\leqslant t<t_{s+1},\end{cases}\tag{4}$$

where $t_s\geqslant 0$, $\sigma_s>0$, $s\in\mathbb{Z}_+$, $0<p<1$, $q>1$. $\mathcal{N}(t)$, $\mathcal{W}(t)$, and $\mathcal{I}(t)$ are continuous functions, and $\mathcal{N}(t)>0$, $\mathcal{W}(t)>0$. Let $\mathcal{I}^+(t)=\max\{\mathcal{I}(t),0\}$. If $\int_{t_0}^{+\infty}\mathcal{I}^+(\varsigma)\mathrm{d}\varsigma<+\infty$, and there exist $\overline{\mathcal{N}}>0$, $\overline{\mathcal{W}}>0$, for any t', t'' such that $\int_{t'}^{t''}\mathcal{N}(\varsigma)\mathrm{d}\varsigma\geqslant\overline{\mathcal{N}}(t''-t')$ and $\int_{t'}^{t''}\mathcal{W}(\varsigma)\mathrm{d}\varsigma\geqslant\overline{\mathcal{W}}(t''-t')$, then $\mathscr{V}(t)=0$ when

$$t\geqslant T_2=t_{s^*}+\frac{1}{\overline{\mathcal{N}}\mathcal{E}_1(1-\chi)(1-p)},\tag{5}$$

where $s^*=\min\{s\in\mathbb{Z}_+:t_s\geqslant T_1=t_0+\frac{1}{\overline{\mathcal{W}}\mathcal{E}_2(1-\chi)(q-1)}\}$, $\chi=\limsup_{s\to\infty}\frac{t_{s+1}-\sigma_s}{t_{s+1}-t_s}$, $\mathcal{E}_1=\mathrm{e}^{-(1-p)\int_{t_0}^{+\infty}\mathcal{I}^+(\varsigma)\mathrm{d}\varsigma}$, and $\mathcal{E}_2=\mathrm{e}^{(1-q)\int_{t_0}^{+\infty}\mathcal{I}^+(\varsigma)\mathrm{d}\varsigma}$.

Proof. The proof process is divided into the following steps.

Step 1. The solution of (2) is Lyapunov stable.

Since $\mathcal{N}(t)>0$ and $\mathcal{W}(t)>0$, one has

$$\dot{\mathscr{V}}(t)\leqslant\mathcal{I}(t)\mathscr{V}(t)\leqslant\mathcal{I}^+(t)\mathscr{V}(t),\ \forall t\in\mathbb{R}_+.\tag{6}$$

Multiplying both sides of (6) by $\mathscr{V}^{-1}(t)$ and integrating from t_0 to t, one deduces

$$\mathscr{V}(t)\leqslant\mathscr{V}(t_0)\mathrm{e}^{\int_{t_0}^{t}\mathcal{I}^+(\varsigma)\mathrm{d}\varsigma}\leqslant\mathcal{K}_2(\|\psi(t_0)\|)\mathrm{e}^{\int_{t_0}^{+\infty}\mathcal{I}^+(\varsigma)\mathrm{d}\varsigma}.\tag{7}$$

By virtue of $\mathscr{V}(t)\geqslant\mathcal{K}_1(\|\psi\|)$,

$$\|\psi(t)\|\leqslant\mathcal{K}_1^{-1}\left(\mathcal{K}_2(\|\psi(t_0)\|)\mathrm{e}^{\int_{t_0}^{+\infty}\mathcal{I}^+(\varsigma)\mathrm{d}\varsigma}\right).\tag{8}$$

For any $\tilde{\epsilon}>0$, we select $\tilde{\delta}=\mathcal{K}_2^{-1}(\mathcal{K}_1(\tilde{\epsilon})\mathrm{e}^{-\int_{t_0}^{+\infty}\mathcal{I}^+(\varsigma)\mathrm{d}\varsigma})$. Then, it can be deduced from (8) that $\|\psi(t)\|<\tilde{\epsilon}$ holds for any initial state $\|\psi(t_0)\|<\tilde{\delta}$. That is, the solution of (2) is Lyapunov stable.

Step 2. When $t\geqslant T_1$, $\mathscr{V}(t)\leqslant 1$ holds.

From (4), it is easy to obtain

$$\dot{\mathscr{V}}(t)\leqslant\begin{cases}\mathcal{I}^+(t)\mathscr{V}(t)-\mathcal{W}(t)\mathscr{V}^q(t), & t_s\leqslant t<\sigma_s,\\ 0, & \sigma_s\leqslant t<t_{s+1}.\end{cases}\tag{9}$$

Multiplying both sides of (9) by $\mathscr{V}^{-q}(t)\mathrm{e}^{-(1-q)\int_{t_0}^{t}\mathcal{I}^+(\varsigma)\mathrm{d}\varsigma}$, we have

$$\dot{\mathcal{Q}}_1(t)\leqslant\begin{cases}-\mathcal{W}(t)\mathrm{e}^{-(1-q)\int_{t_0}^{t}\mathcal{I}^+(\varsigma)\mathrm{d}\varsigma}, & t_s\leqslant t<\sigma_s,\\ 0, & \sigma_s\leqslant t<t_{s+1},\end{cases}\tag{10}$$

where $\mathcal{Q}_1(t)=\mathscr{V}^{1-q}(t)\mathrm{e}^{-(1-q)\int_{t_0}^{t}\mathcal{I}^+(\varsigma)\mathrm{d}\varsigma}/(1-q)$.

When $t_s\leqslant t<\sigma_s$,

$$\begin{aligned}\mathcal{Q}_1(t)-\mathcal{Q}_1(t_0)&\leqslant-\sum_{\imath=0}^{s-1}\int_{t_\imath}^{\sigma_\imath}\mathcal{W}(\varsigma)\mathrm{e}^{-(1-q)\int_{t_0}^{\varsigma}\mathcal{I}^+(\hbar)\mathrm{d}\hbar}\mathrm{d}\varsigma-\int_{t_s}^{t}\mathcal{W}(\varsigma)\mathrm{e}^{-(1-q)\int_{t_0}^{\varsigma}\mathcal{I}^+(\hbar)\mathrm{d}\hbar}\mathrm{d}\varsigma\\&\leqslant-\sum_{\imath=0}^{s-1}\int_{t_\imath}^{\sigma_\imath}\mathcal{W}(\varsigma)\mathrm{d}\varsigma-\int_{t_s}^{t}\mathcal{W}(\varsigma)\mathrm{d}\varsigma\\&\leqslant-\overline{\mathcal{W}}\left[\sum_{\imath=0}^{s-1}(\sigma_\imath-t_\imath)+t-t_s\right].\end{aligned}\tag{11}$$

Based on the definition of χ, one deduces

$$\sum_{\imath=0}^{s-1}(\sigma_\imath-t_\imath)+t-t_s=t-t_0-\sum_{\imath=0}^{s-1}(t_{\imath+1}-\sigma_\imath)\geqslant t-t_0-\chi\sum_{\imath=0}^{s-1}(t_{\imath+1}-t_\imath)\geqslant(1-\chi)(t-t_0). \tag{12}$$

On the other hand,

$$\mathcal{Q}_1(t)-\mathcal{Q}_1(t_0)=\frac{\mathscr{V}^{1-q}(t)\mathrm{e}^{-(1-q)\int_{t_0}^{t}\mathcal{I}^+(\varsigma)\mathrm{d}\varsigma}-\mathscr{V}^{1-q}(t_0)}{1-q}\geqslant\frac{\mathscr{V}^{1-q}(t)}{\mathcal{E}_2(1-q)}. \tag{13}$$

Substituting (12) and (13) into (11) yields

$$\frac{\mathscr{V}^{1-q}(t)}{\mathcal{E}_2(1-q)}\leqslant-\overline{\mathcal{W}}(1-\chi)(t-t_0). \tag{14}$$

If $t\geqslant T_1$, then

$$\frac{\mathscr{V}^{1-q}(t)}{\mathcal{E}_2(1-q)}\leqslant\frac{1}{\mathcal{E}_2(1-q)}. \tag{15}$$

Note that $q>1$. Thence, $\mathscr{V}(t)\leqslant 1$ holds if $t\geqslant T_1$.

When $\sigma_s\leqslant t<t_{s+1}$, it can be acquired from (10) that

$$\mathcal{Q}_1(t)-\mathcal{Q}_1(t_0)\leqslant-\sum_{\imath=0}^{s}\int_{t_\imath}^{\sigma_\imath}\mathcal{W}(\varsigma)\mathrm{e}^{-(1-q)\int_{t_0}^{\varsigma}\mathcal{I}^+(\hbar)\mathrm{d}\hbar}\mathrm{d}\varsigma\leqslant-\overline{\mathcal{W}}\sum_{\imath=0}^{s}(\sigma_\imath-t_\imath). \tag{16}$$

Moreover,

$$\sum_{\imath=0}^{s}(\sigma_\imath-t_\imath)=t_{s+1}-t_0-\sum_{\imath=0}^{s}(t_{\imath+1}-\sigma_\imath)\geqslant t_{s+1}-t_0-\chi\sum_{\imath=0}^{s}(t_{\imath+1}-t_\imath)\geqslant(1-\chi)(t-t_0). \tag{17}$$

Similar to (13)–(15), $\mathscr{V}(t)\leqslant 1$ holds if $t\geqslant T_1$.

Step 3. When $t\geqslant T_2$, $\mathscr{V}(t)=0$ holds.

From (4), it can be obtained that

$$\dot{\mathscr{V}}(t)\leqslant\begin{cases}\mathcal{I}^+(t)\mathscr{V}(t)-\mathcal{N}(t)\mathscr{V}^p(t), & t_s\leqslant t<\sigma_s,\\ 0, & \sigma_s\leqslant t<t_{s+1}.\end{cases} \tag{18}$$

Multiplying $\mathscr{V}^{-p}(t)\mathrm{e}^{-(1-p)\int_{t_0}^{t}\mathcal{I}^+(\varsigma)\mathrm{d}\varsigma}$ on both sides of (18), one deduces

$$\dot{\mathcal{Q}}_2(t)\leqslant\begin{cases}-\mathcal{N}(t)\mathrm{e}^{-(1-p)\int_{t_0}^{t}\mathcal{I}^+(\varsigma)\mathrm{d}\varsigma}, & t_s\leqslant t<\sigma_s,\\ 0, & \sigma_s\leqslant t<t_{s+1},\end{cases} \tag{19}$$

where $\mathcal{Q}_2(t)=\mathscr{V}^{1-p}(t)\mathrm{e}^{-(1-p)\int_{t_0}^{t}\mathcal{I}^+(\varsigma)\mathrm{d}\varsigma}/(1-p)$.

When $t_s\leqslant t<\sigma_s$, the proof is divided into two cases.

Case I. $t_\jmath<T_1<\sigma_\jmath$, $\jmath<s$, and $\jmath\in\mathbb{Z}_+$.

From (19), it yields

$$\begin{aligned}\mathcal{Q}_2(t)-\mathcal{Q}_2(T_1)\leqslant&-\int_{T_1}^{\sigma_\jmath}\mathcal{N}(\varsigma)\mathrm{e}^{-(1-p)\int_{t_0}^{\varsigma}\mathcal{I}^+(\hbar)\mathrm{d}\hbar}\mathrm{d}\varsigma-\sum_{\imath=\jmath+1}^{s-1}\int_{t_\imath}^{\sigma_\imath}\mathcal{N}(\varsigma)\mathrm{e}^{-(1-p)\int_{t_0}^{\varsigma}\mathcal{I}^+(\hbar)\mathrm{d}\hbar}\mathrm{d}\varsigma\\&-\int_{t_s}^{t}\mathcal{N}(\varsigma)\mathrm{e}^{-(1-p)\int_{t_0}^{\varsigma}\mathcal{I}^+(\hbar)\mathrm{d}\hbar}\mathrm{d}\varsigma\\\leqslant&-\mathcal{E}_1\overline{\mathcal{N}}\left[\sigma_\jmath-T_1+\sum_{\imath=\jmath+1}^{s-1}(\sigma_\imath-t_\imath)+t-t_s\right].\end{aligned} \tag{20}$$

Besides,

$$\sigma_\jmath - T_1 + \sum_{\imath=\jmath+1}^{s-1}(\sigma_\imath - t_\imath) + t - t_s \geqslant t - t_{\jmath+1} - \chi\sum_{\imath=\jmath+1}^{s-1}(t_{\imath+1} - t_\imath) \geqslant (1-\chi)(t - t_{\jmath+1}). \tag{21}$$

Therefore,

$$\mathcal{Q}_2(t) \leqslant \mathcal{Q}_2(T_1) - \mathcal{E}_1\overline{\mathcal{N}}(1-\chi)(t - t_{\jmath+1}). \tag{22}$$

That is

$$\mathscr{V}^{1-p}(t) \leqslant \frac{[1 - \mathcal{E}_1\overline{\mathcal{N}}(1-\chi)(1-p)(t-t_{\jmath+1})]}{\mathrm{e}^{-(1-p)\int_{t_0}^{t}\mathcal{I}^+(\varsigma)\mathrm{d}\varsigma}}. \tag{23}$$

Obviously, $\mathscr{V}(t) = 0$ holds if $t \geqslant t_{\jmath+1} + \frac{1}{\overline{\mathcal{N}}\mathcal{E}_1(1-\chi)(1-p)}$. From the definition of s^*, one gets $s^* = \jmath + 1$. That is, $\mathscr{V}(t) = 0$ holds if $t \geqslant T_2$.

Case II. $\sigma_\jmath \leqslant T_1 \leqslant t_{\jmath+1}$, $\jmath < s$, and $\jmath \in \mathbb{Z}_+$.

Take the integral from T_1 to t on both sides of (19). Then

$$\mathcal{Q}_2(t) - \mathcal{Q}_2(T_1) \leqslant -\mathcal{E}_1\overline{\mathcal{N}}\left[\sum_{\imath=\jmath+1}^{s-1}(\sigma_\imath - t_\imath) + t - t_s\right] \leqslant -\mathcal{E}_1\overline{\mathcal{N}}(1-\chi)(t - t_{\jmath+1}). \tag{24}$$

Similar to (22) and (23), $\mathscr{V}(t) = 0$ holds if $t \geqslant T_2$.

When $\sigma_s \leqslant t < t_{s+1}$, it can be proved in two cases.

Case I. $t_\jmath < T_1 < \sigma_\jmath$, $\jmath < s$, and $\jmath \in \mathbb{Z}_+$.

It can be deduced from (19) that

$$\mathcal{Q}_2(t) - \mathcal{Q}_2(T_1) \leqslant -\mathcal{E}_1\overline{\mathcal{N}}\left[\sigma_\jmath - T_1 + \sum_{\imath=\jmath+1}^{s}(\sigma_\imath - t_\imath)\right] \leqslant -\mathcal{E}_1\overline{\mathcal{N}}(1-\chi)(t - t_{\jmath+1}). \tag{25}$$

Similar to (22) and (23), $\mathscr{V}(t) = 0$ holds if $t \geqslant T_2$.

Case II. $\sigma_\jmath \leqslant T_1 \leqslant t_{\jmath+1}$, $\jmath < s$, and $\jmath \in \mathbb{Z}_+$.

Take the integral from T_1 to t on both sides of (19). Then

$$\mathcal{Q}_2(t) - \mathcal{Q}_2(T_1) \leqslant -\mathcal{E}_1\overline{\mathcal{N}}\sum_{\imath=\jmath+1}^{s}(\sigma_\imath - t_\imath) \leqslant -\mathcal{E}_1\overline{\mathcal{N}}(1-\chi)(t - t_{\jmath+1}). \tag{26}$$

Similar to (22) and (23), $\mathscr{V}(t) = 0$ holds if $t \geqslant T_2$.

From the proof above, it is obvious that when $t \geqslant T_2$, $\mathscr{V}(t) = 0$ is true.

Remark 2. To implement FXS utilizing the aperiodically switching control method, the novel fixed-time convergence Lemma 2 is established. In the interval $[t_s, \sigma_s)$, $\dot{\mathscr{V}}(t) \leqslant \mathcal{I}(t)\mathscr{V}(t) - \mathcal{N}(t)\mathscr{V}^p(t) - \mathcal{W}(t)\mathscr{V}^q(t)$ holds, where $\mathcal{I}(t)\mathscr{V}(t)$ is to guarantee that system (2) achieves Lyapunov stability, $-\mathcal{W}(t)\mathscr{V}^q(t)$ is to make the states of system (2) converge to one in a fixed time T_1, and $-\mathcal{N}(t)\mathscr{V}^p(t)$ is to further assure that the states of system (2) converge to zero within a fixed time T_2. In the interval $[\sigma_s, t_{s+1})$, $\dot{\mathscr{V}}(t) \leqslant 0$ holds, which means that it is only necessary to guarantee that system (2) is Lyapunov stable when $t \in [\sigma_s, t_{s+1})$.

Remark 3. Recently, a generalized Lyapunov method was proposed in [30] to achieve FXS. It should be noted that the form of the fixed-time convergence lemma proposed in [30] implies that the control input needs to be activated continuously. Compared with the continuous control method, the discontinuous control method is more economical and practical and can reduce the amount of information transmission. To further reduce the control cost, fixed-time convergence methods based on discontinuous control have drawn wide attention. However, the results presented in [26, 27] require the derivative of the Lyapunov function to be non-positive, and restrict the parameters of the fixed-time convergence lemma to be constant, which is unconducive to dealing with SNNs with time-varying coefficients. Based on the above considerations, this paper develops a novel fixed-time convergence lemma based on aperiodically switching control.

Remark 4. Notably, in the newly proposed fixed-time convergence lemma, $\mathcal{I}(t)$, $\mathcal{N}(t)$, and $\mathcal{W}(t)$ are all time-varying, which is beneficial to dealing with the time-varying coefficients of system (1). Moreover, Lemma 2 does not restrict $\mathcal{I}(t)$ to be negative definite. That is, $\dot{\mathscr{V}}(t)$ can be non-negative at certain times. Consequently, Lemma 2 relaxes the requirement for $\mathscr{V}(t)$ to be monotonically decreasing in the fixed-time convergence lemmas in [26, 27]. It is worth pointing out that if $\mathcal{I}(t) = 0$, $\mathcal{N}(t) = \alpha$, and $\mathcal{W}(t) = \beta$, Lemma 2 degenerates into Lemma 5 in [27], and Lemma 4 in [26] if further $t_s = sT$ and $\sigma_s = (s + \theta)T$ with $T > 0$, $0 < \theta < 1$. Besides, if we further set $\mathcal{W}(t) = 0$, Lemma 2 can be transformed into the finite-time convergence lemmas based on the switching control in [28, 29]. In summary, the fixed-time convergence lemma demonstrated in this paper incorporates and generalizes the work in [26–29].

Remark 5. On the other hand, the novel fixed-time convergence lemma can be reduced to Theorem 3 in [30] if $\sigma_s = t_{s+1}$, and to Theorem 1 in [30] if further $\mathcal{W}(t) = 0$. Thence, the novel fixed-time convergence lemma is also suitable for the analysis of fixed-time and finite-time stabilization based on the continuous feedback control scheme. Whereas, it should be highlighted that the constraints on $\mathcal{I}(t)$, $\mathcal{N}(t)$, and $\mathcal{W}(t)$ in Lemma 2 are still relatively harsh. How to further relax the constraints of time-varying coefficients in the fixed-time convergence lemma is a valuable and challenging problem, which will be considered deeply in subsequent studies.

3 Main results

This section proposes a novel aperiodically switching control scheme and discusses the FXS of SNNs.

To stabilize system (1), the aperiodically switching controller is given as follows:

$$\mathfrak{V}_i(t,\upsilon) = \begin{cases} \left.\begin{array}{ll} \mathfrak{V}_i^{\natural}(t,\upsilon), & \|\varpi(t,\cdot)\| \neq 0, \\ 0, & \|\varpi(t,\cdot)\| = 0, \end{array}\right\} & t_s \leqslant t < \sigma_s, \\ \mathfrak{V}_i^{\sharp}(t,\upsilon), & \sigma_s \leqslant t < t_{s+1}, \end{cases} \tag{27}$$

where

$$\begin{aligned} \mathfrak{V}_i^{\natural}(t,\upsilon) =& \left(\varrho^{-1}\mathcal{I}(t) - \mathfrak{n}_i(t)\right)\varpi_i(t,\upsilon) - \mathfrak{e}_i(t)\mathrm{sign}(\varpi_i(t,\upsilon)) - \sum_{\iota=1}^{n} \mathfrak{s}_{i\iota}(t)\mathrm{sign}(\varpi_i(t,\upsilon))|\varpi_\iota(t-\tau_\iota(t),\upsilon)| \\ &- \frac{\mathcal{N}(t)\varpi_i(t,\upsilon)}{\varrho\|\varpi(t,\cdot)\|^{\varrho(1-p)}} - \frac{\mathcal{W}(t)\varpi_i(t,\upsilon)}{\varrho\|\varpi(t,\cdot)\|^{\varrho(1-q)}}, \\ \mathfrak{V}_i^{\sharp}(t,\upsilon) =& -\mathfrak{n}_i(t)\varpi_i(t,\upsilon) - \mathfrak{e}_i(t)\mathrm{sign}(\varpi_i(t,\upsilon)) - \sum_{\iota=1}^{n} \mathfrak{s}_{i\iota}(t)\mathrm{sign}(\varpi_i(t,\upsilon))|\varpi_\iota(t-\tau_\iota(t),\upsilon)|, \end{aligned}$$

with $i = 1, 2, \ldots, n$, $s \in \mathbb{Z}_+$, $0 < p < 1$, $q > 1$, $\varrho \geqslant 2$, $\mathcal{I}(t)$, $\mathcal{N}(t)$, and $\mathcal{W}(t)$ are the same as defined well in Lemma 2, and the parameters $\mathfrak{n}_i(t)$, $\mathfrak{e}_i(t)$ and $\mathfrak{s}_{i\iota}(t)$ are to be designed.

Remark 6. Actually, controller (27) is an aperiodically switching control scheme. When controller (27) is in the strong control phase $t \in [t_s, \sigma_s)$, the state trajectories of the controlled system converge to zero within a fixed time. When controller (27) is in the weak control phase $t \in [\sigma_s, t_{s+1})$, the controlled system achieves Lyapunov stability. In general, controller $\mathfrak{V}_i^{\natural}(t,\upsilon)$ has three more feedback control parameters $\mathcal{I}(t)$, $\mathcal{N}(t)$, and $\mathcal{W}(t)$ than controller $\mathfrak{V}_i^{\sharp}(t,\upsilon)$. That is, the form of the controller in the weak control phase is more concise than that in the strong control phase.

Remark 7. In contrast to the continuous control scheme with constant control strength, the controller (27) can effectively reduce control costs. Compared with the periodically switching control, the controller (27) does not need to transmit the feedback information of the controlled system periodically, and it has a wider application range. Moreover, the controller (27) degenerates into the control scheme with constant control strength if $\sigma_s = t_{s+1}$. If the control widths of the strong and weak control phases are constant, the controller (27) becomes the periodically switching control. Consequently, the presented control strategy is more general.

Theorem 1. Letting (A1) and (A2) hold, system (1) can realize FXS if there exist positive real numbers γ_ℓ, λ_ℓ meeting $\sum_{\ell=1}^{\varrho}\gamma_\ell = \sum_{\ell=1}^{\varrho}\lambda_\ell = 1$, $\varrho \geqslant 2$, and the parameters of controller (27) satisfy $\beth_\flat \leqslant 0$,

$\ni = 1, 2, 3$, where

$$\beth_1 = \zeta_i|c_{ii}(t)| - b_i(t) - \mathfrak{n}_i(t) - \sum_{j=1}^{m}\frac{4(\varrho-1)a_{ij}(t)}{\varrho^2\nu_j^2} + \frac{1}{\varrho}\sum_{\iota\neq i,\iota=1}^{n}\left(\sum_{\ell=1}^{\varrho-1}\zeta_\iota^{\varrho\gamma_\ell}|c_{i\iota}(t)|^{\varrho\lambda_\ell} + \zeta_i^{\varrho\gamma_\varrho}|c_{\iota i}(t)|^{\varrho\lambda_\varrho}\right), \tag{28}$$

$$\beth_2 = -\,\mathfrak{e}_i(t) + \sum_{\iota=1}^{n}\xi_\iota\left(|d_{i\iota}(t)| + |c_{i\iota}(t)|\right), \tag{29}$$

$$\beth_3 = -\sum_{\iota=1}^{n}\left(\mathfrak{s}_{i\iota}(t) - \zeta_\iota|d_{i\iota}(t)|\right). \tag{30}$$

Moreover, the ST is estimated as $T^\dagger = t_{s^*} + \frac{1}{\overline{\mathcal{N}}\mathcal{E}_1(1-\chi)(1-p)}$, $s^* = \min\{s\in\mathbb{Z}_+ : t_s \geqslant t_0 + \frac{1}{\overline{\mathcal{W}}\mathcal{E}_2(1-\chi)(q-1)}\}$.

Proof. Choose the Lyapunov function below:

$$\mathscr{V}(t) = \sum_{i=1}^{n}\int_\Xi|\varpi_i(t,\upsilon)|^\varrho \mathrm{d}\upsilon. \tag{31}$$

When $t_s \leqslant t < \sigma_s$ and $\|\varpi(t,\cdot)\| \neq 0$, the control input $\mathfrak{V}_i^\natural(t,\upsilon)$ is activated. Based on the chain rule [43], one has

$$\begin{aligned}\dot{\mathscr{V}}(t) =& \sum_{i=1}^{n}\int_\Xi \varrho|\varpi_i(t,\upsilon)|^{\varrho-2}\varpi_i(t,\upsilon)\frac{\partial\varpi_i(t,\upsilon)}{\partial t}\mathrm{d}\upsilon\\ \leqslant& \sum_{i=1}^{n}\int_\Xi\Bigg[-\varrho\left(b_i(t)+\mathfrak{n}_i(t)\right)|\varpi_i(t,\upsilon)|^\varrho + \varrho|\varpi_i(t,\upsilon)|^{\varrho-2}\varpi_i(t,\upsilon)\sum_{j=1}^{m}\frac{\partial}{\partial\upsilon_j}\left(a_{ij}(t)\frac{\partial\varpi_i(t,\upsilon)}{\partial\upsilon_j}\right)\\ &+\sum_{\iota=1}^{n}\varrho|d_{i\iota}(t)||\varpi_i(t,\upsilon)|^{\varrho-1}\left(\zeta_\iota|\varpi_\iota(t-\tau_\iota(t),\upsilon)|+\xi_\iota\right) + \sum_{\iota=1}^{n}\varrho|c_{i\iota}(t)||\varpi_i(t,\upsilon)|^{\varrho-1}\left(\zeta_\iota|\varpi_\iota(t,\upsilon)|+\xi_\iota\right)\\ &-\sum_{\iota=1}^{n}\varrho\mathfrak{s}_{i\iota}(t)|\varpi_i(t,\upsilon)|^{\varrho-1}|\varpi_\iota(t-\tau_\iota(t),\upsilon)| - \varrho\mathfrak{e}_i(t)|\varpi_i(t,\upsilon)|^{\varrho-1} + \mathcal{I}(t)|\varpi_i(t,\upsilon)|^\varrho\Bigg]\mathrm{d}\upsilon\\ &-\mathcal{N}(t)\|\varpi(t,\cdot)\|^{\varrho p} - \mathcal{W}(t)\|\varpi(t,\cdot)\|^{\varrho q}.\end{aligned} \tag{32}$$

By virtue of Lemma 1, one derives

$$\begin{aligned}&\int_\Xi\varrho|\varpi_i(t,\upsilon)|^{\varrho-2}\varpi_i(t,\upsilon)\sum_{j=1}^{m}\frac{\partial}{\partial\upsilon_j}\left(a_{ij}(t)\frac{\partial\varpi_i(t,\upsilon)}{\partial\upsilon_j}\right)\mathrm{d}\upsilon\\ &\quad= -\varrho(\varrho-1)\sum_{j=1}^{m}a_{ij}(t)\int_\Xi|\varpi_i(t,\upsilon)|^{\varrho-2}\left(\frac{\partial\varpi_i(t,\upsilon)}{\partial\upsilon_j}\right)^2\mathrm{d}\upsilon\\ &\quad\leqslant -\sum_{j=1}^{m}\frac{4(\varrho-1)a_{ij}(t)}{\varrho\nu_j^2}\int_\Xi|\varpi_i(t,\upsilon)|^\varrho\mathrm{d}\upsilon.\end{aligned} \tag{33}$$

From the generalization of Young's inequality, that is, $\prod_{\ell=1}^{\varrho}\mathscr{X}_\ell \leqslant \sum_{\ell=1}^{\varrho}\frac{\mathscr{X}_\ell^{\varepsilon_\ell}}{\varepsilon_\ell}$, where $\mathscr{X}_\ell > 0$, $\varepsilon_\ell > 1$, and $\sum_{\ell=1}^{\varrho}\frac{1}{\varepsilon_\ell} = 1$, one deduces

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{\iota=1}^{n}\varrho|c_{i\iota}(t)||\varpi_i(t,\upsilon)|^{\varrho-1}\zeta_\iota|\varpi_\iota(t,\upsilon)|\\ &\quad=\sum_{i=1}^{n}\sum_{\iota\neq i,\iota=1}^{n}\varrho\zeta_\iota|c_{i\iota}(t)||\varpi_i(t,\upsilon)|^{\varrho-1}|\varpi_\iota(t,\upsilon)| + \sum_{i=1}^{n}\varrho\zeta_i|c_{ii}(t)||\varpi_i(t,\upsilon)|^\varrho\\ &\quad\leqslant\sum_{i=1}^{n}\sum_{\iota\neq i,\iota=1}^{n}\left(\sum_{\ell=1}^{\varrho-1}\zeta_\iota^{\varrho\gamma_\ell}|c_{i\iota}(t)|^{\varrho\lambda_\ell}|\varpi_i(t,\upsilon)|^\varrho + \zeta_\iota^{\varrho\gamma_\varrho}|c_{i\iota}(t)|^{\varrho\lambda_\varrho}|\varpi_\iota(t,\upsilon)|^\varrho\right) + \sum_{i=1}^{n}\varrho\zeta_i|c_{ii}(t)||\varpi_i(t,\upsilon)|^\varrho\end{aligned}$$

$$= \sum_{i=1}^{n} \sum_{\iota \neq i, \iota=1}^{n} \left(\sum_{\ell=1}^{\varrho-1} \zeta_{\iota}^{\varrho\gamma_\ell} |c_{i\iota}(t)|^{\varrho\lambda_\ell} + \zeta_{i}^{\varrho\gamma_\varrho} |c_{\iota i}(t)|^{\varrho\lambda_\varrho} \right) |\varpi_i(t,\upsilon)|^{\varrho} + \sum_{i=1}^{n} \varrho\zeta_i |c_{ii}(t)||\varpi_i(t,\upsilon)|^{\varrho}, \tag{34}$$

where the positive real numbers γ_ℓ, λ_ℓ meet $\sum_{\ell=1}^{\varrho} \gamma_\ell = \sum_{\ell=1}^{\varrho} \lambda_\ell = 1$.

Substituting (33) and (34) into (36) gets

$$\begin{aligned} \dot{\mathscr{V}}(t) \leqslant & \sum_{i=1}^{n} \int_{\Xi} \varrho \Big[\beth_1 |\varpi_i(t,\upsilon)|^{\varrho} + \beth_2 |\varpi_i(t,\upsilon)|^{\varrho-1} + \beth_3 |\varpi_i(t,\upsilon)|^{\varrho-1} |\varpi_\iota(t-\tau_\iota(t),\upsilon)| \Big] \mathrm{d}\upsilon \\ & + \mathcal{I}(t)\mathscr{V}(t) - \mathcal{N}(t)\mathscr{V}^p(t) - \mathcal{W}(t)\mathscr{V}^q(t) \\ \leqslant & \, \mathcal{I}(t)\mathscr{V}(t) - \mathcal{N}(t)\mathscr{V}^p(t) - \mathcal{W}(t)\mathscr{V}^q(t). \end{aligned} \tag{35}$$

Moreover, when $\sigma_s \leqslant t < t_{s+1}$, the control input $\mathfrak{V}_i^{\sharp}(t,\upsilon)$ is activated. Similarly, we can obtain

$$\dot{V}(t) \leqslant \sum_{i=1}^{n} \int_{\Xi} \varrho \Big[\beth_1 |\varpi_i(t,\upsilon)|^{\varrho} + \beth_2 |\varpi_i(t,\upsilon)|^{\varrho-1} + \beth_3 |\varpi_i(t,\upsilon)|^{\varrho-1} |\varpi_\iota(t-\tau_\iota(t),\upsilon)| \Big] \mathrm{d}\upsilon \leqslant 0. \tag{36}$$

According to Lemma 2, it can be concluded that $\mathscr{V}(t) = 0$ if $t \geqslant T^{\dagger}$. Thence, system (1) realizes FXS via control strategy (27).

Remark 8. Based on the proposed aperiodically switching control and fixed-time convergence lemma, the conditions to ensure the FXS of SNNs are given in Theorem 1. Note that the classical fixed-time convergence lemma was employed in [22, 31, 32] to discuss the problem of fixed-time control for SNNs. That is, if $\dot{\mathscr{V}}(t) \leqslant -\mathcal{N}\mathscr{V}^p(t) - \mathcal{W}\mathscr{V}^q(t)$ with $\mathcal{N}$, $\mathcal{W} > 0$ holds for $t \in [t_0, +\infty)$, and then $\mathscr{V}(t) = 0$ if $t \geqslant \frac{1}{\mathcal{N}(1-p)} + \frac{1}{\mathcal{W}(q-1)}$. Evidently, if $\mathcal{I}(t) = 0$, $\mathcal{N}(t) = \mathcal{N}$, $\mathcal{W}(t) = \mathcal{W}$, and $\sigma_s = t_{s+1}$ in control scheme (27), the deduced results can be extended for [22,31,32]. Furthermore, the problem of finite-time control for SNNs was investigated in [33, 34] by utilizing the classical finite-time convergence lemma. Namely, if $\dot{\mathscr{V}}(t) \leqslant -\mathcal{N}\mathscr{V}^p(t)$ with $\mathcal{N} > 0$, then the state trajectories of SNNs can converge to zero within the finite time $T = \frac{\mathscr{V}^{1-p}(t_0)}{\mathcal{N}(1-p)}$. In fact, if $\mathcal{W}(t) = 0$ in Lemma 2, inequality (23) can be altered to $\mathscr{V}^{1-p}(t) \leqslant \mathrm{e}^{(1-p)\int_{t_0}^{t} \mathcal{I}^+(\varsigma)\mathrm{d}\varsigma} [\mathscr{V}^{1-p}(t_0) - \mathcal{E}_1\overline{\mathcal{N}}(1-\chi)(1-p)(t-t_0)]$, and then $\mathscr{V}(t) = 0$ if $t \geqslant \frac{\mathscr{V}^{1-p}(t_0)}{\mathcal{E}_1\overline{\mathcal{N}}(1-\chi)(1-p)}$. Consequently, if $\mathcal{I}(t) = \mathcal{W}(t) = 0$, $\mathcal{N}(t) = \mathcal{N}$, and $\sigma_s = t_{s+1}$ in controller (27), the theoretical results presented in this paper are applicable to [33, 34]. Generally, the control method and the FXS conditions proposed in this paper are more flexible and universal.

Obviously, if the effect of spatial diffusion is ignored, system (1) simplifies to NNs with discontinuous activations as follows:

$$\dot{\varpi}_i(t) = -\, b_i(t)\varpi_i(t) + \sum_{\iota=1}^{n} c_{i\iota}(t)\varphi_\iota(\varpi_\iota(t)) + \sum_{\iota=1}^{n} d_{i\iota}(t)\varphi_\iota(\varpi_\iota(t-\tau_\iota(t))). \tag{37}$$

Furthermore, controller (27) becomes the following control scheme:

$$\widehat{\mathfrak{V}}_i(t) = \begin{cases} \widehat{\mathfrak{V}}_i^{\natural}(t), & t_s \leqslant t < \sigma_s, \\ \widehat{\mathfrak{V}}_i^{\sharp}(t), & \sigma_s \leqslant t < t_{s+1}, \end{cases} \tag{38}$$

where

$$\begin{aligned} \widehat{\mathfrak{V}}_i^{\natural}(t) =& \left(\mathcal{I}(t) - \mathfrak{n}_i(t)\right)\varpi_i(t) - \mathfrak{e}_i(t)\mathrm{sign}(\varpi_i(t)) - \sum_{\iota=1}^{n} \mathfrak{S}_{i\iota}(t)\mathrm{sign}(\varpi_i(t))|\varpi_\iota(t-\tau_\iota(t))| \\ & - \left(\mathcal{N}(t)|\varpi_i(t)|^p + \mathcal{W}(t)|\varpi_i(t)|^q\right)\mathrm{sign}(\varpi_i(t)), \\ \widehat{\mathfrak{V}}_i^{\sharp}(t) =& -\mathfrak{n}_i(t)\varpi_i(t) - \mathfrak{e}_i(t)\mathrm{sign}(\varpi_i(t)) - \sum_{\iota=1}^{n} \mathfrak{S}_{i\iota}(t)\mathrm{sign}(\varpi_i(t))|\varpi_\iota(t-\tau_\iota(t))|. \end{aligned}$$

Similar to the proof of Theorem 1, we derive Corollary 1 to ensure the FXS of system (37).

Corollary 1. Letting (A1) and (A2) hold, system (37) can achieve FXS if the control parameters of controller (38) satisfy $\daleth_{\backepsilon} \leqslant 0,\ \backepsilon = 1, 2, 3$, where

$$\daleth_1 = -b_i(t) - \text{ɳ}_i(t) + \sum_{\iota=1}^{n} \zeta_\iota |c_{i\iota}(t)|, \tag{39}$$

$$\daleth_2 = -\text{ȼ}_i(t) + \sum_{\iota=1}^{n} \xi_\iota \left(|d_{i\iota}(t)| + |c_{i\iota}(t)|\right), \tag{40}$$

$$\daleth_3 = -\sum_{\iota=1}^{n} \left(\text{ẟ}_{i\iota}(t) - \zeta_\iota |d_{i\iota}(t)|\right). \tag{41}$$

Remark 9. In [30], the FXS of system (37) was investigated. If $\sigma_s = t_{s+1}$, then aperiodically switching control strategy (38) degenerates to the controller designed in [30]. Therefore, the presented method includes the one in [30] as a special case. Noteworthy, the controller proposed in [30] is always in the strong control phase, that is, for any $t \in [t_0, +\infty)$, $\widehat{\mathfrak{V}}_i(t) = \widehat{\mathfrak{V}}_i^{\natural}(t)$. Compared with [30], the presented aperiodically switching control scheme has lower control costs.

Remark 10. In [27], an adaptive aperiodically switching strategy was presented to explore the fixed-time control for memristive NNs with discontinuous activations. Indeed, the obtained results still hold when the time-varying coefficients in system (37) are switching memristive coefficients. Thus, when $\mathcal{I}(t) = 0$ and appropriate adaptive control parameters are chosen, the FXS criterion obtained in this paper can be applied to [27].

4 Numerical simulations

This section presents several simulations to display the validity of the designed controller and the deduced criteria.

Example 1. Consider the following spatiotemporal system:

$$\begin{aligned}\frac{\partial \varpi_i(t,\upsilon)}{\partial t} =& \frac{\partial}{\partial \upsilon}\left(a_i(t)\frac{\partial \varpi_i(t,\upsilon)}{\partial \upsilon}\right) - b_i(t)\varpi_i(t,\upsilon) \\ &+ \sum_{\iota=1}^{2} c_{i\iota}(t)\varphi_\iota(\varpi_\iota(t,\upsilon)) + \sum_{\iota=1}^{2} d_{i\iota}(t)\varphi_\iota(\varpi_\iota(t-\tau_\iota(t),\upsilon)),\end{aligned} \tag{42}$$

where $i = 1, 2$, $\upsilon \in \Xi = [-5, 5]$, $a_1(t) = a_2(t) = 0.1|\sin(t)|$, $b_1(t) = 0.5|\sin(t)|$, $b_2(t) = 0.5|\cos(t)|$, $c_{11}(t) = 2\sin(t)$, $c_{22}(t) = -0.3\cos(t)$, $c_{12}(t) = c_{21}(t) = -1$, $d_{11}(t) = -2$, $d_{12}(t) = 0.1\sin(t)$, $d_{21}(t) = -1.2\cos(t)$, $d_{22}(t) = -1.5$, $\tau_\iota(t) = \frac{\mathrm{e}^t}{1+\mathrm{e}^t}$, and $\varphi_\iota(\varpi)$ is selected to be $\tanh(\varpi)$. Apparently, $\varphi_\iota(\varpi)$ satisfies (A1) and (A2) with $\zeta_\iota = 1$ and $\xi_\iota = 0$. Figure 1 depicts the spatiotemporal evolutions of system (42) with initial values $\varpi_1(\theta,\upsilon) = -0.6$, $\varpi_2(\theta,\upsilon) = 0.5$, $(\theta,\upsilon) \in [-1, 0) \times \Xi$. From Figure 1, it is easy to see that system (42) is unstable.

To achieve the FXS of system (42), we design the controller (27) with the following control parameters: $\text{ɳ}_1(t) = 2.5|\sin(t)| + 3$, $\text{ɳ}_2(t) = 0.8|\cos(t)| + 2$, $\text{ȼ}_1(t) = \text{ȼ}_2(t) = 0$, $\text{ẟ}_{11}(t) = 2$, $\text{ẟ}_{22}(t) = 1.5$, $\text{ẟ}_{12}(t) = 0.1|\sin(t)|$, $\text{ẟ}_{21}(t) = 1.2|\cos(t)|$, $\mathcal{I}(t) = \frac{1}{1+t^2}$, $\mathcal{N}(t) = \mathcal{W}(t) = 1$, $\varrho = 2$, $p = \frac{1}{2}$, $q = 2$, $t_s = 4s$, $\sigma_s = 4(s+0.7)$, $s \in \mathbb{Z}_+$. It is easy to calculate that the above parameters satisfy Theorem 1 with $\chi = 0.3$, $\overline{\mathcal{N}} = \overline{\mathcal{W}} = 1$, $\overline{\mathcal{E}}_1 = \mathrm{e}^{-\frac{\pi}{4}}$, $\overline{\mathcal{E}}_2 = \mathrm{e}^{-\frac{\pi}{2}}$. The spatiotemporal evolutions of system (42) with control are shown in Figure 2, which confirms that system (42) achieves FXS with the ST $T^\dagger = 14.8721$.

Remark 11. From the simulation results of Example 1, it can be seen that the designed switching controller can guarantee the realization of FXS of SNNs, and the ST of SNNs is estimated. Nevertheless, only asymptotic or exponential stability of SNNs can be achieved in [12, 36–38]. Additionally, compared with the studies on the finite-time or fixed-time stability of SNNs in [31–34], the results in this paper can be reduced to finite-time stability, and the conditions imposed on the fixed-time convergence lemma are relaxed. Accordingly, this paper extends the related results in [12, 31–34, 36–38] on the stability of SNNs.

Example 2. Consider the following system:

$$\dot{\varpi}_i(t) = -b_i\varpi_i(t) + \sum_{\iota=1}^{2} c_{i\iota}\varphi_\iota(\varpi_\iota(t)) + \sum_{\iota=1}^{2} d_{i\iota}\varphi_\iota(\varpi_\iota(t-\tau_\iota(t))), \tag{43}$$

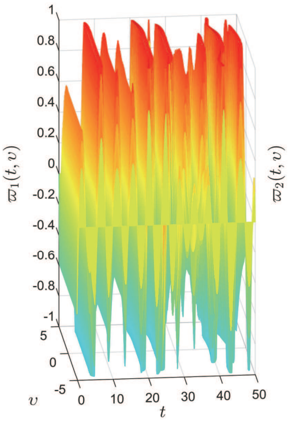

Figure 1 (Color online) Spatiotemporal evolutions of system (42).

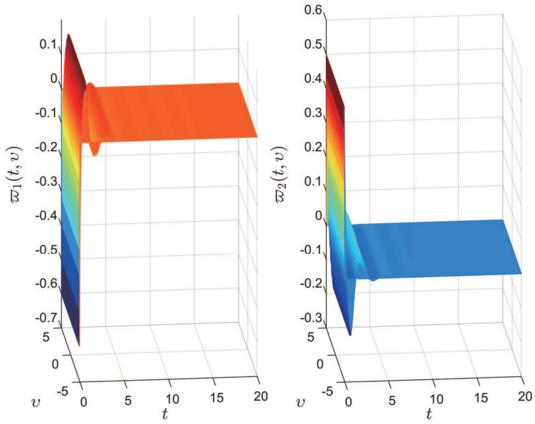

Figure 2 (Color online) Spatiotemporal evolutions of system (42) with control.

where $i = 1, 2$, $b_1 = b_2 = 1$, $c_{11} = 2$, $c_{12} = -0.1$, $c_{21} = -5$, $c_{22} = 3$, $d_{11} = -1.5$, $d_{12} = -0.1$, $d_{21} = -0.2$, $d_{22} = -2.5$. $\tau_\iota(t) = \frac{\mathrm{e}^t}{1+\mathrm{e}^t}$, and

$$\varphi_\iota(x) = \begin{cases} \tanh(x) + 0.02, & x \geqslant 0, \\ \tanh(x) - 0.01, & x < 0. \end{cases}$$

Apparently, $\varphi(x)$ meets (A1) and (A2) with $\zeta_\iota = 1$ and $\xi_\iota = 0.03$. The evolution processes of system (43) with initial values $\varpi_1(\theta) = -0.2$, $\varpi_2(\theta) = 0.5$, $\theta \in [-1, 0)$ are depicted in Figures 3 and 4, which illustrates that system (43) is unstable.

To stabilize system (43), the parameters of controller (38) are chosen as $t_s = s$, $\sigma_s = s + 0.8$, $\mathcal{I}(t) = 0$, $\mathfrak{n}_1(t) = 2$, $\mathfrak{n}_2(t) = 3$, $\mathfrak{e}_1(t) = 3$, $\mathfrak{e}_2(t) = 4$, $\mathfrak{d}_{11}(t) = 2$, $\mathfrak{d}_{12}(t) = 0.5$, $\mathfrak{d}_{21}(t) = 0.5$, $\mathfrak{d}_{22}(t) = 4$, $\mathcal{N}(t) = \mathcal{W}(t) = 1$, $p = \frac{1}{2}$, and $q = 2$. Based on simple calculations, it can be confirmed that the conditions in Corollary 1 hold with $\chi = 0.2$, $\overline{\mathcal{N}} = \overline{\mathcal{W}} = 1$, $\overline{\mathcal{E}}_1 = \overline{\mathcal{E}}_2 = 1$. That is, system (43) via controller (38) can achieve FXS. To intuitively illustrate the effectiveness of the theoretical results, the state trajectories of system (43) via controller (38) with 30 sets of random initial values in $[-1, 1]$ and $[-100, 100]$ are described in Figures 5 and 6, respectively. From Figures 5 and 6, the state trajectories of system (43) via controller (38) can converge to zero within $T^\dagger = 4.25$.

Remark 12. In [28, 29], the finite-time stabilization of NNs via switching control was investigated, which means that the ST is influenced by the initial value of the system. However, the initial values of several practical systems may be obtained inaccurately or known in advance. Different from [28, 29], the FXS results of NNs are obtained in this paper. As can be seen from Figures 5 and 6, the states of the system can converge to the origin within a fixed time under the randomly set initial value, which further indicates that the ST of FXS is independent of the initial value.

Remark 13. First, numerical examples are provided only to confirm the feasibility and effectiveness of the obtained theoretical results. Moreover, time-varying coefficients, discontinuous activations, and reaction-diffusion terms are simultaneously incorporated into NNs in this paper. It is worth highlighting that system (1) degenerates to SNNs in [12] if the activation functions are continuous. If the time-varying coefficients are all constants, system (1) becomes SNNs in [36, 37], and SNNs in [38] if the time delay is further ignored. Besides, system (1) can be transformed to NNs in [13, 30] if the reaction-diffusion terms are neglected. Thence, the system considered in this paper is more general and includes the models in [12, 13, 30, 36–38] as special cases. That is, the theoretical results derived in this paper are also applicable to the numerical examples in [12, 13, 30, 36–38].

Remark 14. Note that the ST calculated in Example 1 is almost 15 s. However, as can be easily seen from Figure 2, the states of system (42) converge to the origin in less than 10 s. That is, the actual convergence time is less than the ST estimated in this paper. The same problem also exists in Figures 5 and 6. This reflects two facts. One is that under the designed aperiodically switching control scheme, the system can converge to the origin within a fixed time, which verifies the feasibility of the method proposed in this paper. The other is that the estimation of ST in this paper is conservative to some extent. In fact,

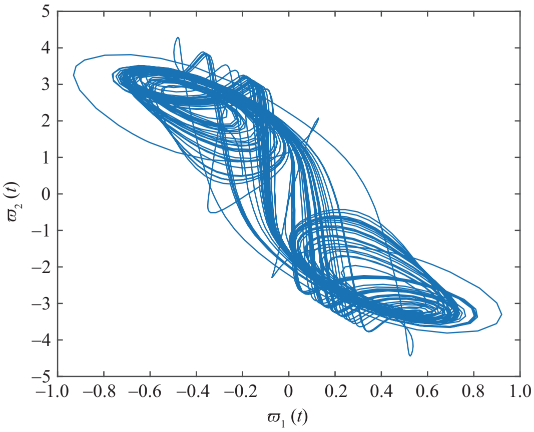

Figure 3 (Color online) Phase plot of system (43).

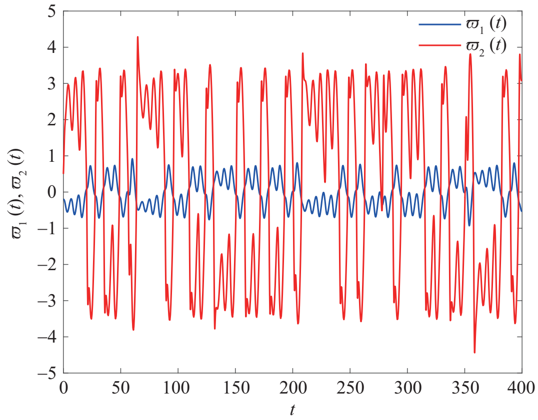

Figure 4 (Color online) State trajectories of system (43).

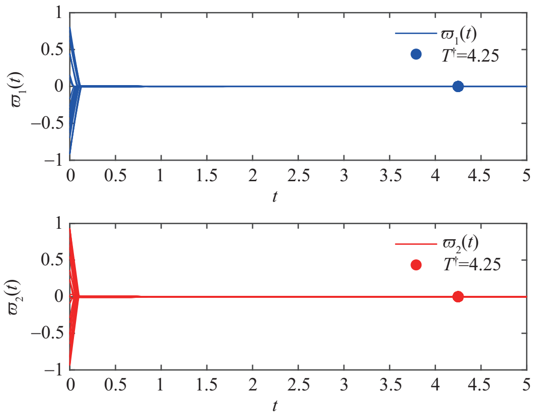

Figure 5 (Color online) State trajectories of system (43) with 30 sets of random initial values in $[-1, 1]$ via control.

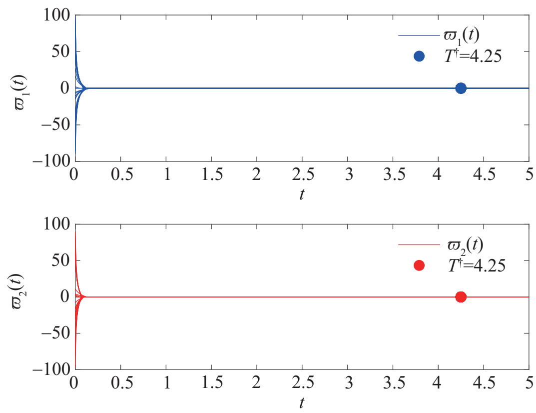

Figure 6 (Color online) State trajectories of system (43) with 30 sets of random initial values in $[-100, 100]$ via control.

the conservatism produced by estimating the ST mainly comes from the treatment of the derivative of the non-negative continuous function $\mathscr{V}(t)$ in the fixed-time convergence lemma. To facilitate the estimation of ST, this paper scales the derivative of the non-negative continuous function satisfying inequality (4). Thence, developing a new method to avoid the conservatism induced by inequality scaling to further increase the accuracy of the estimation of ST is an essential issue [44, 45], which will be a part of our future work.

5 Conclusion

In this paper, the FXS of SNNs with discontinuous activations and time-varying coefficients has been investigated. To handle the time-varying coefficients of SNNs, a novel fixed-time convergence lemma has been proved, which contains some existing results as special cases. To stabilize SNNs within a fixed time, a more economical aperiodically switching controller has been designed. By means of the fixed-time convergence technique via aperiodically switching control, several more universal conditions have been deduced to ensure the FXS of SNNs. The feasibility of the established method has been illustrated by numerical simulations.

It is worth noting that the control strategy proposed in this paper is semi-intermittent, which means that a part of the control input requires to be activated constantly. How to design a strictly intermittent control scheme to achieve FXS of SNNs remains a challenging and open problem. In future work, we will focus on developing a method for FXS based on strictly intermittent control. In addition, the criteria for FXS in this paper are established for SNNs. A natural question is whether the derived theoretical results are still applicable to other general systems, which will be deeply considered in our future research work.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant No. 62076229).

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