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• **Supplementary File** •

Stability analysis of a class of systems with periodically varying delay via looped-functional-based Lyapunov functional

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Appendix A Preliminaries

Firstly, the notations to be used are listed in Table A1.

Table A1 Notations

Notations	Explanations
X^T	the transpose of matrix X
X^{-1}	the inverse of matrix X
$\mathbb{R}^n$	the set of n-dimensional vectors
$\mathbb{R}^{n\times m}$	the space of $n\times m$-dimensional matrices
$\mathbb{S}^n(\mathbb{S}^n_+)$	the set of real $n\times n$ symmetric (positive definite) matrices
$col\{z_0,z_1,\cdots,z_n\}$	$[z_0^T,z_1^T,\cdots,z_n^T]^T$
$\mathrm{diag}\{A,B\}$	$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$
$\begin{bmatrix} A & B \\ * & D \end{bmatrix}$	$\begin{bmatrix} A & B \\ B^T & D \end{bmatrix}$
$\mathrm{Sym}\{X\}$	$X+X^T$

Secondly, for the convenience of subsequent description, we recall the concerned system as given below

$$\begin{cases} \dot{x}(t) =A_0x(t)+A_1x(t-d(t)) \\ x(\theta) =\phi(\theta), \theta\in[-h_2,0] \end{cases} \tag{A1}$$

where $A_0\in\mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ and $A_1\in\mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ are system matrices, $x(t)\in\mathbb{R}^{\mathrm{n}}$ is the system state, the initial condition $\phi(\theta)$ is a continuous function defined on $[-h_2,0]$, $d(t)$ is a continuous bounded function satisfying

$$d(t)\in[h_1,h_2],\ |\dot{d}(t)|\leqslant\mu<1 \tag{A2}$$

where h_1 and h_2 are the minimum and maximum of the delay, respectively. It is assumed that the delay is periodically varying between h_1 and h_2, and each period is consisted of one monotone increasing interval and one monotone decreasing interval.

Thirdly, the following lemma to be used is given as follows.

Lemma 1. [1, 4] Let ω be a continuous differentiable function:$[\alpha,\beta]\to\mathbb{R}^n$ and $\bar{\xi}\in\mathbb{R}^m$. For any matrices $R\in\mathbb{R}^{n\times n}>0$, $N\in\mathbb{R}^{3n\times m}$, the following inequalities hold:

$$-\int_\alpha^\beta \dot{\omega}^T(s)R\dot{\omega}(s)\mathrm{d}s\leqslant-\frac{1}{\beta-\alpha}\tilde{\xi}^T\tilde{\Pi}^T\tilde{R}\tilde{\Pi}\tilde{\xi} \tag{A3}$$

$$-\int_\alpha^\beta \dot{\omega}^T(s)R\dot{\omega}(s)\mathrm{d}s\leqslant\ 2\tilde{\xi}^T\tilde{\Pi}^TN\bar{\xi}+(\beta-\alpha)\bar{\xi}^TN^T\tilde{R}^{-1}N\bar{\xi} \tag{A4}$$

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where

$$\tilde{\xi} = col\left\{\omega(\beta), \omega(\alpha), \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\omega(s)\mathrm{d}s, \frac{1}{(\beta-\alpha)^2}\int_{\alpha}^{\beta}\int_{\theta}^{\beta}\omega(s)\mathrm{d}s\mathrm{d}\theta\right\},$$

$$\begin{aligned}
\tilde{\Pi} &= \left[\tilde{e}_1^T - \tilde{e}_2^T \quad \tilde{e}_1^T + \tilde{e}_2^T - 2\tilde{e}_3^T \quad \tilde{e}_1^T - \tilde{e}_2^T + 6\tilde{e}_3^T - 12\tilde{e}_4^T\right]^T,\\
\tilde{R} &= diag\{R, 3R, 5R\},\\
\tilde{e}_j &= \left[0_{n\times(j-1)n} \ I_n \ 0_{n\times(4-j)n}\right], \quad j = 1, 2, \cdots, 4.
\end{aligned}$$

Appendix B Proof and discussion of Theorem 1

The part gives the detailed proof of Theorem 1 and a brief discussion of the advantages. For simplifying the presentation, we define the following notations.

$$\begin{aligned}
&\delta_1(t) = x(t-h_2) - x(t-d(t)), \quad \delta_2(t) = x(t-d(t)) - x(t-h_1), \quad \delta_3(t) = \int_{t-h_1}^{t} x(s)\mathrm{d}s\\
&\delta_4(t) = \int_{t-d(t)}^{t-h_1} x(s)\mathrm{d}s, \quad \delta_5(t) = \int_{t-h_2}^{t-d(t)} x(s)\mathrm{d}s, \quad \delta_6(t) = \frac{1}{h_1}\int_{t-h_1}^{t}\int_{\theta}^{t} x(s)\mathrm{d}s\mathrm{d}\theta\\
&\delta_7(t) = \frac{1}{d(t)-h_1}\int_{t-d(t)}^{t-h_1}\int_{\theta}^{t-h_1} x(s)\mathrm{d}s\mathrm{d}\theta, \quad \delta_8(t) = \frac{1}{h_2-d(t)}\int_{t-h_2}^{t-d(t)}\int_{\theta}^{t-d(t)} x(s)\mathrm{d}s\mathrm{d}\theta\\
&\eta_1(t) = \left[\ (d(t)-h_1)\delta_1^T(t) \quad (h_2-d(t))\delta_2^T(t)\ \right]^T\\
&\eta_2(t) = \left[\ x^T(t) \quad x^T(t-h_1) \quad x^T(t-d(t)) \quad x^T(t-h_2)\ \right]^T\\
&\eta_3(t) = \left[\ \delta_3^T(t) \quad \delta_4^T(t) \quad \delta_5^T(t) \quad \delta_6^T(t) \quad \delta_7^T(t) \quad \delta_8^T(t)\ \right]^T\\
&\eta_4(t) = \left[\ \eta_2^T(t) \quad \eta_3^T(t)\ \right]^T, \ \eta_5(s) = \left[\ x^T(s) \quad \dot{x}^T(s)\ \right]^T\\
&\xi(t) = \left[\ \eta_2^T(t) \quad \xi_1^T(t) \quad \xi_2^T(t)\ \right]^T\\
&\xi_1(t) = \left[\ \dot{x}^T(t-h_1) \quad \dot{x}^T(t-d(t)) \quad \dot{x}^T(t-h_2) \quad \tfrac{1}{h_1}\delta_3^T(t) \quad \tfrac{1}{d(t)-h_1}\delta_4^T(t)\ \right]^T\\
&\xi_2(t) = \left[\ \tfrac{1}{h_2-d(t)}\delta_5^T(t) \quad \tfrac{1}{h_1}\delta_6^T(t) \quad \tfrac{1}{d(t)-h_1}\delta_7^T(t) \quad \tfrac{1}{h_2-d(t)}\delta_8^T(t)\ \right]^T\\
&e_i = \left[0_{n\times(i-1)n} \quad I_n \quad 0_{n\times(13-i)n}\right], \quad i = 1, 2, \cdots, 13.
\end{aligned}$$

The following stability criterion is obtained by employing a looped-functional-based Lyapunov functional.

Theorem 1. For given scalars $\mu \in [0, 1)$ and $h_2 > h_1 \geqslant 0$, system (A1) is asymptotically stable if there exist $P \in \mathbb{S}_+^{10n}$, $W_1, W_2, W_3 \in \mathbb{S}_+^{2n}$, $U_1, U_2 \in \mathbb{S}_+^{n}$, $P_1, Q_1 \in \mathbb{R}^{2n\times 4n}$, $R_1, R_2, Z_1, Z_2 \in \mathbb{S}^n$, and $E_1, E_2, F_1, F_2 \in \mathbb{R}^{3n\times 13n}$, such that (B1)-(B4) are feasible,

$$\begin{bmatrix}\Phi(h_1, \dot{d}(t)) & \sqrt{h_2-h_1}E_1^T\\ * & -\hat{U}_{Z1}(\dot{d}(t))\end{bmatrix}_{\dot{d}(t)\in[0,\mu]} \leqslant 0, \tag{B1}$$

$$\begin{bmatrix}\Phi(h_2, \dot{d}(t)) & \sqrt{h_2-h_1}E_2^T\\ * & -\hat{U}_{Z2}(\dot{d}(t))\end{bmatrix}_{\dot{d}(t)\in[0,\mu]} \leqslant 0, \tag{B2}$$

$$\begin{bmatrix}\Psi(h_1, \dot{d}(t)) & \sqrt{h_2-h_1}F_1^T\\ * & -\hat{U}_{R1}(\dot{d}(t))\end{bmatrix}_{\dot{d}(t)\in[-\mu,0]} \leqslant 0, \tag{B3}$$

$$\begin{bmatrix}\Psi(h_2, \dot{d}(t)) & \sqrt{h_2-h_1}F_2^T\\ * & -\hat{U}_{R2}(\dot{d}(t))\end{bmatrix}_{\dot{d}(t)\in[-\mu,0]} \leqslant 0, \tag{B4}$$

where

$$\begin{aligned}
\Phi(d(t), \dot{d}(t)) &= \Xi_0(d(t), \dot{d}(t)) + \Xi_1(d(t), \dot{d}(t)),\\
\Xi_0(d(t), \dot{d}(t)) &= Sym\{E_1^T M_1 + E_2^T M_2 + \lambda_1^T Q_1 \Pi_2 + \Pi_1^T Q_1 \lambda_2 + \Pi_3^T P \lambda_3\},\\
\Xi_1(d(t), \dot{d}(t)) &= (d(t)-h_1)((1-\dot{d}(t))e_6^T Z_1 e_6 - e_7^T Z_1 e_7) + (d(t)-h_2)(e_5^T Z_2 e_5 - (1-\dot{d}(t))e_6^T Z_2 e_6)\\
&\quad + (1-\dot{d}(t))\Pi_6^T(W_1 - W_2)\Pi_6 - \Pi_7^T W_1 \Pi_7 + \Pi_5^T(W_2 - W_3)\Pi_5 + \Pi_4^T W_3 \Pi_4\\
&\quad + (h_2-h_1)e_0^T U_1 e_0 + \ h_1 e_0^T U_2 e_0 - \frac{1}{h_1} M_3^T \hat{U}_2 M_3,\\
\Psi(d(t), \dot{d}(t)) &= \Omega_0(d(t), \dot{d}(t)) + \Omega_1(d(t), \dot{d}(t)),\\
\Omega_0(d(t), \dot{d}(t)) &= Sym\{F_1^T M_1 + F_2^T M_2 + \lambda_1^T P_1 \Pi_2 + \Pi_1^T P_1 \lambda_2 + \Pi_3^T P \lambda_3\},
\end{aligned}$$

$$\begin{aligned}\Omega_1(d(t),\dot d(t)) = &(d(t)-h_1)((1-\dot d(t))e_6^TR_1e_6 - e_7^TR_1e_7) + (d(t)-h_2)(e_5^TR_2e_5 - (1-\dot d(t))e_6^TR_2e_6)\\ &+(1-\dot d(t))\Pi_6^T(W_1-W_2)\Pi_6 - \Pi_7^TW_1\Pi_7 + \Pi_5^T(W_2-W_3)\Pi_5 + \Pi_4^TW_3\Pi_4\\ &+(h_2-h_1)e_0^TU_1e_0 + \ h_1e_0^TU_2e_0 - \frac{1}{h_1}M_3^T\hat U_2M_3,\end{aligned}$$

with

$$\Pi_1 = \begin{bmatrix}(d(t)-h_1)(e_4^T-e_3^T) & (h_2-d(t))(e_3^T-e_2^T)\end{bmatrix}^T,\ \Pi_2 = \begin{bmatrix}e_1^T & e_2^T & e_3^T & e_4^T\end{bmatrix}^T,$$

$$\Pi_3 = \begin{bmatrix}e_1^T & e_2^T & e_3^T & e_4^T & h_1e_8^T & (d(t)-h_1)e_9^T & (h_2-d(t))e_{10}^T & h_1e_{11}^T & (d(t)-h_1)e_{12}^T & (h_2-d(t))e_{13}^T\end{bmatrix}^T,$$

$$\Pi_4 = \begin{bmatrix}e_1^T & e_0^T\end{bmatrix}^T,\ \Pi_5 = \begin{bmatrix}e_2^T & e_5^T\end{bmatrix}^T,\ \Pi_6 = \begin{bmatrix}e_3^T & e_6^T\end{bmatrix}^T,\ \Pi_7 = \begin{bmatrix}e_4^T & e_7^T\end{bmatrix}^T,$$

$$\lambda_1 = \begin{bmatrix}\tau_1^T & \tau_2^T\end{bmatrix}^T,\ \lambda_2 = \begin{bmatrix}e_0^T & e_5^T & (1-\dot d(t))e_6^T & e_7^T\end{bmatrix}^T,\ \lambda_3 = \begin{bmatrix}\tau_3^T & \tau_4^T\end{bmatrix}^T,$$

$$\tau_1 = \dot d(t)(e_4-e_3) + (d(t)-h_1)(e_7-(1-\dot d(t))e_6),$$

$$\tau_2 = -\dot d(t)(e_3-e_2) + (h_2-d(t))((1-\dot d(t))e_6 - e_5),$$

$$\tau_3 = \begin{bmatrix}e_0^T & e_5^T & (1-\dot d(t))e_6^T & e_7^T & e_1^T-e_2^T & e_2^T-(1-\dot d(t))e_3^T & (1-\dot d(t))e_3^T-e_4^T\end{bmatrix}^T,$$

$$\tau_4 = \begin{bmatrix}e_1^T-e_8^T & e_2^T-(1-\dot d(t))e_9^T-\dot d(t)e_{12}^T & (1-\dot d(t))e_3^T - e_{10}^T + \dot d(t)e_{13}^T\end{bmatrix},$$

$$M_1 = \begin{bmatrix}e_3^T-e_4^T & e_3^T+e_4^T-2e_{10}^T & e_3^T-e_4^T+6e_{10}^T-12e_{13}^T\end{bmatrix}^T,$$

$$M_2 = \begin{bmatrix}e_2^T-e_3^T & e_2^T+e_3^T-2e_9^T & e_2^T-e_3^T+6e_9^T-12e_{12}^T\end{bmatrix}^T,$$

$$M_3 = \begin{bmatrix}e_1^T-e_2^T & e_1^T+e_2^T-2e_8^T & e_1^T-e_2^T+6e_8^T-12e_{11}^T\end{bmatrix}^T,$$

$$\hat U_2 = diag\{U_2, 3U_2, 5U_2\},\ e_0 = A_0e_1 + A_1e_3,$$

$$\hat U_{Zi}(\dot d(t)) = diag\{U_{Zi}(\dot d(t)), 3U_{Zi}(\dot d(t)), 5U_{Zi}(\dot d(t))\},$$

$$\hat U_{Ri}(\dot d(t)) = diag\{U_{Ri}(\dot d(t)), 3U_{Ri}(\dot d(t)), 5U_{Ri}(\dot d(t))\},$$

$$U_{Zi}(\dot d(t)) = U_1 - \dot d(t)Z_i, U_{Ri}(\dot d(t)) = U_1 - \dot d(t)R_i,\ i=1,2.$$

Proof: Choose a Lyapunov functional candidate as follows,

$$V(t) = \begin{cases} V_c(t)+V_I(t), & t\in[t_{2k-1},t_{2k})\\ V_c(t)+V_D(t), & t\in[t_{2k},t_{2k+1})\end{cases} \tag{B5}$$

where

$$\begin{aligned}V_c(t) =&\eta_4^T(t)P\eta_4(t) + \int_{t-h_2}^{t-d(t)}\eta_5^{\mathrm T}(s)W_1\eta_5(s)\mathrm{d}s + \int_{t-d(t)}^{t-h_1}\eta_5^{\mathrm T}(s)W_2\eta_5(s)\mathrm{d}s + \int_{t-h_1}^{t}\eta_5^{\mathrm T}(s)W_3\eta_5(s)\mathrm{d}s\\ &+\int_{-h_2}^{-h_1}\int_{t+\theta}^{t}\dot x^T(s)U_1\dot x(s)\mathrm{d}s\mathrm{d}\theta + \int_{-h_1}^{0}\int_{t+\theta}^{t}\dot x^T(s)U_2\dot x(s)\mathrm{d}s\mathrm{d}\theta\\ V_I(t) =&2\eta_1^T(t)Q_1\eta_2(t) + (d(t)-d(t_{2k-1}))\int_{t-d(t_{2k})}^{t-d(t)}\dot x^T(s)Z_1\dot x(s)\mathrm{d}s + (d(t)-d(t_{2k}))\int_{t-d(t)}^{t-d(t_{2k-1})}\dot x^T(s)Z_2\dot x(s)\mathrm{d}s\\ V_D(t) =&2\eta_1^T(t)P_1\eta_2(t) + (d(t)-d(t_{2k+1}))\int_{t-d(t_{2k})}^{t-d(t)}\dot x^T(s)R_1\dot x(s)\mathrm{d}s + (d(t)-d(t_{2k}))\int_{t-d(t)}^{t-d(t_{2k+1})}\dot x^T(s)R_2\dot x(s)\mathrm{d}s\end{aligned} \tag{B6}$$

and $P\in\mathbb{S}_+^{10n}$, $W_1,W_2,W_3\in\mathbb{S}_+^{2n}$, $U_1,U_2\in\mathbb{S}_+^n$, $P_1,Q_1\in\mathbb{R}^{2n\times 4n}$, and $R_1,R_2,Z_1,Z_2\in\mathbb{S}^n$ are to be determined.

Notice that $V_c(t)\geqslant 0$, $V_I(t_{2k-1}) = \lim_{t\to t_{2k}} V_I(t) = 0$ and $V_D(t_{2k}) = \lim_{t\to t_{2k+1}} V_D(t) = 0$. Thus, the Lyapunov functional (B5) is continuous in time and satisfies $V(t_{2k-1})\geqslant 0, V(t_{2k})\geqslant 0, k=1,2,\cdots$.

Firstly, we consider the case of $t\in[t_{2k-1},t_{2k})$, i.e. $\dot d(t)\in[0,\mu]$. Taking the derivative of $V(t)$ along the trajectories of system (A1) yields

$$\dot V(t) = \dot V_c(t) + \dot V_I(t) \tag{B7}$$

where

$$\begin{aligned}\dot V_c(t) =& 2\eta_4^T(t)P\dot\eta_4(t) - \eta_5^T(t-h_2)W_1\eta_5(t-h_2) + (1-\dot d(t))\eta_5^T(t-d(t))(W_1-W_2)\eta_5(t-d(t))\\ &+\eta_5^T(t-h_1)(W_2-W_3)\eta_5(t-h_1) + \eta_5^T(t)W_3\eta_5(t) + (h_2-h_1)\dot x^T(t)U_1\dot x(t) + h_1\dot x^T(t)U_2\dot x(t) + J_1 + J_2,\\ \dot V_I(t) =& 2\dot\eta_1^T(t)Q_1\eta_2(t) + 2\eta_1^T(t)Q_1\dot\eta_2(t) + (d(t)-h_1)((1-\dot d(t))e_6^TZ_1e_6 - e_7^TZ_1e_7)\end{aligned}$$

$$+ (d(t) - h_2)(e_5^T Z_2 e_5 - (1 - \dot{d}(t)) e_6^T Z_2 e_6) + J_3 + J_4$$

with

$$J_1 = -\int_{t-h_2}^{t-h_1} \dot{x}^T(s) U_1 \dot{x}(s) \mathrm{d}s, \; J_2 = -\int_{t-h_1}^{t} \dot{x}^T(s) U_2 \dot{x}(s) \mathrm{d}s, \; J_3 = \dot{d}(t) \int_{t-h_2}^{t-d(t)} \dot{x}^T(s) Z_1 \dot{x}(s) \mathrm{d}s, \; J_4 = \dot{d}(t) \int_{t-d(t)}^{t-h_1} \dot{x}^T(s) Z_2 \dot{x}(s) \mathrm{d}s.$$

Applying Lemma 1, we get

$$J_2 \leqslant -\frac{1}{h_1} \xi^T(t) M_3^T \hat{U}_2 M_3 \xi(t) \tag{B8}$$

$$\begin{aligned} J_1 + J_3 + J_4 = & -\int_{t-h_2}^{t-d(t)} \dot{x}^T(s) U_{Z1}(\dot{d}(t)) \dot{x}(s) \mathrm{d}s - \int_{t-d(t)}^{t-h_1} \dot{x}^T(s) U_{Z2}(\dot{d}(t)) \dot{x}(s) \mathrm{d}s \\ & \leqslant \xi^T(t) \{ Sym\{E_1^T M_1 + E_2^T M_2\} + (h_2 - d(t)) E_1^T \hat{U}_{Z1}^{-1}(\dot{d}(t)) E_1 \\ & \quad + (d(t) - h_1) E_2^T \hat{U}_{Z2}^{-1}(\dot{d}(t)) E_2 \} \xi(t) \end{aligned} \tag{B9}$$

for $U_{Z1}(\dot{d}(t)) = U_1 - \dot{d}(t) Z_1 > 0$ and $U_{Z2}(\dot{d}(t)) = U_1 - \dot{d}(t) Z_2 > 0$, where $\hat{U}_{Z1}(\dot{d}(t))$, and $\hat{U}_{Z2}(\dot{d}(t))$ are defined in Theorem 1.

From (B7) - (B9), we have

$$\dot{V}(t) \leqslant \xi^T(t) \{ \Phi(d(t), \dot{d}(t)) + (h_2 - d(t)) E_1^T \hat{U}_{Z1}^{-1}(\dot{d}(t)) E_1 + (d(t) - h_1) E_2^T \hat{U}_{Z2}^{-1}(\dot{d}(t)) E_2 \} \xi(t) \tag{B10}$$

where $\Phi(d(t), \dot{d}(t))$ is defined in Theorem 1.

It follows from (B10) that, there exists $\rho_1 > 0$ such that $\dot{V}(t) < -\rho_1 \|x(t)\|^2$ if $\Phi(h_1, \dot{d}(t)) + (h_2 - h_1) E_1^T \hat{U}_{Z1}^{-1}(\dot{d}(t)) E_1 < 0$ and $\Phi(h_2, \dot{d}(t)) + (h_2 - h_1) E_2^T \hat{U}_{Z2}^{-1}(\dot{d}(t)) E_2 < 0$ for $d(t) \in [h_1, h_2]$ and $\dot{d}(t) \in [0, \mu]$, which are equivalent to (B1) and (B2), respectively.

For the case of $t \in [t_{2k}, t_{2k+1})$, i.e. $\dot{d}(t) \in [-\mu, 0]$, employing a similar procedure, we can obtain that

$$\dot{V}(t) \leqslant \xi^T(t) \{ \Psi(d(t), \dot{d}(t)) + (h_2 - d(t)) F_1^T \hat{U}_{R1}^{-1}(\dot{d}(t)) F_1 + (d(t) - h_1) F_2^T \hat{U}_{R2}^{-1}(\dot{d}(t)) F_2 \} \xi(t) \tag{B11}$$

Thus, we get that, there exists $\rho_2 > 0$ such that $\dot{V}(t) < -\rho_2 \|x(t)\|^2$ if (B3) and (B4) are satisfied. Noted that $V(t)$ is continuous function in time, it is concluded that system (A1) is stable. This completes the proof.

Remark 1. Notice that $V(t_{2k-1}) \geqslant 0, V(t_{2k}) \geqslant 0, V(t_{2k+1}) \geqslant 0$ and $\dot{V}(t) < -min\{\rho_1, \rho_2\} \|x(t)\|^2, t \in [t_{2k-1}, t_{2k+1}]$ imply $V(t) \geqslant 0, \forall t \in [t_{2k-1}, t_{2k+1}]$. Therefore, it is only required $V_c(t) \geqslant 0$, while $V_I(t)$ and $V_D(t)$ are not required to be positive. In Lyapunov functional (B5), two groups of Lyapunov matrices are chosen separately for the monotone decreasing intervals and the monotone increasing intervals, which further relax the derived stability condition, hence greatly reduce the conservativeness of the obtained results.

Appendix C Corollary 1 and Corollary 2

For the case of $h_1 = 0$, based on Theorem 1, the following corollary can be obtained.

Corollary 1. For given scalars $\mu \in [0, 1)$ and $h_2 \geqslant 0$, system (A1) is asymptotically stable if there exist $\bar{P} \in \mathbb{S}_+^{7n}$, $W_1, W_2 \in \mathbb{S}_+^{2n}$, $U_1 \in \mathbb{S}_+^n$, $\bar{P}_1, \bar{Q}_1 \in \mathbb{R}^{2n \times 3n}$, $R_1, R_2, Z_1, Z_2 \in \mathbb{S}^n$, and $\bar{E}_1, \bar{E}_2, \bar{F}_1, \bar{F}_2 \in \mathbb{R}^{3n \times 9n}$, such that (C1)-(C4) are feasible,

$$\begin{bmatrix} \bar{\Phi}(0, \dot{d}(t)) & \sqrt{h_2} \bar{E}_1^T \\ * & -\hat{U}_{Z1}(\dot{d}(t)) \end{bmatrix}_{\dot{d}(t) \in [0, \mu]} \leqslant 0, \tag{C1}$$

$$\begin{bmatrix} \bar{\Phi}(h_2, \dot{d}(t)) & \sqrt{h_2} \bar{E}_2^T \\ * & -\hat{U}_{Z2}(\dot{d}(t)) \end{bmatrix}_{\dot{d}(t) \in [0, \mu]} \leqslant 0, \tag{C2}$$

$$\begin{bmatrix} \bar{\Psi}(0, \dot{d}(t)) & \sqrt{h_2} \bar{F}_1^T \\ * & -\hat{U}_{R1}(\dot{d}(t)) \end{bmatrix}_{\dot{d}(t) \in [-\mu, 0]} \leqslant 0, \tag{C3}$$

$$\begin{bmatrix} \bar{\Psi}(h_2, \dot{d}(t)) & \sqrt{h_2} \bar{F}_2^T \\ * & -\hat{U}_{R2}(\dot{d}(t)) \end{bmatrix}_{\dot{d}(t) \in [-\mu, 0]} \leqslant 0, \tag{C4}$$

where

$$\begin{aligned} \bar{\Phi}(d(t), \dot{d}(t)) &= \bar{\Xi}_0(d(t), \dot{d}(t)) + \bar{\Xi}_1(d(t), \dot{d}(t)), \\ \bar{\Xi}_0(d(t), \dot{d}(t)) &= Sym\{\bar{E}_1^T \bar{M}_1 + \bar{E}_2^T \bar{M}_2 + \bar{\lambda}_1^T \bar{Q}_1 \bar{\Pi}_2 + \bar{\Pi}_1^T \bar{Q}_1 \bar{\lambda}_2 + \bar{\Pi}_3^T \bar{P} \bar{\lambda}_3\}, \\ \bar{\Xi}_1(d(t), \dot{d}(t)) &= d(t)((1 - \dot{d}(t)) \bar{e}_4^T Z_1 \bar{e}_4 - \bar{e}_5^T Z_1 \bar{e}_5) + (d(t) - h_2)(\bar{e}_0^T Z_2 \bar{e}_0 - (1 - \dot{d}(t)) \bar{e}_4^T Z_2 \bar{e}_4) \\ &\quad + (1 - \dot{d}(t)) \bar{\Pi}_5^T (W_1 - W_2) \bar{\Pi}_5 - \bar{\Pi}_6^T W_1 \bar{\Pi}_6 + \bar{\Pi}_4^T W_2 \bar{\Pi}_4 + h_2 \bar{e}_0^T U_1 \bar{e}_0, \\ \bar{\Psi}(d(t), \dot{d}(t)) &= \bar{\Omega}_0(d(t), \dot{d}(t)) + \bar{\Omega}_1(d(t), \dot{d}(t)), \\ \bar{\Omega}_0(d(t), \dot{d}(t)) &= Sym\{\bar{F}_1^T \bar{M}_1 + \bar{F}_2^T \bar{M}_2 + \bar{\lambda}_1^T \bar{P}_1 \bar{\Pi}_2 + \bar{\Pi}_1^T \bar{P}_1 \bar{\lambda}_2 + \bar{\Pi}_3^T \bar{P} \bar{\lambda}_3\}, \\ \bar{\Omega}_1(d(t), \dot{d}(t)) &= d(t)((1 - \dot{d}(t)) \bar{e}_4^T R_1 \bar{e}_4 - \bar{e}_5^T R_1 \bar{e}_5) + (d(t) - h_2)(\bar{e}_0^T R_2 \bar{e}_0 - (1 - \dot{d}(t)) \bar{e}_4^T R_2 \bar{e}_4) \\ &\quad + (1 - \dot{d}(t)) \bar{\Pi}_5^T (W_1 - W_2) \bar{\Pi}_5 - \bar{\Pi}_6^T W_1 \bar{\Pi}_6 + \bar{\Pi}_4^T W_2 \bar{\Pi}_4 + h_2 \bar{e}_0^T U_1 \bar{e}_0. \end{aligned}$$

with

$$\bar{\Pi}_1 = \begin{bmatrix} d(t)(\bar{e}_3^T - \bar{e}_2^T) & (h_2 - d(t))(\bar{e}_2^T - \bar{e}_1^T) \end{bmatrix}^T, \ \bar{\Pi}_2 = \begin{bmatrix} \bar{e}_1^T & \bar{e}_2^T & \bar{e}_3^T \end{bmatrix}^T,$$

$$\bar{\Pi}_3 = \begin{bmatrix} \bar{e}_1^T & \bar{e}_2^T & \bar{e}_3^T & d(t)\bar{e}_6^T & (h_2 - d(t))\bar{e}_7^T & d(t)\bar{e}_8^T & (h_2 - d(t))\bar{e}_9^T \end{bmatrix}^T, \ \bar{\Pi}_4 = \begin{bmatrix} \bar{e}_1^T & \bar{e}_0^T \end{bmatrix}^T, \ \bar{\Pi}_5 = \begin{bmatrix} \bar{e}_2^T & \bar{e}_4^T \end{bmatrix}^T,$$

$$\bar{\Pi}_6 = \begin{bmatrix} \bar{e}_3^T, \ \bar{e}_5^T \end{bmatrix}^T, \ \bar{\lambda}_1 = \begin{bmatrix} \dot{d}(t)(\bar{e}_3^T - \bar{e}_2^T) + d(t)(\bar{e}_5^T - (1 - \dot{d}(t))\bar{e}_4^T) & -\dot{d}(t)(\bar{e}_2^T - \bar{e}_1^T) + (h_2 - d(t))((1 - \dot{d}(t))\bar{e}_4^T - \bar{e}_0^T) \end{bmatrix}^T,$$

$$\bar{\lambda}_2 = \begin{bmatrix} \bar{e}_0^T & (1 - \dot{d}(t))\bar{e}_4^T & \bar{e}_5^T \end{bmatrix}^T, \quad \bar{\lambda}_3 = \begin{bmatrix} \bar{\tau}_1^T & \bar{\tau}_2^T \end{bmatrix}^T, \ \bar{\tau}_1 = \begin{bmatrix} \bar{e}_0^T & (1 - \dot{d}(t))\bar{e}_4^T & \bar{e}_5^T & \bar{e}_1^T - (1 - \dot{d}(t))\bar{e}_2^T & (1 - \dot{d}(t))\bar{e}_2^T - \bar{e}_3^T \end{bmatrix}^T,$$

$$\bar{\tau}_2 = \begin{bmatrix} \bar{e}_1^T - (1 - \dot{d}(t))\bar{e}_6^T - \dot{d}(t)\bar{e}_8^T & (1 - \dot{d}(t))\bar{e}_2^T - \bar{e}_7^T + \dot{d}(t)\bar{e}_9^T \end{bmatrix}^T,$$

$$\bar{M}_1 = \begin{bmatrix} \bar{e}_2^T - \bar{e}_3^T & \bar{e}_2^T + \bar{e}_3^T - 2\bar{e}_7^T & \bar{e}_2^T - \bar{e}_3^T + 6\bar{e}_7^T - 12\bar{e}_9^T \end{bmatrix}^T, \bar{M}_2 = \begin{bmatrix} \bar{e}_1^T - \bar{e}_2^T & \bar{e}_1^T + \bar{e}_2^T - 2\bar{e}_6^T & \bar{e}_1^T - \bar{e}_2^T + 6\bar{e}_6^T - 12\bar{e}_8^T \end{bmatrix}^T,$$

$$\bar{e}_0 = A_0\bar{e}_1 + A_1\bar{e}_2, \bar{e}_j = \begin{bmatrix} 0_{n\times(j-1)n} & I_n & 0_{n\times(9-j)n} \end{bmatrix}, \quad j = 1, 2, \cdots, 9,$$

and $\hat{U}_{Zi}(\dot{d}(t)), \hat{U}_{Ri}(\dot{d}(t)), i = 1, 2$, are defined as the same in Theorem 1.

To reduce the computational complexity, the following stability condition is derived by employing the sample free-matrix-based inequality presented in Lemma 1 of [6].

Corollary 2. For given scalars $\mu \in [0, 1)$ and $h_2 \geqslant 0$, system (A1) is asymptotically stable if there exist $\bar{P} \in \mathbb{S}_+^{7n}$, $W_1, W_2 \in \mathbb{S}_+^{2n}$, $U_1 \in \mathbb{S}_+^n$, $\bar{P}_1, \bar{Q}_1 \in \mathbb{R}^{2n\times 3n}$, $R_1, R_2, Z_1, Z_2 \in \mathbb{S}^n$, and $\hat{E}_1, \hat{E}_2, \hat{F}_1, \hat{F}_2 \in \mathbb{R}^{3n\times 4n}$, such that (C1)-(C4) with $\bar{E}_i = \hat{E}_i\Gamma_i, \bar{F}_i = \hat{F}_i\Gamma_i, i = 1, 2$, are feasible, where

$$\Gamma_1 = \begin{bmatrix} \bar{e}_2^T & \bar{e}_3^T & \bar{e}_7^T & \bar{e}_9^T \end{bmatrix}^T, \ \Gamma_2 = \begin{bmatrix} \bar{e}_1^T & \bar{e}_2^T & \bar{e}_6^T & \bar{e}_8^T \end{bmatrix}^T.$$

Appendix D Numerical examples

This part provides two benchmark examples to show the effectiveness of the proposed method and the reduced conservativeness in comparison with other methods. The system matrices are shown in Table D1.

Table D1 System matrices for given examples

	A_0	A_1
Example 1	$\begin{bmatrix} -2.0 & 0.0 \\ 0.0 & -0.9 \end{bmatrix}$	$\begin{bmatrix} -1.0 & 0.0 \\ -1.0 & -1.0 \end{bmatrix}$
Example 2	$\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$

For comparison with some recent stability criteria, we calculate the maximum allowable upper bounds h_2 with $h_1 = 0$ by using Corollary 1 and Corollary 2. The calculated results for Example 1 and Example 2 are summarized in Table D2 and Table D3, respectively. For comparison, Table D2 also lists the number of decision variables (NDV) corresponding to these methods. It is observed in the tables that our approach can obtain the lest conservative result for different μ. It is worth mentioning that, in comparison with recent literature [7, 8, 10, 11], Corollary 2 has improvement in the reduction of both conservativeness and computational complexity.

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Table D2 Allowable upper bounds of h for different μ (Example 1)

μ	**0.1**	**0.5**	**0.8**	**NDV**
[3]	4.910	3.233	2.789	231
[4]	4.921	3.221	2.792	480
[5]	4.93	3.09	2.66	263
[7]	4.945	3.314	2.882	419
[8]	4.939	3.298	2.869	446
[9]	4.966	3.395	2.983	514
[10]	5.026	3.428	2.997	1111
[11]	5.044	3.443	2.983	1008
Corollary 1	5.309	3.853	3.432	620
Corollary 2	5.249	3.700	3.256	380

Table D3 Allowable upper bounds of h_2 with $h_1 = 0$ for different μ (Example 2)

μ	**0.1**	**0.2**	**0.5**	**0.8**
[2]	7.176	4.543	2.496	1.922
[3]	7.230	4.556	2.509	1.940
[4]	7.308	4.670	2.664	2.072
[8]	7.401	4.765	2.709	2.091
[9]	7.572	4.947	2.801	2.137
[10]	7.651	4.936	2.764	2.114
[11]	7.685	4.969	2.774	2.117
Corollary 1	14.105	9.982	6.220	3.512
Corollary 2	13.929	9.821	5.992	3.345