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GE-semigroup method for controllability of stochastic descriptor linear systems

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Appendix A Proof of Theorem 1

Proof. According to Definitions 3, 4 and Proposition 1, we can obtain that system (1) is exactly controllable on [0,T] if and only if $\mathrm{ran} N_0^T = L^2(F_T, P, \overline{P_1})$.

Appendix B Proof of Theorem 2

Proof. (a) $\Rightarrow$ (b) Suppose (a) holds. Then for all $y \in L^2(F_T, P, \overline{P_1})$ and for all $\gamma \geqslant 0$,

$$\langle y, (\gamma I + O_c^T)y\rangle_{L^2(F_T,P,\overline{P_1})} \geqslant (\gamma+\beta)\|y\|^2_{L^2(F_T,P,\overline{P_1})}. \tag{B1}$$

From (B1), we get that $\gamma I + O_c^T$ is injective. Therefore, $(\gamma I + O_c^T)^{-1} : \mathrm{ran}(\gamma I + O_c^T) \to L^2(F_T, P, \overline{P_1})$ is well defined. For all $y \in \mathrm{ran}(\gamma I + O_c^T)$, we have

$$\|(\gamma I + O_c^T)^{-1}y\|_{L^2(F_T,P,\overline{P_1})} \leqslant \frac{1}{\gamma+\beta}\|y\|_{L^2(F_T,P,\overline{P_1})} \leqslant \frac{1}{\beta}\|y\|_{L^2(F_T,P,\overline{P_1})}. \tag{B2}$$

According (B2) we have that $(\gamma I + O_c^T)^{-1}$ is a bounded linear operator on $\mathrm{ran}(\gamma I + O_c^T)$. Therefore, $\mathrm{ran}(\gamma I + O_c^T)$ is closed. In the following we prove that $\mathrm{ran}(\gamma I + O_c^T)$ is dense in $L^2(F_T, P, \overline{P_1})$. Let $y \in \ker(\gamma I + O_c^{T*}) = (\mathrm{ran}(\gamma I + O_c^T))^\perp$. According to $\gamma I + O_c^T = \gamma I + O_c^{T*}$, we can obtain

$$(\mathrm{ran}(\gamma I + O_c^T))^\perp = \ker(\gamma I + O_c^{T*}) = \ker(\gamma I + O_c^T).$$

Since $\gamma I + O_c^T$ is injective, we have $\ker(\gamma I + O_c^T) = \{0\}$. Therefore, $\mathrm{ran}(\gamma I + O_c^T)$ is dense in $L^2(F_T, P, \overline{P_1})$. Thus $\mathrm{ran}(\gamma I + O_c^T) = L^2(F_T, P, \overline{P_1})$ and $(\gamma I + O_c^T)^{-1} \in B(L^2(F_T, P, \overline{P_1}), L^2(F_T, P, \overline{P_1}))$ is well defined, and according to (B2),

$$\|(\gamma I + O_c^T)^{-1}\|_{B(L^2(F_T,P,\overline{P_1}),L^2(F_T,P,\overline{P_1}))} \leqslant \frac{1}{\beta}.$$

This means that $\|(\gamma I + O_c^T)^{-1}\|_{B(L^2(F_T,P,\overline{P_1}),L^2(F_T,P,\overline{P_1}))}$ is uniformly bounded in respect to $\gamma \geqslant 0$. Since

$$\begin{aligned}
&\|(\gamma I + O_c^T)^{-1} - (O_c^T)^{-1}\|_{B(L^2(F_T,P,\overline{P_1}),L^2(F_T,P,\overline{P_1}))} \\
&= \|(O_c^T)^{-1}(O_c^T - \gamma I - O_c^T)(\gamma I + O_c^T)^{-1}\|_{B(L^2(F_T,P,\overline{P_1}),L^2(F_T,P,\overline{P_1}))} \\
&\leqslant \gamma\|(O_c^T)^{-1}\|_{B(L^2(F_T,P,\overline{P_1}),L^2(F_T,P,\overline{P_1}))}\times \\
&\quad \|(\gamma I + O_c^T)^{-1}\|_{B(L^2(F_T,P,\overline{P_1}),L^2(F_T,P,\overline{P_1}))} \leqslant \frac{\gamma}{\beta^2},
\end{aligned}$$

we can obtain that (b) is true.
(b)$\Rightarrow$(c). The proof is obvious.
(c)$\Rightarrow$(a). Suppose (c) holds. Then $\lim_{\gamma\to 0^+} \gamma^{1/2}\|(\gamma I + O_c^T)^{-1/2}\|_{B(L^2(F_T,P,\overline{P_1}),L^2(F_T,P,\overline{P_1}))} = 0$. Therefore, there exists $\beta > 0$ such that $\beta^{1/2}\|(\beta I + O_c^T)^{-1/2}\|_{B(L^2(F_T,P,\overline{P_1}),L^2(F_T,P,\overline{P_1}))} \leqslant 2^{-1/2}$. Therefore, for all $y \in L^2(F_T, P, \overline{P_1})$, we have

$$\begin{aligned}
\|y\|^2_{L^2(F_T,P,\overline{P_1})} &= \|\beta^{1/2}(\beta I + O_c^T)^{-1/2}\beta^{-1/2}(\beta I + O_c^T)^{1/2}y\|^2_{L^2(F_T,P,\overline{P_1})} \\
&\leqslant \frac{1}{2}\|\beta^{-1/2}(\beta I + O_c^T)^{1/2}y\|^2_{L^2(F_T,P,\overline{P_1})} \\
&= \frac{1}{2}\beta^{-1}\langle(\beta I + O_c^T)y, y\rangle_{L^2(F_T,P,\overline{P_1})}.
\end{aligned}$$

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This implies that (a) is true.
(a)⇒(d) Notice that (a) implies that N_0^{T*} is one to one, i.e., $\ker N_0^{T*}=\{0\}$. Similar to the proof of Theorem 4.1.7 of [1], we can prove that C_0^{T*} has closed range.
(d)⇒(a). Similar to the proof of Theorem 4.1.7 of [1], we have that $(N_0^{T*})^{-1}$ is bounded on $\mathrm{ran} C_0^{T*}$, i.e., existence $\beta>0$ makes

$$\|(N_0^{T*})^{-1}\xi\|^2_{L^2(F_T,P,\overline{P_1})}\leqslant \frac{1}{\beta}\|\xi\|^2_{L^2([0,T],\Omega,X_2)}$$

for every $\xi\in \mathrm{ran} N_0^{T*}$. Substituting $\xi=N_0^{T*}y$ proves (a).

Similar to the proof of Theorem 3.3 of [2], we have that (a) is equivalent to exact controllability of stochastic descriptor linear system (1).

Appendix C Theorem C:

Stochastic descriptor linear system (1) is exactly controllable if and only if the stochastic descriptor linear system

$$M_1du(t)=(M_2-\gamma_0M_1)u(t)dt+M_3v(t)dt+M_4dw(t), u(0)=u_0, t\geqslant 0 \tag{C1}$$

for every $\gamma_0\in\mathbb{C}$, is exactly controllable.
Proof. According to the definition of $V(t)$, we have

$$[\gamma M_1-(M_2-\gamma_0M_1)]^{-1}=\int_0^{+\infty}e^{\gamma t}e^{-\gamma_0 t}V(t)dt, \mathrm{Re}\gamma>-\mathrm{Re}\gamma_0+\omega.$$

It is obvious that $V_1(t)=e^{-\gamma_0 t}V(t)$ is a GE-semigroup induced by M_1 with generator $M_2-\gamma_0M_1$. From

$$N_1^Tv=\int_0^T V_1(T-t)M_3v(t)dt=e^{-\gamma_0 T}\int_0^T V(T-t)Ce^{\gamma_0 t}v(t)dt,$$

we obtain that $\mathrm{ran} N_0^T=\mathrm{ran} N_1^T$.

Appendix D Proof of Theorem 3

Proof. Similar to the proof of Theorem 3.5 of [2], we can obtain that Theorem 3 is true.

Appendix E Proof of Theorem 4

Proof. (a)⇒(b). Suppose (a) holds, but existence $\xi\neq 0$ makes

$$\lim_{\gamma\to 0^+}\langle\gamma(\gamma I+O_c^T)^{-1}u,y\rangle_{L^2(F_T,P,\overline{P_1})}=\langle\xi,y\rangle_{L^2(F_T,P,\overline{P_1})}$$

for all $y\in L^2(F_T,P,\overline{P_1})$. Let $\gamma(\gamma I+O_c^T)^{-1}u=\xi_\gamma$. Then $\gamma u=(\gamma I+O_c^T)\xi_\gamma$,

$$\begin{aligned}\gamma\langle u,y\rangle_{L^2(F_T,P,\overline{P_1})}&=\gamma\langle\xi_\gamma,y\rangle_{L^2(F_T,P,\overline{P_1})}+\langle O_c^T\xi_\gamma,y\rangle_{L^2(F_T,P,\overline{P_1})}\\&=\gamma\langle\xi_\gamma,y\rangle_{L^2(F_T,P,\overline{P_1})}+\langle\xi_\gamma,O_c^Ty\rangle_{L^2(F_T,P,\overline{P_1})}\end{aligned}$$

for all $y\in L^2(F_T,P,\overline{P_1})$. This implies that

$$\langle\xi,O_c^Ty\rangle_{L^2(F_T,P,\overline{P_1})}=\lim_{\gamma\to 0^+}\langle\xi_\gamma,O_c^Ty\rangle_{L^2(F_T,P,\overline{P_1})}=0$$

for all $y\in L^2(F_T,P,\overline{P_1})$, i.e., $O_c^T\xi=0$, which contradicts the positivity of O_c^T.
(b)⇒(c). Suppose (b) holds. Since

$$\begin{aligned}\|\gamma(\gamma I+O_c^T)^{-1}y\|^2_{L^2(F_T,P,\overline{P_1})}\leqslant\|\gamma(\gamma I+O_c^T)^{-1}\|_{B(L^2(F_T,P,\overline{P_1}),L^2(F_T,P,\overline{P_1}))}\times\\ \langle\gamma(\gamma I+O_c^T)^{-1}y,y\rangle_{L^2(F_T,P,\overline{P_1})},\end{aligned}$$

this implies that (c) holds.
(c)⇒(a). Suppose (c) holds, but existence $\xi\in L^2(F_T,P,\overline{P_1})$ and $\xi\neq 0$ makes

$$\langle O_c^T\xi,\xi\rangle_{L^2(F_T,P,\overline{P_1})}=0.$$

Then

$$\langle O_c^T\xi,\xi\rangle_{L^2(F_T,P,\overline{P_1})}=\langle((O_c^T)^{1/2})^2\xi,\xi\rangle_{L^2(F_T,P,\overline{P_1})}=\|(O_c^T)^{1/2}\xi\|^2_{L^2(F_T,P,\overline{P_1})}=0.$$

This implies that $(O_c^T)^{1/2}\xi=0$, i.e., $O_c^T\xi=0$. Thus $\xi=\gamma(\gamma I+O_c^T)^{-1}\xi$ and

$$\lim_{\gamma\to 0^+}\|\gamma(\gamma I+O_c^T)^{-1}\xi\|_{L^2(F_T,P,\overline{P_1})}=\|\xi\|_{L^2(F_T,P,\overline{P_1})}\neq 0,$$

which contradicts (c).

Now we prove that the necessary and sufficient condition for the stochastic descriptor linear system (1) to be approximate controllability is that (a) is true. It is obvious that (a) is true if and only if (10) holds. By Theorem 3, we have that the necessary and sufficient condition for the stochastic descriptor linear system (1) to be approximately controllable is that (a) holds.

Appendix F Observability and examples

Let $v(t)=0$ in system (1) and Proposition 1 hold for the following systems. Then we obtain

$$M_1du(t)=M_2u(t)dt+M_4dw(t), u(0)=u_0, t\geqslant 0, x(t)=M_5u(t). \tag{F1}$$

The dual system of (F1) is

$$M_1^*du(t)=M_2^*u(t)dt+M_5^*v(t)dt+M_4dw(t), u(0)=u_0, t\geqslant 0. \tag{F2}$$

The operator

$$Q_0^T: L^2(F_T,P,\overline{P_1})\to L^2([0,T],\Omega,X_4)$$

is defined by $Q_0^Ty=M_5V(T-t)E(y|F_t)$, and $G_o^T=Q_0^{T*}Q_0^T$.

Definition 1 (F). If Q_0^T is one to one and $(Q_0^T)^{-1}$ is bounded on ranQ_0^T, we say that system (F1) is exactly observable on $[0,T]$.

Definition 2 (F). If Q_0^T is injective, we say that system (F1) is approximately observable on $[0,T]$.

The following dual principle can be obtained immediately.

Theorem 1 (F). The necessary and sufficient condition for the stochastic descriptor linear system (F1) to be exactly and approximately observable on $[0,T]$, respectively, is that the dual stochastic descriptor linear system (F2) is exactly and approximately controllable on $[0,T]$, respectively.

Proof. Similar to the proof of Theorem 4.3 of [2], we can obtain that Theorem 1 (F) is true.

According to Theorems 2, 4 and Theorem 1 (F) we can get the following corollaries.

Corollary 1 (F). The necessary and sufficient condition for the stochastic descriptor linear system (F1) to be exactly observable on $[0,T]$ is that one of the following conditions holds for some $\beta>0$ and for very $y\in L^2(F_T,P,\overline{P_1})$:

(a) $\langle G_o^Ty,y\rangle_{L^2(F_T,P,\overline{P_1})}\geqslant\beta\|y\|^2_{L^2(F_T,P,\overline{P_1})}$ for all $y\in L^2(F_T,P,\overline{P_1})$.

(b) $\lim_{\gamma\to 0}\|(\gamma I+G_o^T)^{-1}-(G_o^T)^{-1}\|_{B(L^2(F_T,P,\overline{P_1}),L^2(F_T,P,\overline{P_1}))}=0$.

(c) $\lim_{\gamma\to 0}\|\gamma(\gamma I+G_o^T)^{-1}\|_{B(L^2(F_T,P,\overline{P_1}),L^2(F_T,P,\overline{P_1}))}=0$.

(d) ker$Q_0^T=\{0\}$ and ranQ_0^T is closed.

Corollary 2 (F). The necessary and sufficient condition for the stochastic descriptor linear system (F1) to be approximately observable on $[0,T]$ is that one of the following conditions is true:

(a) $\langle G_o^Ty,y\rangle_{L^2(F_T,P,\overline{P_1}}>0$ for all $y\in L^2(F_T,P,\overline{P_1}), y\neq 0$.

(b) $\lim_{\gamma\to 0^+}\langle\gamma(\gamma I+G_o^T)^{-1}u,y\rangle_{L^2(F_T,P,\overline{P_1})}=0$ for all $u,y\in L^2(F_T,P,\overline{P_1})$.

(c) $\lim_{\gamma\to 0^+}\|\gamma(\gamma I+G_o^T)^{-1}y\|_{L^2(F_T,P,\overline{P_1})}=0$ for all $y\in L^2(F_T,P,\overline{P_1})$.

(d) ker$Q_0^T=\{0\}$.

Example 1 (F). Consider the stochastic descriptor linear system described by stochastic descriptor heat conduction equation

$$M_1\frac{\partial u(t,\xi)}{\partial t}dt=\frac{\partial^2Lu(t,\xi)}{\partial\xi^2}dt+v(t,\xi)dt+dw(t), 0<t<T, 0<\xi<1,$$

$$u(0,\xi)=u_0(\xi), 0\leqslant\xi\leqslant 1, u(t,0)=u(t,1)=0, 0\leqslant t\leqslant T. \tag{F3}$$

Let $X_1=X_2=X_3=L^2(0,1), (B_1u)(\xi)=\frac{d^2u(\xi)}{d\xi^2}, D(B_1)=H^2(0,1)\cap H_0^1(0,1)$. Then $\lambda_n=-n^2\pi^2$ is the eigenvalue of operator B_1, and the corresponding eigenvector is $\phi_n(\xi)=2^{1/2}\sin n\pi\xi, n=1,2,\cdots$, all $\{\phi_n\}$ constitute the standard orthogonal basis of X_1. Let $H_1=\text{span}\{\phi_{2i-1}: i=1,2,\cdots\}, H_2=\text{span}\{\phi_{2i}: i=1,2,\cdots\}; M_1: X_1\to X_1, M_1\phi_{2i-1}=\phi_i, M_1\phi_{2i}=0, i=1,2,\cdots; M_1|_{H_1}=L: H_1\to X_1, L\phi_{2i-1}=\phi_i, L^{-1}: X_1\to H_1, L^{-1}\phi_i=\phi_{2i-1}, i=1,2,\cdots; M_2=B_1L: D(B_1L)\subset H_1\to X_1, (M_2u)(\xi)=(B_1Lu)(\xi)$. It is obvious that $X_1=H_1\oplus H_2, \|M_1\|_{B(X_1,X_1)}=1, \|L\|_{B(H_1,X_1)}=1, \|L^{-1}\|_{B(M_1,H_1)}=1$, ran$M_1=X_1$, ker$M_1=H_2, H_1=\overline{P_1}$. System (F3) is equivalent to

$$M_1du(t)=M_2u(t)dt+M_3v(t)dt+M_4dw(t), u(0)=u_0,$$

where $M_3=M_4=I$.

Here we discuss the approximate controllability of (F3).

Since $\lambda M_1-M_2=\lambda L-B_1L=(\lambda I-B_1)L$, we have that

$$(\lambda M_1-M_2)^{-1}=L^{-1}(\lambda I-B_1)^{-1}=\int_0^{+\infty}L^{-1}V_1(t)e^{-\lambda t}dt, \text{Re}\lambda>\omega,$$

where

$$(\lambda I-B_1)^{-1}=\int_0^{+\infty}V_1(t)e^{-\lambda t}dt, \text{Re}\lambda>\omega, \|V_1(t)\|_{B(X_1,X_1)}\leqslant Ce^{\alpha t}(C\geqslant 1,\alpha>0),$$

$$V_1(t)u=\sum_{n=1}^{+\infty}e^{-n^2\pi^2t}\langle u,\phi_n\rangle_{X_1}\phi_n, u\in X_1$$

(see Example 6.1.6 of [3]). By [4] and Definition 2, we obtain that $V(t)=L^{-1}V_1(t)$ is a GE-semigroup induced by M_1 with generator M_2. For $y\in L^2(F_T,P,\overline{P_1})$, since

$$N_0^{T*}y=V_1^*(T-\tau)(L^{-1})^*E(y|F_\tau)=\sum_{n=1}^{+\infty}e^{-n^2\pi^2(T-\tau)}\langle(L^{-1})^*E(y|F_\tau),\phi_n\rangle_{L^2(F_\tau,P,X_1)}\phi_n,$$

we have that $y=0$ if and only if $N_0^{T*}y=0$ for any $T>0, \tau\in[0,T]$, i.e., $\ker N_0^{T*}|_{L^2(F_T,P,\overline{P_1})}=\{0\}$. By Theorem 3, (F3) is approximate controllable on $[0,T]$ for any $T>0$.

Example 2 (F). According to [5], the stochastic version of economics Leontief dynamic input-output model can be written as following

$$M_1du(t) = M_2u(t)dt + M_3v(t)dt + M_4dw(t), x(t) = M_5u(t), \tag{F4}$$

which is the same with (1), where $X_1 = X_2 = X_3 = X_4$.

If $v(t) = 0$ in (F4), we can obtain:

$$M_1du(t) = M_2u(t)dt + M_4dw(t), x(t) = M_5u(t), \tag{F5}$$

which is the form of (F1).

Next we take the data as follows:

$$M_1 = \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix}, M_2 = \begin{bmatrix} 3I_1 & 0 \\ 0 & 7I_2 \end{bmatrix}, M_4 = \begin{bmatrix} 5I_1 & 0 \\ 0 & 0 \end{bmatrix}, M_5 = \begin{bmatrix} 4I_1 & 0 \\ 0 & 0 \end{bmatrix},$$

where I_1, I_2 are identical operators in Hilbert spaces X_{11}, X_{12} respectively. System (F5) is the form

$$\begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} du_1(t) \\ du_2(t) \end{bmatrix} = \begin{bmatrix} 3I_1 & 0 \\ 0 & 7I_2 \end{bmatrix} \begin{bmatrix} u_1(t)dt \\ u_2(t)dt \end{bmatrix} + \begin{bmatrix} 5I_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} dw_1(t) \\ dw_2(t) \end{bmatrix}$$

$$x(t) = \begin{bmatrix} 4I_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \tag{F6}$$

where $\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} \in X_{11} \oplus X_{12} = X_1$. According to Theorem 1.5.4 of [6], we have that $\overline{P_1} = X_{11}$ and

$$V(t) = \begin{bmatrix} e^{3t}I_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

If $\begin{bmatrix} y \\ 0 \end{bmatrix} \in L^2(F_T, P, \overline{P_1})$, and

$$Q_0^T \begin{bmatrix} y \\ 0 \end{bmatrix} = M_5V(T-t)E(\begin{bmatrix} y \\ 0 \end{bmatrix} |F_t) = 0, t \in [0, T],$$

then

$$M_5V(T-T)E(\begin{bmatrix} y \\ 0 \end{bmatrix} |F_T) = 4 \begin{bmatrix} y \\ 0 \end{bmatrix} = 0.$$

Hence $\ker Q_0^T = \{0\}$, i.e., stochastic descriptor system (F6) is approximately observable by Corollary 2 (F).

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