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Parametric control of quasi-linear second-order systems with partitioned eigenstructure assignment by output feedback

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Abstract In this paper, a parametric design approach for stabilizing a quasi-linear second-order system with partitioned eigenstructure assignment (PESA) is investigated through output feedback control. The PESA approach is established by partitioning the desired eigenvalue matrix into two parts to separate the associated right and left eigenvectors into a subset of the generalized eigenvectors simultaneously. A parametric controller is established by solving two second-order generalized Sylvester matrix equations, and a certain form with the desired eigenstructure can be derived with the established quasi-linear output feedback controller. Unlike the prevailing approach that assigns the entire set of generalized eigenvectors, which is difficult to satisfy a large number of complicated constraints in practical systems by the normalized pair of right and left eigenvector matrices, a subset of the generalized eigenvectors is considered. In addition, the proposed PESA approach provides less computational load and is easy to use. A numerical example and application in spacecraft rendezvous are provided to verify the numerical economy and high efficiency of the proposed approach.

Keywords quasi-linear second-order systems, partitioned eigenstructure assignment, parametric approach, Sylvester matrix equation, output feedback

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1 Introduction

There are many second-order practical systems subject to the laws of physics, e.g., Kirchhoff's law and Lagrangian equation. In the last few years, we have witnessed the developed theoretical advances in second-order systems [1–5] and wide implementation in many fields, such as multi-agent systems [6–9], robotic manipulators [10–12] and hemispherical resonator gyroscope [13]. It is worth noting that most physical systems are quasi-linear, meaning that a nonlinear dynamical system could be expressed as a linear form.

Even though nonlinear systems have stirred considerable research attention and seminal approaches have been proposed to handle nonlinearity, such as feedback linearization, backstepping, passivity-based control and Lyapunov redesign, the dynamic performance is seldom discussed. Most nonlinear approaches only guarantee stability, and the dynamic performance may not be easy to obtain. Meanwhile, the linear system has been widely investigated, and the stability and transient responses are characterized by the closed-loop eigenstructure. Thus, quasi-linear systems are proposed, which are indeed nonlinear but can be expressed in linear forms. Many real systems can be modeled as quasi-linear systems, such as the fields of mechanics [14, 15], robotics [16, 17], and missile [18]. Thus, a wide range of theoretical advances of the quasi-linear systems has been developed [19–25]. It is worth noting that the strict-feedback and nonstrict-feedback nonlinear systems have been investigated by considering various conditions [26, 27].

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Particularly, the output feedback control method is considered with prescribed performance for uncertain strict-feedback nonlinear systems [28] and switched non-strict-feedback nonlinear systems [29].

Recently, a branch of parametric approaches has been proposed to deal with the quasi-linear system with the time-varying coefficient matrices, including state variables and time-varying parameter vectors. A certain form with the desired eigenstructure can be derived using the established quasi-linear state feedback or output feedback controller. The controller can be constructed by solving the generalized Sylvester matrix equations. Even though the parametric approach scheme seems like feedback linearization techniques, which derived a closed-loop linear system, they are different in several ways. Feedback linearization techniques deal with some special types of nonlinear systems. It is achieved by introducing a feedback mechanism and a change of variables to transform the nonlinear systems into equivalent linear systems. At the same time, the region of attraction of the origin is unknown. By solving a type of parameter-varying generalized Sylvester matrix equations, complete parametrization of the quasi-linear state feedback or output feedback controller can be established. Then, linear systems can be obtained using the predesigned eigenstructure. In addition, feedback linearization may need to solve partial differential equations that define the input-state linearizing transformations. Besides, it lacks a systematic analytical approach compared with the parametric approach.

The seminal parametric approach was first proposed by Duan [30]. It has greatly enriched some other theories [2,31–35] and has received successful applications [36,37]. Recently, Duan extended this parametric approach to quasi-linear systems through state feedback [20], output feedback [21], and applications in flight vehicle control [38, 39]. Moreover, Gu et al. extended the parametric approach to quasi-linear second-order systems [22], quasi-linear high-order systems [24,40] and descriptor quasi-linear systems [25].

However, the existing approaches assign the entire set of generalized eigenvectors and should be satisfied with complicated constraints, which is not feasible in many applications. Meanwhile, it is unnecessary to assign the entire set of generalized eigenvectors since it will be sufficient to assign a subset of the generalized eigenvectors. Thus, introducing a scheme that includes less complicated constraints is of theoretical importance and practical significance. In this study, we will make a dedicated effort to tackle these difficulties. Inspired by the above studies, we propose a partitioned eigenstructure assignment (PESA) approach, which greatly reduces a large number of complicated constraints. The proposed approach reduces the difficulty of the controller design and makes it easier to utilize the method in practical systems.

To the best of our knowledge, this is the first time that the PESA approach for the quasi-linear systems is established by partitioning the desired eigenvalue matrix into two parts to separate the associated right and left eigenvectors into a subset of the generalized eigenvectors simultaneously. The products of partial right and left eigenvectors have low dimensions and largely reduce the number of complicated constraints. The proposed PESA approach by output feedback acting on a quasi-linear second-order system has the following features.

- A parametric controller is established, and a certain form with the desired eigenstructure can be derived using the established quasi-linear output feedback controller.
- The proposed PESA approach partitions the desired eigenvalue matrix into two parts and constructs a low dimension constraint by assigning only a subset of the left and right generalized eigenvectors instead of the entire set, making it numerically economical and efficient.

The remainder of the paper is organized as follows. In Section 2, we introduce the quasi-linear second-order system and key assumptions. The motivation statement and steps of the proposed PESA approach to reduce complicated constraints are presented in Section 3. Section 4 states the main results of the parametric solution, where the proposed parametric solution of the output feedback controller is presented with arbitrary and diagonal cases. Section 5 provides a numerical example and application in spacecraft rendezvous to illustrate the validity of the proposed approach. Finally, Section 6 presents the conclusion.

2 System description

Consider the following description of the quasi-linear second-order system:

$$A_2(\vartheta, z, \dot{z})\ddot{z} + A_1(\vartheta, z, \dot{z})\dot{z} + A_0(\vartheta, z, \dot{z})z = B(\vartheta, z, \dot{z})u, \tag{1a}$$

$$y_0 = C_0(\vartheta, z, \dot{z})z, \tag{1b}$$

$$y_1 = C_1(\vartheta, z, \dot{z})\dot{z}, \tag{1c}$$

where $z \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^r$ is the control vector, $y_0 \in \mathbb{R}^{m_0}, y_1 \in \mathbb{R}^{m_1}$ and $m_0 + m_1 = m$ are the measured output and derivative output vectors, $A_2(\vartheta, z, \dot{z}), A_1(\vartheta, z, \dot{z}), A_0(\vartheta, z, \dot{z}) \in \mathbb{R}^{n\times n}$, $B(\vartheta, z, \dot{z}) \in \mathbb{R}^{n\times r}$, $C_0(\vartheta, z, \dot{z}) \in \mathbb{R}^{m_0\times n}$ and $C_1(\vartheta, z, \dot{z}) \in \mathbb{R}^{m_1\times n}$ represent coefficient matrices of the quasi-linear system which are piecewise continuous functions in respect of $\vartheta, z, \dot{z}$. In addition, ϑ is a parameter vector within a compact set Ω which is time-varying and satisfies

$$\vartheta = \vartheta(t) \in \Omega \subset \mathbb{R}^l. \tag{2}$$

The system (1a)–(1c) also satisfies the following assumptions.

Assumption 1 (See the study of Duan [21] and Gu et al. [22])**.** $B(\vartheta, z, \dot{z})$, $C_0(\vartheta, z, \dot{z})$ and $C_1(\vartheta, z, \dot{z})$ are uniformly bounded in respect of $\vartheta, z, \dot{z}$.

Remark 1. $B(\vartheta, z, \dot{z})$, $C_0(\vartheta, z, \dot{z})$ and $C_1(\vartheta, z, \dot{z})$ are coefficient matrices of the quasi-linear system, and it is reasonable to suppose them to be uniformly bounded in respect of $\vartheta, z, \dot{z}$. This requirement is utilized for the coprime factorization introducing later, and this is a restriction of parametric control for the quasi-linear systems.

Assumption 2 (See the study of Gu et al. [22])**.** $\mathrm{rank} A_2(\vartheta, z, \dot{z}) = n$.

A controller is designed in order to control the above system (1a)–(1c):

$$\begin{aligned} u &= \sum_{i=0}^{1} K_i(\vartheta, z, \dot{z}) y_i(\vartheta, z, \dot{z}) + v \\ &= \begin{bmatrix} K_0(\vartheta, z, \dot{z}) C_0(\vartheta, z, \dot{z}) & K_1(\vartheta, z, \dot{z}) C_1(\vartheta, z, \dot{z}) \end{bmatrix} \begin{bmatrix} z \\ \dot{z} \end{bmatrix} + v, \end{aligned} \tag{3}$$

where $K_0(\vartheta, z, \dot{z}) \in \mathbb{R}^{r\times m_0}, K_1(\vartheta, z, \dot{z}) \in \mathbb{R}^{r\times m_1}$ are gains of the feedback controller and piece-wisely continuous in respect of $\vartheta, z, \dot{z}$, and v is an external signal. Thus, a closed-loop system is derived by this controller.

$$A_2(\vartheta, z, \dot{z}) \ddot{z} + A_{c1}(\vartheta, z, \dot{z}) \dot{z} + A_{c0}(\vartheta, z, \dot{z}) z = B(\vartheta, z, \dot{z}) v, \tag{4}$$

with

$$A_{c0}(\vartheta, z, \dot{z}) = A_0(\vartheta, z, \dot{z}) - B(\vartheta, z, \dot{z}) K_0(\vartheta, z, \dot{z}) C_0(\vartheta, z, \dot{z}), \tag{5a}$$

$$A_{c1}(\vartheta, z, \dot{z}) = A_1(\vartheta, z, \dot{z}) - B(\vartheta, z, \dot{z}) K_1(\vartheta, z, \dot{z}) C_1(\vartheta, z, \dot{z}). \tag{5b}$$

The closed-loop system (4) is transformed into a quasi-linear first-order form by defining $x = [\, z \;\; \dot{z} \,]^{\mathrm{T}}$:

$$E(\vartheta, x) \dot{x} = A_c(\vartheta, x) x + B_c(\vartheta, x) v, \tag{6}$$

with

$$E(\vartheta, x) = \begin{bmatrix} I_n & 0 \\ 0 & A_2(\vartheta, x) \end{bmatrix}, \tag{7a}$$

$$A_c(\vartheta, x) = \begin{bmatrix} 0 & I_n \\ -A_{c0}(\vartheta, x) & -A_{c1}(\vartheta, x) \end{bmatrix}, \tag{7b}$$

$$B_c(\vartheta, x) = \begin{bmatrix} 0 \\ B(\vartheta, x) \end{bmatrix}. \tag{7c}$$

This approach aims to turn the closed-loop system (6), which is transformed by (1a)–(1c), into a linear system with a predesigned eigenstructure by the designing controller (3).

3 Motivation and steps of the proposed PESA to reduce complicated constraints

First of all, the description of existing generalized eigenstructure assignment [22] in the quasi-linear second-order systems is presented. Namely, it seeks the gain matrices $K_0(\vartheta, z, \dot{z}) \in \mathbb{R}^{r\times m_0}, K_1(\vartheta, z, \dot{z}) \in$

$\mathbb{R}^{r\times m_1}$ and two corresponding matrices ${T_g}^{\mathrm{T}}(\vartheta,x)\in\mathbb{R}^{2n\times 2n}$, $V_g(\vartheta,x)\in\mathbb{R}^{2n\times 2n}$ satisfying

$$ {T_g}^{\mathrm{T}}(\vartheta,x)\,A_c(\vartheta,x)=F{T_g}^{\mathrm{T}}(\vartheta,x)\,E(\vartheta,x)\,, \tag{8}$$
$$ A_c(\vartheta,x)\,V_g(\vartheta,x)=E(\vartheta,x)\,V_g(\vartheta,x)\,F, \tag{9}$$

and

$$ {T_g}^{\mathrm{T}}(\vartheta,x)\,E(\vartheta,x)\,V_g(\vartheta,x)=I_{2n}, \tag{10}$$

where $F\in\mathbb{R}^{2n\times 2n}$ is similar to the matrix $A_c(\vartheta,x)$ with desired eigenvalues. The existing scheme assigns the entire set of generalized eigenvector and needs to satisfy the normalized pair (10). However, it is not necessary to assign the entire set of generalized eigenvector columns of V_g and T_g while it would be sufficient to assign a subset of the generalized eigenvector columns of them. If F is partitioned into two parts, the associated right and left eigenvectors are separated simultaneously. The products of partial right and left eigenvectors have low dimensions and this can greatly reduce the number of constraints. Such a fact naturally inspires us to propose the following PESA problem.

Remark 2. It is worth noting that this PESA approach does not mean only partial eigenvalues are assigned and just F, which includes the information of eigenvalues, is separated into two parts, so the associated right and left eigenvectors are separated simultaneously which naturally gives partial right and left eigenvectors. If F is in a Jordan form or diagonal, it can be directly separated as

$$ F=\begin{bmatrix} F_l & \\ & F_r \end{bmatrix}. \tag{11}$$

If F is an arbitrary matrix, it requires that the structure of F_r and F_l has a common Jordan form with F after some similarity transformation.

The subsets of right and left eigenvectors of the $(E(\vartheta,x)\,,A_c(\vartheta,x))$ of the proposed PESA-based parametric approach are established as

$$ {T_o}^{\mathrm{T}}(\vartheta,x)\,A_c(\vartheta,x)=F_l{T_o}^{\mathrm{T}}(\vartheta,x)\,E(\vartheta,x)\,, \tag{12}$$
$$ A_c(\vartheta,x)\,V_o(\vartheta,x)=E(\vartheta,x)\,V_o(\vartheta,x)\,F_r, \tag{13}$$

and

$$ {T_o}^{\mathrm{T}}(\vartheta,x)\,E(\vartheta,x)\,V_o(\vartheta,x)=0_{(2n-m)\times m}, \tag{14}$$

where $T_o^{\mathrm{T}}\in\mathbb{R}^{(2n-m)\times 2n}$, $V_o\in\mathbb{R}^{2n\times m}$, $F_l\in\mathbb{R}^{(2n-m)\times(2n-m)}$ and $F_r\in\mathbb{R}^{m\times m}$.

Remark 3. The prevailing approach that assigns the entire set of generalized eigenvectors in [22] surely holds the relation (10) between right and left eigenvector matrices since the approach deals with the same F and $T_g^{\mathrm{T}}=V_g^{-1}$. However, the proposed PESA approach has a new relation (14) since F is partitioned into two parts, and the products of the partial right and left eigenvector matrices equal zero.

Remark 4. In fact, it is difficult to satisfy a large number of complicated constraints (10) in practical systems by the normalized pair of right and left eigenvector matrices. Thus, the idea of PESA is proposed and provides less computational load to make it easier to be utilized than the prevailing approach that assigns the entire set of generalized eigenvectors. If the full eigenvectors assignment is utilized, the constraint needs to satisfy (10) and it includes $2n\times 2n$ equations in this condition. However, the constraint (14) of the proposed PESA approach only needs to satisfy $(2n-m)\,m$ equations, which greatly reduces the number of constraint equations and is more numerically economical and efficient.

In order to unfold the following analysis, some transformations about the partial set of left and right generalized eigenvectors in (12) and (13) are introduced. In order to simplify the analysis, the partial left closed-loop eigenvector matrix is rewritten as

$$ T_o^{\mathrm{T}}(\vartheta,x)=\left[\,T_{o0}^{\mathrm{T}}(\vartheta,x)\;\,T_{o1}^{\mathrm{T}}(\vartheta,x)\,\right], \tag{15}$$

where $T_{o0}^{\mathrm{T}}\in\mathbb{R}^{(2n-m)\times n}$ and $T_{o1}^{\mathrm{T}}\in\mathbb{R}^{(2n-m)\times n}$. Then Eq. (12) is rewritten as

$$ \left[\,T_{o0}^{\mathrm{T}}(\vartheta,x)\;\,T_{o1}^{\mathrm{T}}(\vartheta,x)\,\right]A_c(\vartheta,x)=F_l\left[\,T_{o0}^{\mathrm{T}}(\vartheta,x)\;\,T_{o1}^{\mathrm{T}}(\vartheta,x)\,\right]E(\vartheta,x)\,. \tag{16}$$

Substituting (7a) and (7b) into the above equation, it follows that

$$F_l{}^2 T^{\mathrm{T}}(\vartheta, x) A_2(\vartheta, x) + F_l T^{\mathrm{T}}(\vartheta, x) A_{c1}(\vartheta, x) + T^{\mathrm{T}}(\vartheta, x) A_{c0}(\vartheta, x) = 0, \tag{17}$$

where

$$T_{o1}^{\mathrm{T}}(\vartheta, x) = T^{\mathrm{T}}(\vartheta, x), \tag{18a}$$

$$T_{o0}^{\mathrm{T}}(\vartheta, x) = F_l T^{\mathrm{T}}(\vartheta, x) A_2(\vartheta, x) + T^{\mathrm{T}}(\vartheta, x) A_{c1}(\vartheta, x). \tag{18b}$$

And the partial right closed-loop eigenvector matrix is also rewritten as

$$V_o(\vartheta, x) = \begin{bmatrix} V_{o0} \\ V_{o1} \end{bmatrix}, \tag{19}$$

where $V_{o0} \in \mathbb{R}^{n\times m}$ and $V_{o1} \in \mathbb{R}^{n\times m}$. Then Eq. (13) is rewritten as

$$A_c(\vartheta, x) \begin{bmatrix} V_{o0}(\vartheta, x) \\ V_{o1}(\vartheta, x) \end{bmatrix} = E(\vartheta, x) \begin{bmatrix} V_{o0}(\vartheta, x) \\ V_{o1}(\vartheta, x) \end{bmatrix} F_r. \tag{20}$$

Substituting (7a) and (7b) into the above equation, it follows that

$$A_2(\vartheta, x) V(\vartheta, x) F_r^2 + A_{c1}(\vartheta, x) V(\vartheta, x) F_r + A_{c0}(\vartheta, x) V(\vartheta, x) = 0, \tag{21}$$

where

$$V_{o0}(\vartheta, x) = V(\vartheta, x), \tag{22a}$$

$$V_{o1}(\vartheta, x) = V(\vartheta, x) F_r. \tag{22b}$$

Problem 1. Consider the system described by (1a)–(1c) under Assumptions 1 and 2. For two matrices $F_r = \mathbb{R}^{m\times m}$ and $F_l = \mathbb{R}^{(2n-m)\times(2n-m)}$, find all the matrices $K_0(\vartheta, z, \dot{z}) \in \mathbb{R}^{r\times m_0}$, $K_1(\vartheta, z, \dot{z}) \in \mathbb{R}^{r\times m_1}$ and all the matrices $T_o \in \mathbb{R}^{2n\times(2n-m)}$ and $V_o(\vartheta, x) \in \mathbb{R}^{2n\times m}$ satisfying (17), (21), and, meanwhile, meeting the constraint (14).

4 Parametric solution

Consider that there exist a pair of polynomial matrices

$$\begin{cases} N(\vartheta, x, s) = \sum\limits_{i=0}^{\omega_r} N_i(\vartheta, x) s^i, \\ D(\vartheta, x, s) = \sum\limits_{i=0}^{\omega_r} D_i(\vartheta, x) s^i, \end{cases} \tag{23}$$

which satisfy the following time-varying right coprime factorization:

$$\left[s^2 A_2(\vartheta, x) + s A_1(\vartheta, x) + A_0(\vartheta, x)\right]^{-1} B(\vartheta, x) = N(\vartheta, x, s) D^{-1}(\vartheta, x, s), \tag{24}$$

where $N(\vartheta, x, s) \in \mathbb{R}^{n\times r}[s]$, $D(\vartheta, x, s) \in \mathbb{R}^{r\times r}[s]$, and $\omega_r = \max\{\deg(d_{ij}(\vartheta, x, s)), i = 1, 2, \ldots, r, j = 1, 2, \ldots, r,\}$ when $D(\vartheta, x, s) = [d_{ij}(\vartheta, x, s)]_{r\times r}$.

Similarly, there exist polynomial matrices:

$$\begin{cases} H(\vartheta, x, s) = \sum\limits_{i=0}^{\omega_l} H_i(\vartheta, x) s^i, \\ L(\vartheta, x, s) = \begin{bmatrix} L_0(\vartheta, x, s) \\ L_1(\vartheta, x, s) \end{bmatrix} = \begin{bmatrix} \sum\limits_{i=0}^{\omega_l} L_{0i}(\vartheta, x) s^i \\ \sum\limits_{i=0}^{\omega_l} L_{1i}(\vartheta, x) s^i \end{bmatrix}, \end{cases} \tag{25}$$

which satisfy the time-varying right coprime factorization:

$$\left[s^2 A_2^{\mathrm{T}}(\vartheta, x) + s A_1^{\mathrm{T}}(\vartheta, x) + A_0^{\mathrm{T}}(\vartheta, x)\right]^{-1} \begin{bmatrix} C_0^{\mathrm{T}}(\vartheta, x) & s C_1^{\mathrm{T}}(\vartheta, x) \end{bmatrix} = H(\vartheta, x, s) L^{-1}(\vartheta, x, s), \tag{26}$$

where $H(\vartheta, x, s) \in \mathbb{R}^{n\times m}[s]$, $L(\vartheta, x, s) \in \mathbb{R}^{m\times m}[s]$, $L_0(\vartheta, x, s) \in \mathbb{R}^{m_0\times m}[s]$, $L_1(\vartheta, x, s) \in \mathbb{R}^{m_1\times m}[s]$ and $\omega_l = \max\{\deg(l_{ij}(\vartheta, x, s)), i = 1, 2, \ldots, r, j = 1, 2, \ldots, r\}$ when $L(\vartheta, x, s) = [l_{ij}(\vartheta, x, s)]_{m\times m}$.

4.1 Case of F_l and F_r being arbitrary

With the above preparation, the general solution to Problem 1 in terms of the above polynomial matrices $N(\vartheta,x,s)$, $D(\vartheta,x,s)$, $H(\vartheta,x,s)$ and $L(\vartheta,x,s)$ is presented.

Theorem 1. Let $N(\vartheta,x,s)$ and $D(\vartheta,x,s)$ be a couple of right coprime polynomial matrices given by (24), and $H(\vartheta,x,s)$ and $L(\vartheta,x,s)$ be a couple of right coprime polynomial matrices given by (26). Then all the solutions to Problem 1 can be stressed as follows.

(1) Problem 1 has a solution if and only if there exist two matrices $Z_l \in \mathbb{R}^{m\times(2n-m)}$ and $Z_r \in \mathbb{R}^{r\times m}$ satisfying

$$\begin{bmatrix} I & (F_l)^{\mathrm{T}} & \cdots & (F_l^{\omega_l})^{\mathrm{T}} \end{bmatrix} \Theta(Z_l,Z_r) \begin{bmatrix} I \\ F_r \\ \vdots \\ F_r^{\omega_r} \end{bmatrix} = 0, \tag{27}$$

where

$$\Theta(Z_l,Z_r) = [\Theta_{ij}(Z_l,Z_r)]_{\omega_l n\times\omega_r n}, \tag{28}$$

with

$$\begin{aligned} \Theta_{ij}(Z_l,Z_r) = {} & F_l Z_l^{\mathrm{T}} H_{i-1}^{\mathrm{T}}(\vartheta,x) A_2(\vartheta,x) N_{j-1}(\vartheta,x) Z_r \\ & + Z_l^{\mathrm{T}} H_{i-1}^{\mathrm{T}}(\vartheta,x) A_1(\vartheta,x) N_{j-1}(\vartheta,x) Z_r \\ & - Z_l^{\mathrm{T}} L_{1i-1}^{\mathrm{T}}(\vartheta,x) C_1(\vartheta,x) N_{j-1}(\vartheta,x) Z_r \\ & + Z_l^{\mathrm{T}} H_{i-1}^{\mathrm{T}}(\vartheta,x) A_2(\vartheta,x) N_{j-1}(\vartheta,x) Z_r F_r, \\ & i=1,2,\ldots,\omega_l+1,\ j=1,2,\ldots,\omega_r+1. \end{aligned} \tag{29}$$

(2) Under the premise of the above solution, the partial left and right eigenvector matrices T_o and V_o are expressed as

$$T_o = \begin{bmatrix} T_{o0} \\ T_{o1} \end{bmatrix} = \begin{bmatrix} A_2^{\mathrm{T}}(\vartheta,x) T(Z_l,F_l) F_l^{\mathrm{T}} \\ +A_1^{\mathrm{T}}(\vartheta,x) T(Z_l,F_l) \\ -C_1^{\mathrm{T}}(\vartheta,x) W_{l1}(Z_l,F_l) \\ T(Z_l,F_l) \end{bmatrix}, \tag{30}$$

and

$$V_o = \begin{bmatrix} V_{o0} \\ V_{o1} \end{bmatrix} = \begin{bmatrix} V(Z_r,F_r) \\ V(Z_r,F_r) F_r \end{bmatrix}, \tag{31}$$

where

$$\begin{cases} V(Z_r,F_r) = \sum\limits_{i=0}^{\omega_r} N_i(\vartheta,x) Z_r F_r^i, \\ T(Z_l,F_l) = \sum\limits_{i=0}^{\omega_l} H_i(\vartheta,x) Z_l F_l^i, \\ W_{l1}(Z_l,F_l) = \sum\limits_{i=0}^{\omega_l} L_{1i}(\vartheta,x) Z_l F_l^i. \end{cases} \tag{32}$$

(3) Thus, the gain matrices for the output feedback controller are parameterized as either

$$K(\vartheta,x) = W_r(Z_r,F_r)\left(C(\vartheta,x) V_o(Z_r,F_r)\right)^{-1} \tag{33}$$

or

$$T^{\mathrm{T}}(Z_l,F_l) B(\vartheta,x) K(\vartheta,x) = W_l^{\mathrm{T}}(Z_l,F_l), \tag{34}$$

where

$$K(\vartheta,x) = \begin{bmatrix} K_0(\vartheta,x) & K_1(\vartheta,x) \end{bmatrix}, \tag{35}$$

$$C(\vartheta,x)=\begin{bmatrix} C_0(\vartheta,x) & 0 \\ 0 & C_1(\vartheta,x)\end{bmatrix}, \tag{36}$$

$$W_l^{\mathrm{T}}(Z_l,F_l)=\begin{bmatrix} W_{l0}^{\mathrm{T}}(Z_l,F_l) & W_{l1}^{\mathrm{T}}(Z_l,F_l)\end{bmatrix}, \tag{37}$$

$$\begin{cases} W_r(Z_r,F_r)=\sum\limits_{i=0}^{\omega_r} D_i(\vartheta,x)\,Z_rF_r^i, \\ W_{l0}(Z_l,F_l)=\sum\limits_{i=0}^{\omega_l} L_{0i}(\vartheta,x)\,Z_lF_l^i. \end{cases} \tag{38}$$

Remark 5. Even though both Eqs. (33) and (34) can be utilized to obtain the gain matrices and they have a common solution, Eq. (33) is simpler and more direct. In addition, Eq. (34) can be turned into $K(\vartheta,x)=((T^{\mathrm{T}}B)^{\mathrm{T}}T^{\mathrm{T}}B)^{-1}(T^{\mathrm{T}}B)^{\mathrm{T}}W_l^{\mathrm{T}}$ if $(T^{\mathrm{T}}B)^{\mathrm{T}}T^{\mathrm{T}}B$ has a full rank.

Proof. The sketch of the proof can be stressed through the following steps.

- Obtain the second-order generalized Sylvester equations by (17) and (21), and then the parametric expressions of (32) and (38) can be derived.
- Under the premise parametric expressions of $V(Z_r,F_r)$, $W_r(Z_r,F_r)$, $T(Z_l,F_l)$, $W_{l0}(Z_l,F_l)$ and $W_{l1}(Z_l,F_l)$, the condition (27) is proven.
- Derive all the corresponding parameterized gain matrices and prove that both the two expressions (33) and (34) are common.

Step 1. Obtain the second-order generalized Sylvester equations and related parametric expressions.

Substituting (5a) and (5b) into (17) and (21), it follows that

$$\begin{aligned} &{F_l}^2T^{\mathrm{T}}(\vartheta,x)\,A_2(\vartheta,x)+F_lT^{\mathrm{T}}(\vartheta,x)\,A_1(\vartheta,x)+T^{\mathrm{T}}(\vartheta,x)\,A_0(\vartheta,x)\\ &=T^{\mathrm{T}}(\vartheta,x)\,B(\vartheta,x)\,K_0(\vartheta,x)\,C_0(\vartheta,x)+F_lT^{\mathrm{T}}(\vartheta,x)\,B(\vartheta,x)\,K_1(\vartheta,x)\,C_1(\vartheta,x), \end{aligned} \tag{39}$$

and

$$\begin{aligned} &A_2(\vartheta,x)\,V(\vartheta,x)\,F_r^2+A_1(\vartheta,x)\,V(\vartheta,x)\,F_r+A_0(\vartheta,x)\,V(\vartheta,x)\\ &=B(\vartheta,x)\,K_0(\vartheta,x)\,C_0(\vartheta,x)\,V(\vartheta,x)+B(\vartheta,x)\,K_1(\vartheta,x)\,C_1(\vartheta,x)\,V(\vartheta,x)\,F_r. \end{aligned} \tag{40}$$

Let

$$W_{l0}^{\mathrm{T}}(\vartheta,x)=T^{\mathrm{T}}(\vartheta,x)\,B(\vartheta,x)\,K_0(\vartheta,x), \tag{41}$$

$$W_{l1}^{\mathrm{T}}(\vartheta,x)=T^{\mathrm{T}}(\vartheta,x)\,B(\vartheta,x)\,K_1(\vartheta,x), \tag{42}$$

and

$$W_r(\vartheta,x)=K_0(\vartheta,x)\,C_0(\vartheta,x)\,V(\vartheta,x)+K_1(\vartheta,x)\,C_1(\vartheta,x)\,V(\vartheta,x)\,F_r. \tag{43}$$

With the help of (31), (35) and (36), Eq. (43) is rewritten as

$$W_r(\vartheta,x)=K(\vartheta,x)\,C(\vartheta,x)\,V_o(\vartheta,x), \tag{44}$$

and then Eqs. (39) and (40) can deduce the following second-order generalized Sylvester equations with the help of (41), (42) and (44), respectively:

$$\begin{aligned} &{F_l}^2T^{\mathrm{T}}(\vartheta,x)\,A_2(\vartheta,x)+F_lT^{\mathrm{T}}(\vartheta,x)\,A_1(\vartheta,x)+T^{\mathrm{T}}(\vartheta,x)\,A_0(\vartheta,x)\\ &=W_{l0}^{\mathrm{T}}(\vartheta,x)\,C_0(\vartheta,x)+F_lW_{l1}^{\mathrm{T}}(\vartheta,x)\,C_1(\vartheta,x), \end{aligned} \tag{45}$$

and

$$A_2(\vartheta,x)\,V(\vartheta,x)\,F_r^2+A_1(\vartheta,x)\,V(\vartheta,x)\,F_r+A_0(\vartheta,x)\,V(\vartheta,x)=B(\vartheta,x)\,W_r^{\mathrm{T}}(\vartheta,x). \tag{46}$$

Thus, the related solutions to these second-order generalized Sylvester equations are obtained as (32) and (38) by utilizing the well established solutions in [30].

Step 2. Prove the condition (27).

Throughout (18a) and (18b), the definitions of A_{c1} in (5b) and W_{l1} in (42), Eq. (30) can be easily obtained. Furthermore, Eq. (31) can be directly obtained by (22a) and (22b).

Substituting (7a), (7b), (30) and (31) into (14), the following formula can be obtained:

$$F_lT^{\mathrm{T}}(Z_l,F_l)\,A_2(\vartheta,x)\,V(\vartheta,x)+T^{\mathrm{T}}(\vartheta,x)\,A_1(\vartheta,x)\,V(\vartheta,x)$$

$$- W_{l1}^{\mathrm{T}}\left(Z_l,F_l\right)C_1\left(\vartheta,x\right)V\left(\vartheta,x\right)+T^{\mathrm{T}}\left(Z_l,F_l\right)A_2\left(\vartheta,x\right)V\left(\vartheta,x\right)F_r=0. \tag{47}$$

With the help of (32) and (38), the above equation can be expressed as

$$\begin{aligned} &F_l\left(\sum_{i=0}^{\omega_l}\left(F_l^i\right)^{\mathrm{T}}Z_l^{\mathrm{T}}H_i^{\mathrm{T}}\left(\vartheta,x\right)\right)A_2\left(\vartheta,x\right)\left(\sum_{j=0}^{\omega_r}N_j\left(\vartheta,x\right)Z_rF_r^j\right)\\ &+\left(\sum_{i=0}^{\omega_l}\left(F_l^i\right)^{\mathrm{T}}Z_l^{\mathrm{T}}H_i^{\mathrm{T}}\left(\vartheta,x\right)\right)A_1\left(\vartheta,x\right)\left(\sum_{j=0}^{\omega_r}N_j\left(\vartheta,x\right)Z_rF_r^j\right)\\ &-\left(\sum_{i=0}^{\omega_l}\left(F_l^i\right)^{\mathrm{T}}Z_l^{\mathrm{T}}L_{1i}^{\mathrm{T}}\left(\vartheta,x\right)\right)C_1\left(\vartheta,x\right)\left(\sum_{j=0}^{\omega_r}N_j\left(\vartheta,x\right)Z_rF_r^j\right)\\ &+\left(\sum_{i=0}^{\omega_l}\left(F_l^i\right)^{\mathrm{T}}Z_l^{\mathrm{T}}H_i^{\mathrm{T}}\left(\vartheta,x\right)\right)A_2\left(\vartheta,x\right)\left(\sum_{j=0}^{\omega_r}N_j\left(\vartheta,x\right)Z_rF_r^j\right)F_r=0. \end{aligned} \tag{48}$$

As a consequence, this equation can be directly rewritten in the form of (27).

Step 3. Derive all the corresponding parameterized gain matrices.

Substituting (41) and (42) into (37) gives

$$W_l^{\mathrm{T}}\left(Z_l,F_l\right)=T^{\mathrm{T}}\left(\vartheta,x\right)B\left(\vartheta,x\right)K\left(\vartheta,x\right). \tag{49}$$

Under the premise steps, the gain matrices can be easily derived by (44) and (49), respectively. However, $T^{\mathrm{T}}B$ is not a square matrix, so both sides of (49) are multiplied by $\left(T^{\mathrm{T}}\left(Z_l,F_l\right)B\left(\vartheta,x\right)\right)^{\mathrm{T}}$ and post-multiplied by the inverse of $((T^{\mathrm{T}}B)^{\mathrm{T}}T^{\mathrm{T}}B)^{-1}$ to obtain $K\left(\vartheta,x\right)=((T^{\mathrm{T}}B)^{\mathrm{T}}T^{\mathrm{T}}B)^{-1}(T^{\mathrm{T}}B)^{\mathrm{T}}W_l^{\mathrm{T}}$ if $(T^{\mathrm{T}}B)^{\mathrm{T}}T^{\mathrm{T}}B$ has a full rank.

With the help of (43) and (49), we can obtain that Eqs. (33) and (34) have a common solution if and only if the following condition holds:

$$T^{\mathrm{T}}\left(\vartheta,x\right)B\left(\vartheta,x\right)W_r\left(\vartheta,x\right)=T^{\mathrm{T}}BKCV_o=W_l^{\mathrm{T}}\left(\vartheta,x\right)C\left(\vartheta,x\right)V_o\left(\vartheta,x\right). \tag{50}$$

In order to prove (50), we need the following preparations.

First of all, substituting (5b) and (42) into (18b), it follows that

$$\begin{aligned} T_{o0}^{\mathrm{T}}\left(\vartheta,x\right)&=F_lT^{\mathrm{T}}\left(\vartheta,x\right)A_2\left(\vartheta,x\right)+T^{\mathrm{T}}\left(\vartheta,x\right)A_{c1}\left(\vartheta,x\right)\\ &=F_lT^{\mathrm{T}}\left(\vartheta,x\right)A_2\left(\vartheta,x\right)+T^{\mathrm{T}}\left(\vartheta,x\right)\left(A_1\left(\vartheta,x\right)-B\left(\vartheta,x\right)K_1\left(\vartheta,x\right)C_1\left(\vartheta,x\right)\right)\\ &=F_lT^{\mathrm{T}}\left(\vartheta,x\right)A_2\left(\vartheta,x\right)+T^{\mathrm{T}}\left(\vartheta,x\right)A_1\left(\vartheta,x\right)-T^{\mathrm{T}}\left(\vartheta,x\right)B\left(\vartheta,x\right)K_1\left(\vartheta,x\right)C_1\left(\vartheta,x\right)\\ &=F_lT^{\mathrm{T}}\left(\vartheta,x\right)A_2\left(\vartheta,x\right)+T^{\mathrm{T}}\left(\vartheta,x\right)A_1\left(\vartheta,x\right)-W_{l1}^{\mathrm{T}}\left(\vartheta,x\right)C_1\left(\vartheta,x\right), \end{aligned} \tag{51}$$

and the above equation gives

$$T^{\mathrm{T}}\left(\vartheta,x\right)A_1\left(\vartheta,x\right)=W_{l1}^{\mathrm{T}}\left(\vartheta,x\right)C_1\left(\vartheta,x\right)+T_{o0}^{\mathrm{T}}\left(\vartheta,x\right)-F_lT^{\mathrm{T}}\left(\vartheta,x\right)A_2\left(\vartheta,x\right). \tag{52}$$

Then substituting (5a), (18b) and (41) into (17) gives

$$\begin{aligned} 0&={F_l}^2T^{\mathrm{T}}\left(\vartheta,x\right)A_2\left(\vartheta,x\right)+F_lT^{\mathrm{T}}\left(\vartheta,x\right)A_{c1}\left(\vartheta,x\right)+T^{\mathrm{T}}\left(\vartheta,x\right)A_{c0}\left(\vartheta,x\right)\\ &=T^{\mathrm{T}}\left(\vartheta,x\right)\left(A_0\left(\vartheta,x\right)-B\left(\vartheta,x\right)K_0\left(\vartheta,x\right)C_0\left(\vartheta,x\right)\right)\\ &\quad+F_l\left(F_lT^{\mathrm{T}}\left(\vartheta,x\right)A_2\left(\vartheta,x\right)+T^{\mathrm{T}}\left(\vartheta,x\right)A_{c1}\left(\vartheta,x\right)\right)\\ &=T^{\mathrm{T}}\left(\vartheta,x\right)\left(A_0\left(\vartheta,x\right)-B\left(\vartheta,x\right)K_0\left(\vartheta,x\right)C_0\left(\vartheta,x\right)\right)+F_lT_{o0}^{\mathrm{T}}\left(\vartheta,x\right)\\ &=T^{\mathrm{T}}\left(\vartheta,x\right)A_0\left(\vartheta,x\right)-T^{\mathrm{T}}\left(\vartheta,x\right)B\left(\vartheta,x\right)K_0\left(\vartheta,x\right)C_0\left(\vartheta,x\right)+F_lT_{o0}^{\mathrm{T}}\left(\vartheta,x\right)\\ &=T^{\mathrm{T}}\left(\vartheta,x\right)A_0\left(\vartheta,x\right)-W_{l0}^{\mathrm{T}}\left(\vartheta,x\right)C_0\left(\vartheta,x\right)+F_lT_{o0}^{\mathrm{T}}\left(\vartheta,x\right), \end{aligned} \tag{53}$$

and the above equation gives

$$T^{\mathrm{T}}\left(\vartheta,x\right)A_0\left(\vartheta,x\right)=W_{l0}^{\mathrm{T}}\left(\vartheta,x\right)C_0\left(\vartheta,x\right)-F_lT_{o0}^{\mathrm{T}}\left(\vartheta,x\right). \tag{54}$$

In addition, substituting (15), (18b) and (31) into (14) gives

$$\begin{aligned} 0 &= T_o{}^{\mathrm{T}}(\vartheta, x) E(\vartheta, x) V_o(\vartheta, x) \\ &= \left[T_{o0}^{\mathrm{T}}(\vartheta, x)\ T_{o1}^{\mathrm{T}}(\vartheta, x) \right] \begin{bmatrix} I_n & 0 \\ 0 & A_2(\vartheta, x) \end{bmatrix} \begin{bmatrix} V(\vartheta, x) \\ V(\vartheta, x) F_r \end{bmatrix} \\ &= T_{o1}^{\mathrm{T}}(\vartheta, x) A_2(\vartheta, x) V(\vartheta, x) F_r + T_{o0}^{\mathrm{T}}(\vartheta, x) V(\vartheta, x). \end{aligned} \tag{55}$$

Finally, combining (46), (52), (54) and (55), it follows that

$$\begin{aligned} T^{\mathrm{T}} B W_r &= T^{\mathrm{T}} \left(A_2 V F_r^2 + A_1 V F_r + A_0 V \right) \\ &= T^{\mathrm{T}} A_2 V F_r^2 + T^{\mathrm{T}} A_1 V F_r + T^{\mathrm{T}} A_0 V \\ &= T^{\mathrm{T}} A_2 V F_r^2 + \left(W_{l1}^{\mathrm{T}} C_1 + T_{o0}^{\mathrm{T}} - F_l T^{\mathrm{T}} A_2 \right) V F_r + \left(W_{l0}^{\mathrm{T}} C_0 - F_l T_{o0}^{\mathrm{T}} \right) V \\ &= T_{o1}^{\mathrm{T}} A_2 V F_r^2 + W_{l1}^{\mathrm{T}} C_1 V F_r + T_{o0}^{\mathrm{T}} V F_r - F_l T_{o1}^{\mathrm{T}} A_2 V F_r + W_{l0}^{\mathrm{T}} C_0 V - F_l T_{o0}^{\mathrm{T}} V \\ &= \left(T_{o1}^{\mathrm{T}} A_2 V F_r + T_{o0}^{\mathrm{T}} V \right) F_r - F_l \left(T_{o1}^{\mathrm{T}} A_2 V F_r + T_{o0}^{\mathrm{T}} V \right) \\ &\quad + \left[W_{l0}^{\mathrm{T}}\ W_{l1}^{\mathrm{T}} \right] \begin{bmatrix} C_0 & 0 \\ 0 & C_1 \end{bmatrix} \begin{bmatrix} V \\ V F_r \end{bmatrix} \\ &= W_l^{\mathrm{T}} C V_o. \end{aligned} \tag{56}$$

4.2 Case of F_l and F_r being diagonal

In many practical scenarios, F is usually taken as a diagonal form and it follows that

$$F_l = \operatorname{diag}(s_1, s_2, \ldots, s_{2n-m}), \tag{57}$$

$$F_r = \operatorname{diag}(s_1, s_2, \ldots, s_m), \tag{58}$$

where $s_1, s_2, \ldots, s_{2n} \in \mathbb{C}^-$ are a group of self-conjugate complex poles. Thus, the solution of the second-order generalized Sylvester equation (45) is expressed as

$$\begin{cases} T = [\, t_1\ t_2\ \cdots\ t_{2n-m} \,], \\ t_i = H(\vartheta, x, s_i) z_{l,i}, \quad i = 1, 2, \ldots, 2n-m, \end{cases} \tag{59}$$

and

$$\begin{cases} W_l = \begin{bmatrix} W_{l0} \\ W_{l1} \end{bmatrix} = \begin{bmatrix} \omega_{l0,1}\ \omega_{l0,2}\ \cdots\ \omega_{l0,2n-m} \\ \omega_{l1,1}\ \omega_{l1,2}\ \cdots\ \omega_{l1,2n-m} \end{bmatrix}, \\ \omega_{l0,i} = L_0(\vartheta, x, s_i) z_{l,i}, \quad i = 1, 2, \ldots, 2n-m, \\ \omega_{l1,i} = L_1(\vartheta, x, s_i) z_{l,i}, \quad i = 1, 2, \ldots, 2n-m. \end{cases} \tag{60}$$

Similarly, the solution of the second-order generalized Sylvester equation (46) gives

$$\begin{cases} V = [\, v_{2n-m+1}\ v_{2n-m+2}\ \cdots\ v_{2n} \,], \\ v_{2n-m+j} = N(\vartheta, x, s_j) z_{r,j}, \ j = 1, 2, \ldots, m, \end{cases} \tag{61}$$

and

$$\begin{cases} W_r = [\, \omega_{r,2n-m+1}\ \omega_{r,2n-m+2}\ \cdots\ \omega_{r,2n-m} \,], \\ \omega_{r,2n-m+j} = D(\vartheta, x, s_j) z_{r,j}, \ j = 1, 2, \ldots, m, \end{cases} \tag{62}$$

where $z_{l,i} \in \mathbb{R}^m, i = 1, 2, \ldots, 2n-m$ and $z_{r,j} \in \mathbb{R}^r, j = 1, 2, \ldots, m$.

With the above preparation, the general solution to Problem 1 in terms of the diagonal matrix F is presented.

Theorem 2. Let $N(\vartheta, x, s)$ and $D(\vartheta, x, s)$ be a couple of right coprime polynomial matrices given by (24), and $H(\vartheta, x, s)$ and $L(\vartheta, x, s)$ be a couple of right coprime polynomial matrices given by (26). Then all the solutions to Problem 1 with diagonal matrix F can be stressed as follows.

(1) Problem 1 has a solution if and only if there exist two parameter vectors $z_{l,i} \in \mathbb{R}^m, i = 1, 2, \ldots, 2n-m$ and $z_{r,j} \in \mathbb{R}^r, j = 1, 2, \ldots, m$ satisfying

$$\begin{aligned} & s_i z_{l,i}^{\mathrm{T}} H^{\mathrm{T}}(\vartheta, x, s_i) A_2(\vartheta, x) N(\vartheta, x, s_j) z_{r,j} \\ & + z_{l,i}^{\mathrm{T}} H^{\mathrm{T}}(\vartheta, x, s_i) A_1(\vartheta, x) N(\vartheta, x, s_j) z_{r,j} \\ & - z_{l,i}^{\mathrm{T}} L_1^{T}(\vartheta, x, s_i) C_1(\vartheta, x) N(\vartheta, x, s_j) z_{r,j} \\ & + z_{l,i}^{\mathrm{T}} H^{\mathrm{T}}(\vartheta, x, s_i) A_2(\vartheta, x) N(\vartheta, x, s_j) z_{r,j} s_{2n-m+j} = 0, \\ & i = 1, 2, \ldots, 2n-m;\ j = 1, 2, \ldots, m. \end{aligned} \tag{63}$$

(2) Under the premise of the above solution, the partial left and right eigenvector matrices T_o and V_o are expressed as

$$T_{oi} = \begin{bmatrix} A_2^{\mathrm{T}}(\vartheta, x) H(\vartheta, x, s_i) z_{l,i} s_i \\ +A_1^{\mathrm{T}}(\vartheta, x) H(\vartheta, x, s_i) z_{l,i} \\ -C_1^{\mathrm{T}}(\vartheta, x) L_1(\vartheta, x, s_i) z_{l,i} \\ H(\vartheta, x, s_i) z_{l,i} \end{bmatrix}, \quad i = 1, 2, \ldots, 2n-m, \tag{64}$$

and

$$V_{oj} = \begin{bmatrix} N(\vartheta, x, s_j) z_{r,j} \\ N(\vartheta, x, s_j) z_{r,j} s_j \end{bmatrix}, \quad j = 1, 2, \ldots, m. \tag{65}$$

In addition, the gain matrices for the output feedback controller are parameterized as either (33) or (34).

Proof. The matrices in (59)–(62) can be directly obtained with the diagonal form of matrices F_l in (57) and F_r in (58). The results are easy to prove and similar with Theorem 1. For ease of presentation, this process is dropped here.

4.3 General procedure

Based on the above parametric solution with the condition of F_l and F_r being arbitrary or diagonal, a general procedure for the proposed PESA approach can be presented as follows.

Step 1. Specify F_l and F_r according to the closed-loop stability and performance requirements which is similar to the desired partitioned closed-loop eigenvalue matrix.

Step 2. Obtain the solution of two pairs of right coprime factorization $N(\vartheta, x, s)$, $D(\vartheta, x, s)$ and $H(\vartheta, x, s)$, $L(\vartheta, x, s)$ by (24) and (26), respectively.

Step 3. Obtain the solution of T_o and V_o by (30) and (31) with the case of F_l and F_r being arbitrary or by (64) and (65) with the case of F_l and F_r being diagonal.

Step 4. Solve the constraint (14) by (27) or (63) to obtain a set of numerical Z_r and Z_l. Compute the output feedback gain matrices through the formula (33) or (34).

5 Examples

5.1 Numerical example

A numerical example in the form of (1a)–(1c) is presented here to verify the proposed PESA approach:

$$A_2(\vartheta, z, \dot{z}) = B(\vartheta, z, \dot{z}) = C_0(\vartheta, z, \dot{z}) = I_2,$$

$$A_1(\vartheta, z, \dot{z}) = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}, \ A_0(\vartheta, z, \dot{z}) = \begin{bmatrix} 1 & z_2 \\ -1 - z_1 & 2z_2 - \vartheta \end{bmatrix}, \ C_1(\vartheta, z, \dot{z}) = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

where $\vartheta = \sin t$.

The time-varying right coprime factorization (24) and (26) can be deduced with the above system matrices as

$$\begin{cases} N(\vartheta, s) = I_2, \\ D(\vartheta, s) = \begin{bmatrix} s^2+1 & -2s + z_2 \\ 2s - z_1 - 1 & s^2 + 2z_2 - \vartheta \end{bmatrix}, \end{cases}$$

and

$$\begin{cases} H(\vartheta,s)=\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix},\\ L(\vartheta,s)=\begin{bmatrix}L_0(\vartheta,s)\\L_1(\vartheta,s)\end{bmatrix}=\begin{bmatrix}s^2+1&2s-z_1-1&-s\\-2s+z_2&s^2+2z_2-\vartheta&0\\0&0&1\end{bmatrix}.\end{cases}$$

Let

$$z_r=\begin{bmatrix}z_{r,11}&z_{r,12}&z_{r,13}\\z_{r,21}&z_{r,22}&z_{r,23}\end{bmatrix},\quad z_l=\begin{bmatrix}z_{l,1}\\z_{l,2}\\z_{l,3}\end{bmatrix},$$

$$F_r=\mathrm{diag}\left(s_{r1},s_{r2},s_{r3}\right),\ F_l=s_{l1},$$

where $s_{ri}=-i, i=1,2,3$ and $s_{l1}=-4$. Then the condition (14) can be expressed as

$$\begin{bmatrix}z_{r,11}(2z_{l,2}-z_{l,3}+s_{l1}z_{l,1})-z_{r,21}(2z_{l,1}-s_{l1}z_{l,2})+s_{r1}z_{l,1}z_{r,11}+s_{r1}z_{l,2}z_{r,21}\\z_{r,12}(2z_{l,2}-z_{l,3}+s_{l1}z_{l,1})-z_{r,22}(2z_{l,1}-s_{l1}z_{l,2})+s_{r2}z_{l,1}z_{r,12}+s_{r2}z_{l,2}z_{r,22}\\z_{r,13}(2z_{l,2}-z_{l,3}+s_{l1}z_{l,1})-z_{r,23}(2z_{l,1}-s_{l1}z_{l,2})+s_{r3}z_{l,1}z_{r,13}+s_{r3}z_{l,2}z_{r,23}\end{bmatrix}=0,$$

which has the following solutions:

$$\begin{cases}z_{l,2}=\dfrac{2}{s_{l1}}z_{l,1},\\ z_{l,3}=2z_{l,2}+s_{l1}z_{l,1}-1,\end{cases}$$

and

$$\begin{cases}z_{r,11}=-s_{r1}z_{l,2},\\ z_{r,21}=1+s_{r1}z_{l,1},\\ z_{r,12}=-s_{r2}z_{l,2},\\ z_{r,22}=1+s_{r2}z_{l,1},\\ z_{r,13}=-s_{r3}z_{l,2},\\ z_{r,23}=1+s_{r3}z_{l,1}.\end{cases}$$

If the entire set of the left and right generalized eigenvectors scheme is utilized, this constraint needs to solve 16 equations while the proposed PESA scheme only needs to solve the above 3 equations. Particularly, if we choose $z_{l,1}=1$ and a specific solution of the above equation can be obtained.

Thus, it follows that

$$T=\begin{bmatrix}1\\-\frac{1}{2}\end{bmatrix},\ T_o=\begin{bmatrix}-1\\0\\1\\-\frac{1}{2}\end{bmatrix},\ V=\begin{bmatrix}-\frac{1}{2}&-1&-\frac{3}{2}\\0&-1&-2\end{bmatrix},\ V_0=\begin{bmatrix}-\frac{1}{2}&-1&-\frac{3}{2}\\0&-1&-2\\\frac{1}{2}&2&\frac{9}{2}\\0&2&6\end{bmatrix},$$

$$W_r=\begin{bmatrix}-1&-z_2-9&-2z_2-27\\\frac{z_1}{2}+\frac{3}{2}&\vartheta+z_1-2z_2+1&2\vartheta+\frac{3z_1}{2}-4z_2-\frac{15}{2}\end{bmatrix},\ W_l=\begin{bmatrix}\frac{z_1+11}{2}\\\frac{\vartheta}{2}\\-4\end{bmatrix}.$$

Then the gain matrices can be derived:

$$K_0=\begin{bmatrix}-8&z_2-3\\-z_1-11&2z_2-\vartheta-6\end{bmatrix},\ K_1=\begin{bmatrix}-10\\-8\end{bmatrix},$$

and the corresponding closed-loop system is expressed as

$$\ddot{z} + A_1^{cl}\dot{z} + A_0^{cl}z = 0,$$

where

$$A_1^{cl} = \begin{bmatrix} 10 & -2 \\ 10 & 0 \end{bmatrix},\ A_0^{cl} = \begin{bmatrix} 9 & 3 \\ 10 & 6 \end{bmatrix}.$$

In addition, the closed-loop eigenvalues in the form of (7b) can be checked as

$$\mathrm{eig}\left(\begin{bmatrix} 0 & I_2 \\ -A_0^{cl} & -A_1^{cl} \end{bmatrix}\right) = \mathrm{diag}(-4,-3,-2,-1).$$

5.2 An example in spacecraft rendezvous problem

In this subsection, the proposed approach is applied for the control of the spacecraft rendezvous problem, and a simpler linear equation for the chaser's relative motion can be derived with a very close chaser and the target spacecraft [22, 41, 42].

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 2k\dot{\vartheta}^{\frac{3}{2}}x + 2\dot{\vartheta}\dot{y} + \dot{\vartheta}^2 x + \ddot{\vartheta}y \\ -k\dot{\vartheta}^{\frac{3}{2}}y - 2\dot{\vartheta}\dot{x} + \dot{\vartheta}^2 y - \ddot{\vartheta}x \\ -k\dot{\vartheta}^{\frac{3}{2}}z \end{bmatrix} + \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}, \tag{66}$$

where x, y and z represent the radial, along-track and out-of-plane components of the position vector of the chaser satellite, ϑ is the true anomaly, and k is a constant value. In fact, the z subsystem is independent with x and y subsystems, so this example only considers the x and y subsystems to avoid complicated process which may distract the reader from the main focus of the proposed PESA approach.

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 2k\dot{\vartheta}^{\frac{3}{2}}x + 2\dot{\vartheta}\dot{y} + \dot{\vartheta}^2 x + \ddot{\vartheta}y \\ -k\dot{\vartheta}^{\frac{3}{2}}y - 2\dot{\vartheta}\dot{x} + \dot{\vartheta}^2 y - \ddot{\vartheta}x \end{bmatrix} + \begin{bmatrix} u_x \\ u_y \end{bmatrix}. \tag{67}$$

The angular velocity $\dot{\vartheta}$ and acceleration of target spacecraft $\ddot{\vartheta}$ in (67) are represented by

$$\dot{\vartheta} = \sqrt{\frac{\mu\,(1 + e\cos\vartheta)}{r_t^3}} \tag{68}$$

and

$$\ddot{\vartheta} = \frac{-2\mu e\sin\vartheta}{r_t^3}, \tag{69}$$

respectively, $e = 0.73074$ is the eccentricity, and $\mu = 3.986 \times 10^{14}$ m^3/s^2 is the gravitational parameter. The constant k in (67) is chosen as 2.267×10^{-2} /s$^{1/2}$, and all the model parameters are frequently used in [22, 41, 42]. Moreover, r_t in (69) is the vector from the center of gravity to the target spacecraft which is described as

$$r_t = \frac{a\left(1 - e^2\right)}{1 + e\cos\vartheta}, \tag{70}$$

where $a = 2.4616 \times 10^7$ m.

The initial values of $\dot{\vartheta}$ and $\ddot{\vartheta}$ are both 0.8, and the related relative position and relative velocity are initialized as $x(0) = y(0) = 50$ m, $\dot{x}(0) = 3$ m/s and $\dot{y}(0) = -3$ m/s. With the above model (67) and related parameters in it, we design our parametric control scheme.

Let

$$z = [\,x\ y\,]^{\mathrm{T}}.$$

The x and y subsystem models in the condition of (1a)–(1c) are expressed as

$$A_2\,(\vartheta,\dot{z},z) = B\,(\vartheta,\dot{z},z) = I_2,$$

$$A_1(\vartheta,\dot{q},q)=\begin{bmatrix}0 & -2\dot{\vartheta}\\ 2\dot{\vartheta} & 0\end{bmatrix},\ A_0(\vartheta,\dot{q},q)=\begin{bmatrix}-2k\dot{\vartheta}^{\frac{3}{2}}-\dot{\vartheta}^2 & -\ddot{\vartheta}\\ \ddot{\vartheta} & k\dot{\vartheta}^{\frac{3}{2}}-\dot{\vartheta}^2\end{bmatrix},$$

$$C_1(\vartheta,\dot{z},z)=\begin{bmatrix}1 & 0\end{bmatrix},\ C_0(\vartheta,\dot{z},z)=I_2.$$

The time-varying right coprime factorization (24) and (26) can be deduced with the above system matrices as

$$\begin{cases}N(\vartheta,s)=I_2,\\ D(\vartheta,s)=\begin{bmatrix}s^2-\dot{\vartheta}^2-2k\dot{\vartheta}^{\frac{3}{2}} & -2\dot{\vartheta}s-\ddot{\vartheta}\\ 2\dot{\vartheta}s+\ddot{\vartheta} & s^2-\dot{\vartheta}^2+k\dot{\vartheta}^{\frac{3}{2}}\end{bmatrix},\end{cases}$$

and

$$\begin{cases}H(\vartheta,s)=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix},\\ L(\vartheta,s)=\begin{bmatrix}L_0(\vartheta,s)\\ L_1(\vartheta,s)\end{bmatrix}=\begin{bmatrix}s^2-\dot{\vartheta}^2-2k\dot{\vartheta}^{\frac{3}{2}} & 2\dot{\vartheta}s+\ddot{\vartheta} & -s\\ -2\dot{\vartheta}s-\ddot{\vartheta} & s^2-\dot{\vartheta}^2+k\dot{\vartheta}^{\frac{3}{2}} & 0\\ 0 & 0 & 1\end{bmatrix}.\end{cases}$$

Choose

$$z_r=\begin{bmatrix}z_{r,11} & z_{r,12} & z_{r,13}\\ z_{r,21} & z_{r,22} & z_{r,23}\end{bmatrix},\ z_l=\begin{bmatrix}z_{l,1}\\ z_{l,2}\\ z_{l,3}\end{bmatrix},$$

and let

$$F_r=\operatorname{diag}(s_{r1},s_{r2},s_{r3}),\ F_l=s_{l1},$$

where $s_{ri}=-i, i=0.1,0.2,0.3$ and $s_{l1}=-0.4$. Then the condition (14) can be expressed as

$$\begin{bmatrix}z_{r,11}(2\dot{\vartheta}z_{l,2}-z_{l,3}+s_{l1}z_{l,1})-z_{r,21}(2\dot{\vartheta}z_{l,1}-s_{l1}z_{l,2})+s_{r1}z_{l,1}z_{r,11}+s_{r1}z_{l,2}z_{r,21}\\ z_{r,12}(2\dot{\vartheta}z_{l,2}-z_{l,3}+s_{l1}z_{l,1})-z_{r,22}(2\dot{\vartheta}z_{l,1}-s_{l1}z_{l,2})+s_{r2}z_{l,1}z_{r,12}+s_{r2}z_{l,2}z_{r,22}\\ z_{r,13}(2\dot{\vartheta}z_{l,2}-z_{l,3}+s_{l1}z_{l,1})-z_{r,23}(2\dot{\vartheta}z_{l,1}-s_{l1}z_{l,2})+s_{r3}z_{l,1}z_{r,13}+s_{r3}z_{l,2}z_{r,23}\end{bmatrix}=0,$$

which has the following solution:

$$\begin{cases}z_{l,2}=\dfrac{1}{s_{l1}}2\dot{\vartheta}z_{l,1},\\ z_{l,3}=2\dot{\vartheta}z_{l,2}+s_{l1}z_{l,1}-3,\end{cases}$$

and

$$\begin{cases}z_{r,11}=-s_{r1}z_{l,2},\\ z_{r,21}=3+s_{r1}z_{l,1},\\ z_{r,12}=-s_{r2}z_{l,2},\\ z_{r,22}=3+s_{r2}z_{l,1},\\ z_{r,13}=-s_{r3}z_{l,2},\\ z_{r,23}=3+s_{r3}z_{l,1}.\end{cases}$$

Particularly, if we choose $z_{l,1}=1$ and a specific solution of the above equation could be obtained.

Thus, it follows that

$$T=\begin{bmatrix}1\\ -5\dot{\vartheta}\end{bmatrix},\ T_o=\begin{bmatrix}3\\ 0\\ 1\\ -5\dot{\vartheta}\end{bmatrix},\ V=\begin{bmatrix}-\frac{\dot{\vartheta}}{2} & -\dot{\vartheta} & -\frac{3\dot{\vartheta}}{2}\\ \frac{29}{10} & \frac{14}{5} & \frac{27}{10}\end{bmatrix},\ V_0=\begin{bmatrix}-\frac{\dot{\vartheta}}{2} & -\dot{\vartheta} & -\frac{3\dot{\vartheta}}{2}\\ \frac{29}{10} & \frac{14}{5} & \frac{27}{10}\\ \frac{\dot{\vartheta}}{20} & \frac{\dot{\vartheta}}{5} & \frac{9\dot{\vartheta}}{20}\\ -\frac{29}{100} & -\frac{14}{25} & -\frac{81}{100}\end{bmatrix},$$

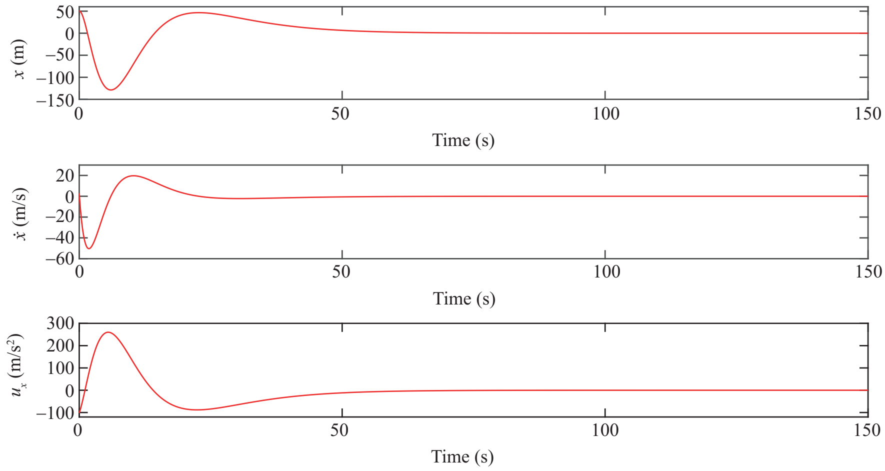

Figure 1 (Color online) Simulation result of relative position x, relative velocity $\dot{x}$, and control signals u_x.

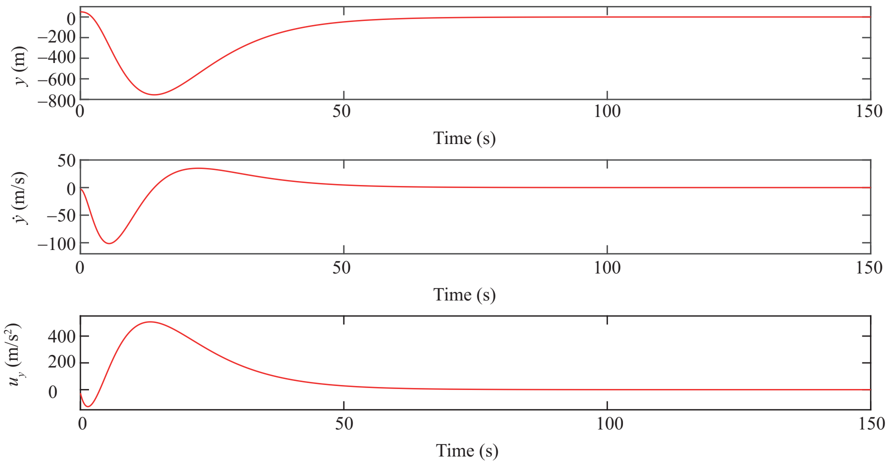

Figure 2 (Color online) Simulation result of relative position y, relative velocity $\dot{y}$, and control signals u_y.

$$W_r = \begin{bmatrix} \frac{29\dot{\vartheta}}{50} - \frac{29\ddot{\vartheta}}{10} + \frac{\dot{\vartheta}}{2}(2k\dot{\vartheta}^{\frac{3}{2}} + \dot{\vartheta}^2 - \frac{1}{100}) & -\frac{14\ddot{\vartheta}}{5} + \frac{28\dot{\vartheta}}{25} + \dot{\vartheta}(2k\dot{\vartheta}^{\frac{3}{2}} + \dot{\vartheta}^2 - \frac{1}{25}) & -\frac{27\ddot{\vartheta}}{10} + \frac{81\dot{\vartheta}}{50} + \frac{3\dot{\vartheta}}{2}(2k\dot{\vartheta}^{\frac{3}{2}} + \dot{\vartheta}^2 - \frac{9}{100}) \\ \frac{29}{10}k\dot{\vartheta}^{\frac{3}{2}} - \frac{\dot{\vartheta}}{2}(\ddot{\vartheta} - \frac{\dot{\vartheta}}{5}) - \frac{29\dot{\vartheta}^2}{10} + \frac{29}{1000} & \frac{14}{5}k\dot{\vartheta}^{\frac{3}{2}} - \dot{\vartheta}(\ddot{\vartheta} - \frac{2}{5}\dot{\vartheta}) - \frac{14}{5}\dot{\vartheta}^2 + \frac{14}{125} & \frac{27}{10}k\dot{\vartheta}^{\frac{3}{2}} - \frac{3\dot{\vartheta}}{2}(\ddot{\vartheta} - \frac{3}{5}\dot{\vartheta}) - \frac{27\dot{\vartheta}^2}{10} + \frac{243}{1000} \end{bmatrix},$$

and

$$W_l = \begin{bmatrix} -5\dot{\vartheta}\left(\ddot{\vartheta} - \frac{4\dot{\vartheta}}{5}\right) - 2k\dot{\vartheta}^{\frac{3}{2}} - 5\dot{\vartheta}^2 - \frac{6}{5} \\ \frac{4}{5}\dot{\vartheta} - \ddot{\vartheta} - \frac{\dot{\vartheta}}{5}\left(k\dot{\vartheta}^{\frac{3}{2}} - \dot{\vartheta}^2 + \frac{4}{25}\right) \\ -10\dot{\vartheta}^2 - \frac{17}{5} \end{bmatrix}.$$

Then the gain matrices can be derived:

$$K_1 = \begin{bmatrix} -1 \\ 2\dot{\vartheta} - \frac{12}{25\dot{\vartheta}} \end{bmatrix}, \quad K_0 = \begin{bmatrix} -2k\dot{\vartheta}^{\frac{3}{2}} - \dot{\vartheta}^2 - \frac{327}{250} & -\ddot{\vartheta} - \frac{\dot{\vartheta}}{100} \\ \ddot{\vartheta} - \frac{27}{1250\dot{\vartheta}} & k\dot{\vartheta}^{\frac{3}{2}} - \dot{\vartheta}^2 - \frac{1}{500} \end{bmatrix}.$$

In addition, the closed-loop eigenvalues in the form of (7b) can be checked as diag$(-0.4, -0.3, -0.2, -0.1)$.

Thus, the simulation results of the relative position x, relative velocity $\dot{x}$, and control signals u_x are shown in Figure 1, while the relative position y, the relative velocity $\dot{y}$, and control signals u_y are shown in Figure 2. It can be observed that both relative positions and velocities quickly converge to zero, which proves the effectiveness of the proposed PESA approach.

6 Conclusion

Herein, a parametric design PESA approach is proposed for stabilizing a type of quasi-linear second-order systems by output feedback control. The PESA approach is established by partitioning the desired eigenvalue matrix into two parts to separate associated right and left eigenvectors into a subset of the generalized eigenvectors simultaneously. Then, a parametric controller is established by solving a pair of second-order generalized Sylvester matrix equations, and a certain form with the desired eigenstructure can be derived using the established quasi-linear output feedback controller. Thus, the proposed subset of generalized eigenvectors is numerically economical and easy to use. A simulation example in general spacecraft rendezvous is provided to verify the numerical economy and high efficiency of the proposed approach. The limitation of the proposed control approach is that the nonlinear systems are assumed to be expressed as a quasi-linear form. Besides, it is not applicable to all kinds of nonlinear systems. The proposed method offers design freedom, which can be further utilized to achieve additional performance and possible extensions for future research to high-order quasi-linear systems.

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