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GE-semigroup method for controllability of stochastic descriptor linear systems

Zhaoqiang GE

School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, China

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Dear editor,

Nowadays, there are more and more researches on stochastic descriptor systems (see [1–7]). For example, Refs. [4,5] studied the exact controllability of a class of stochastic descriptor systems by a strongly continuous semigroup in Hilbert spaces. In this study, the exact controllability and approximate controllability of the following stochastic descriptor system are studied by using the GE-semigroup theory, respectively:

$$M_1\mathrm{d}u(t) = M_2u(t)\mathrm{d}t + M_3v(t)\mathrm{d}t + M_4\mathrm{d}w(t),$$

$$u(0) = u_0, \quad t \geqslant 0,$$

$$x(t) = M_5u(t), \tag{1}$$

where $u(t) \in X_1$, $v(t) \in X_2$, $w(t)$ is the stand Wiener process on $X_3, u_0 \in X_1$, $x(t) \in X_4$; X_1, X_2, X_3, X_4 are Hilbert spaces; M_1, M_2, M_3, M_4 and M_5 are described in the notations below. Firstly, the mild solution to (1) is studied by the GE-semigroup theory, and the existence and uniqueness condition is given. Secondly, the necessary and sufficient conditions of the exact controllability and approximate controllability of (1) are given by the GE-semigroup theory, respectively. Thirdly, necessary and sufficient conditions of the exact observability and approximate observability of (1) are given by the GE-semigroup theory, respectively, and the dual principle is proved to be true. At last, two examples are given to illustrate the theoretical results.

Notations. $B(X_2, X_1) = \{A : X_2 \to X_1, A \text{ is a bounded linear operator}\}$. $M_1 \in B(X_1, X_1)$, $M_3 \in B(X_2, X_1)$, $M_4 \in B(X_3, X_1)$, $M_5 \in B(X_1, X_4)$, $M_2 : D(M_2) \subseteq X_1 \to X_1$ is a closed linear operator; (Ω, F, F_t, P) denotes a complete probability space with a filtration F_t satisfying the usual condition; all processes defined on (Ω, F, F_t, P) are F_t-adapted; $w(t)$ is defined on (Ω, F, F_t, P); E denotes the mathematical expectation; $\langle\cdot,\cdot\rangle_{X_1}$ denotes the inner product in X_1, $\|\cdot\|_{X_1}$ denotes the norm in X_1 according to $\langle\cdot,\cdot\rangle_{X_1}$ in X_1; $L^2(F_t, P, X_1) = \{u \in X_1 : u \text{ is defined on } (\Omega, F, F_t, P),\ E(\|u\|^2_{X_1}) < +\infty\}$; $L^2([0,T], F_t, X_1) = \{u(t) \in L^2(F_t, P, X_1) : t \in [0,T]\}$; $L^2([0,T], \Omega, X_1) = \{u(t) \in L^2([0,T], F_t, X_1) : \|u\|_{L^2([0,T],\Omega,X_1)} = (\int_0^T E(\|u(t)\|^2_{X_1})\mathrm{d}t)^{1/2} < +\infty\}$.

Mild solution. In the following, the mild solution of system (1) is studied by the GE-semigroup theory.

Definition 1 ([8, 9]). Let $V(t) : X_1 \to X_1, t \geqslant 0$ be a family of bounded linear operators. If

$$V(t+s) = V(t)M_1V(s), \quad t,\ s \geqslant 0, \tag{2}$$

then we say that $\{V(t) : t \geqslant 0\}$ is a GE-semigroup induced by M_1.

If

$$\lim_{t\to 0^+} \|V(t)u - V(0)u\|_{X_1} = 0 \tag{3}$$

for arbitrary $u \in X_1$, then we say that $V(t)$ is strongly continuous on X_1.

Lemma 1 ([8]). Let $V(t)$ be strongly continuous. Then there exist $C \geqslant 1$ and $\alpha > 0$ such that

$$\|V(t)\|_{B(X_1,X_1)} \leqslant Ce^{\alpha t}, \quad t \geqslant 0. \tag{4}$$

In this case, $V(t)$ is called to be exponentially bounded.

Definition 2 ([9]). If $V(t)$ is a strongly continuous GE-semigroup induced by M_1 and

$$M_2u = \lim_{h\to 0^+} \frac{M_1V(h)M_1 - M_1V(0)M_1}{h}u, \tag{5}$$

for every $u \in P_1$, where $P_1 = \{u : u \in D(M_2) \subseteq X_1, V(0)M_1u = u,\ \exists \lim_{h\to 0^+} \frac{M_1V(h)M_1 - M_1V(0)M_1}{h}u\}$, then we say that M_2 is a generator of $V(t)$.

Definition 3. Let M_2 be the generator of $V(t)$. If $v(t) \in L^2([0,T], \Omega, X_2)$ and $u_0 \in L^2(F_0, P, \overline{P_1})$, then we say that

$$u(t, u_0) = V(t)M_1u_0 + \int_0^t V(t-\tau)M_3v(\tau)\mathrm{d}\tau + \int_0^t V(t-\tau)M_4\mathrm{d}w(\tau) \tag{6}$$

is a mild solution to system (1) on $[0,T]$.

From Definitions 2 and 3, we can get the following proposition.

Email: gezqjd@mail.xjtu.edu.cn

Proposition 1. Let M_2 be the generator of $V(t)$. If $v(t) \in L^2([0,T],\Omega,X_2)$, $u_0 \in L^2(F_0,P,\overline{P_1})$; $M_3v(t), M_4\mathrm{d}w(t) \in M_1(L^2([0,T],\Omega,\overline{P_1}))$, then system (1) has a unique mild solution, which is given by (6).

Controllability. Here we consider controllability of system (1). Suppose Proposition 1 holds. In order to obtain the necessary and sufficient condition of the controllability of system (1), we introduce the following operators. $N_0^T : L^2([0,T],\Omega,X_2) \to L^2(F_T,P,\overline{P_1})$, $O_c^T : L^2(F_T,P,\overline{P_1}) \to L^2(F_T,P,\overline{P_1})$ are defined as

$$N_0^T v = \int_0^T V(T-t)M_3 v(t)\mathrm{d}t, \tag{7}$$

$$O_c^T y = \int_0^T V(T-t)M_3M_3^*V^*(T-t)E(y|F_t)\mathrm{d}t, \tag{8}$$

respectively. Obviously $N_0^T \in B(L^2([0,T],\Omega,X_2), L^2(F_T,P,\overline{P_1}))$, $O_c^T \in B(L^2(F_T,P,\overline{P_1}), L^2(F_T,P,\overline{P_1}))$, $N_0^{T*} : L^2(F_T,P,\overline{P_1}) \to L^2([0,T],\Omega,X_2)$ is defined by $N_0^{T*}y = M_3^*V^*(T-\tau)E(y|F_\tau)$, and $O_c^T = N_0^T N_0^{T*}$, where N_0^{T*} denotes the adjoint operator of N_0^T.

Definition 4. If for all $u_0 \in L^2(F_0,P,\overline{P_1})$, $u_T \in L^2(F_T,P,\overline{P_1})$, there exists $v(t) \in L^2([0,T],\Omega,X_2)$ such that the mild solution $u(t,u_0)$ to system (1) satisfies $u(T,u_0) = u_T$, then we say that system (1) is exactly controllable on $[0,T]$.

Theorem 1. The necessary and sufficient condition for system (1) to be exactly controllable on $[0,T]$ is $\mathrm{ran}N_0^T = L^2(F_T,P,\overline{P_1})$.

For the proof of Theorem 1, see Appendix A.

Theorem 2. The necessary and sufficient condition for system (1) to be exactly controllable on $[0,T]$ is that one of the following conditions is true:

(a) $\langle O_c^T y, y\rangle_{L^2(F_T,P,\overline{P_1})} \geqslant \beta\|y\|^2_{L^2(F_T,P,\overline{P_1})}$ for some $\beta > 0$ and all $y \in L^2(F_T,P,\overline{P_1})$.

(b) $\lim_{\gamma\to 0^+} \|(\gamma I + O_c^T)^{-1} - (O_c^T)^{-1}\|_{B(L^2(F_T,P,\overline{P_1}),L^2(F_T,P,\overline{P_1}))} = 0$.

(c) $\lim_{\gamma\to 0^+} \|\gamma(\gamma I + O_c^T)^{-1}\|_{B(L^2(F_T,P,\overline{P_1}),L^2(F_T,P,\overline{P_1}))} = 0$.

(d) $\ker N_0^{T*} = \{0\}$ and $\mathrm{ran}N_0^{T*}$ is closed.

For the proof of Theorem 2, see Appendix B. For other results of exact controllability, see Appendix C.

Definition 5. If for any state $u_T \in L^2(F_T,P,\overline{P_1})$, any initial state $u_0 \in L^2(F_0,P,\overline{P_1})$, and any $\epsilon > 0$, there exists $v \in L^2([0,T],\Omega,X_2)$ such that the mild solution $u(t,u_0)$ to system (1) satisfies

$$\|u(T,u_0) - u_T\|_{L^2(F_T,P,\overline{P_1})} < \epsilon,$$

we say that system (1) is approximately controllable on $[0,T]$.

According to Definition 5, we have that the necessary and sufficient condition for system (1) to be approximately controllable on $[0,T]$ is

$$\overline{\mathrm{ran}N_0^T} = L^2(F_T,P,\overline{P_1}). \tag{9}$$

Theorem 3. The necessary and sufficient condition for system (1) to be approximately controllable on $[0,T]$ is

$$\ker N_0^{T*} = \{0\}. \tag{10}$$

For the proof of Theorem 3, see Appendix D.

Theorem 4. The necessary and sufficient condition for system (1) to be approximately controllable on $[0,T]$ is that one of the following conditions is true:

(a) $\langle O_c^T y, y\rangle_{L^2(F_T,P,\overline{P_1})} > 0$ for all $y \in L^2(F_T,P,\overline{P_1}), y \neq 0$.

(b) $\lim_{\gamma\to 0^+} < \gamma(\gamma I + O_c^T)^{-1}u, y >_{L^2(F_T,P,\overline{P_1})} = 0$ for all $u, y \in L^2(F_T,P,\overline{P_1})$.

(c) $\lim_{\gamma\to 0^+} \|\gamma(\gamma I + O_c^T)^{-1}y\|_{L^2(F_T,P,\overline{P_1})} = 0$ for all $y \in L^2(F_T,P,\overline{P_1})$.

For the proof of Theorem 4, see Appendix E.

Observability. The concepts of exact observability and approxiamate observability are very important for stochastic descriptor linear system (1). Here the necessary and sufficient conditions for these two concepts are given, respectively, and the dual principle is proved to be true.

For the detail of observability and examples of system (1), see Appendix F.

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Supporting information Appendixes A–F. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

References

1 Melnikova I V, Filinkov A I, Anufrieva U A. Abstract stochastic equations. I. classical and distributional solutions. J Math Sci, 2002, 111: 3430–3475

2 Vlasenko L A, Rutkas A G. Stochastic impulse control of parabolic systems of Sobolev type. Diff Equat, 2011, 47: 1498–1507

3 Kuttler K L, Li J. Generalized stochastic evolution equations. J Differ Equ, 2014, 257: 816–842

4 Liaskos K B, Pantelous A A, Stratis I G. Linear stochastic degenerate Sobolev equation and application. Int J Control, 2015, 88: 2538–2553

5 Liaskos K B, Stratis I G, Pantelous A A. Stochastic degenerate Sobolev equations: well posedness and exact controllability. Math Meth Appl Sci, 2018, 41: 1025–1032

6 Ge Z Q. Impulse controllability and impulse observability of stochastic singular systems. J Syst Sci Complex, 2021, 34: 899–911

7 Ge Z Q, Ge X C. An exact null controllability of stochastic singular systems. Sci China Inf Sci, 2021, 64: 179202

8 Ge Z Q, Zhu G T, Feng D X. Generalized operator semigroup and well-posedness of singular distributed parameter systems (in Chinese). Sci Sin Math, 2010, 40: 477–495

9 Ge Z Q, Feng D X. Well-posed problem of nonlinear singular distributed parameter systems and nonlinear GE-semigroup. Sci China Inf Sci, 2013, 56: 128201