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Quantum algorithm for kernelized correlation filter

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Appendix A Review of classical KCF algorithm

In this section, we will review and supplement the original classical KCF algorithm [1, 2] to make it easier to understand the quantum algorithm proposed in this paper. First, we introduce the concept of dense sampling in Appendix A.1. The derivation of the filtering parameters is given in Appendix A.2. Then, in Appendix A.3, a brief review of the response detection phase is given.

Appendix A.1 Dense sampling

In general, the training data have a large amount of redundancy. This means that their structure may not be exploited efficiently. In order to solve this problem, Henriques et al. [1] gave an approach called dense sampling.

Considering an $N \times 1$ training base sample vector $\mathbf{x} = [x_0, x_1, \cdots, x_{N-1}]^T$,the dense sampling for generating virtual samples is defined as

$$\mathbf{v}_i = P^i \mathbf{x}, \ \forall i = 0, 1, \cdots, N-1, \tag{A1}$$

where P is the cyclic shift operator (permutation matrix) that cyclically shifts a vector by one element. Now an $N \times N$ circulant matrix $C(\mathbf{x})$ is obtained by using the set of virtual samples from Eq. (A1) as the rows of a data matrix X:

$$X = C(\mathbf{x}) = \begin{bmatrix} (\mathbf{v}_0)^T \\ (\mathbf{v}_1)^T \\ (\mathbf{v}_2)^T \\ \vdots \\ (\mathbf{v}_{N-1})^T \end{bmatrix} = \begin{bmatrix} (P^0\mathbf{x})^T \\ (P^1\mathbf{x})^T \\ (P^2\mathbf{x})^T \\ \vdots \\ \left(P^{N-1}\mathbf{x}\right)^T \end{bmatrix} = \begin{bmatrix} x_0 & x_1 & x_2 & \cdots & x_{N-1} \\ x_{N-1} & x_0 & x_1 & \cdots & x_{N-2} \\ x_{N-2} & x_{N-1} & x_0 & \cdots & x_{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & x_3 & \cdots & x_0 \end{bmatrix}. \tag{A2}$$

As we know, circulant matrix have some interesting properties [3]. For a circulant matrix $C(\mathbf{x})$, the eigenvalue and the corresponding eigenvector can be expressed as

$$\lambda_j = \sum_i x_i e^{-2\pi \mathrm{i} \frac{ij}{N}}, \tag{A3}$$

$$\frac{1}{\sqrt{N}}[1, e^{-2\pi \mathrm{i} \frac{j}{N}}, e^{-2\pi \mathrm{i} \frac{2j}{N}}, \cdots, e^{-2\pi \mathrm{i} \frac{(N-1)j}{N}}]^T. \tag{A4}$$

Here, $\mathrm{i} = \sqrt{-1}$ is an imaginary number. This means that the eigenvalue decomposition of the circulant matrix can be expressed as

$$C(\mathbf{x}) = F \mathrm{diag}(\hat{\mathbf{x}}) F^\dagger, \tag{A5}$$

where $F^\dagger$ is the Hermitian transpose of the Discrete Fourier Transform (DFT) matrix F, and $\hat{\mathbf{x}}$ denotes the DFT of the vector $\mathbf{x}$, i.e., $\hat{\mathbf{x}} = \mathcal{F}(\mathbf{x}) = \sqrt{N} F \mathbf{x}$.

Likewise, it can be extended to two dimensions (2D). Given a $M \times N$ training base sample $\mathbf{x}$, dense sampling is performed on it by cyclically shifting horizontally and vertically, respectively. Then the virtual sample set matrix is obtained:

$$V = \begin{bmatrix} \mathbf{v}_{00} & \mathbf{v}_{01} & \cdots & \mathbf{v}_{0(N-1)} \\ \mathbf{v}_{10} & \mathbf{v}_{11} & \cdots & \mathbf{v}_{1(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{v}_{(M-1)0} & \mathbf{v}_{(M-1)1} & \cdots & \mathbf{v}_{(M-1)(N-1)} \end{bmatrix} = \begin{bmatrix} P^0\mathbf{x}Q^0 & P^0\mathbf{x}Q^1 & \cdots & P^0\mathbf{x}Q^{N-1} \\ P^1\mathbf{x}Q^0 & P^1\mathbf{x}Q^1 & \cdots & P^1\mathbf{x}Q^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ P^{M-1}\mathbf{x}Q^0 & P^{M-1}\mathbf{x}Q^1 & \cdots & P^{M-1}\mathbf{x}Q^{N-1} \end{bmatrix}, \tag{A6}$$

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where $P^i\mathbf{x}$ and $\mathbf{x}Q^j$ mean performing i horizontal shifts and j vertical shifts on $\mathbf{x}$, respectively. By concatenating all vectorized virtual samples, a block circulant matrix with circulant blocks (BCCB, i.e., a matrix that is circulant at the block level, and is composed of blocks each of which is also circulant) [2,4,5] can be obtained:

$$X = \begin{bmatrix} [\mathrm{vec}(\mathbf{v}_{00})]^T \\ [\mathrm{vec}(\mathbf{v}_{01})]^T \\ \vdots \\ \left[\mathrm{vec}\left(\mathbf{v}_{(M-1)(N-1)}\right)\right]^T \end{bmatrix} = BCCB(\mathbf{x}) = F2^* \mathrm{diag}(\mathrm{vec}(\hat{\mathbf{x}}))F2. \tag{A7}$$

To make it clear, an example is given in Appendix C.

Appendix A.2 Parameter training

Ridge regression (RR) is a regularized form of least-squares, where the objective function used to learn the filter $r\,()$ can be expressed as

$$\min_{\mathbf{w}} \sum_i \left(r(\mathbf{v}_i) - y_i\right)^2 + \delta \left\|\mathbf{w}\right\|^2, \tag{A8}$$

where δ is a regularization parameter, and y_i is a regression target of $\mathbf{v}_i$. Its closed-form solution is given by [6]:

$$\mathbf{w} = \left(X^\dagger X + \delta I\right)^{-1} X^\dagger \mathbf{y}, \tag{A9}$$

where the data matrix X is a circulant matrix or BCCB matrix and I is an identity matrix. Combining with the special properties of the circulant matrix, the above formula can be further transformed into the following (a detailed derivation can be found in Appendix D):

$$\hat{\mathbf{w}} = (\hat{\mathbf{x}} \odot \hat{\mathbf{y}}) \ominus (\hat{\mathbf{x}}^* \odot \hat{\mathbf{x}} + \delta), \tag{A10}$$

where $\odot$ denotes an element-wise product and $\ominus$ denotes an element-wise division.

In the case of non-linearity, the kernel trick can be used to map the input of the non-linearity problem into the high-dimensional feature-space $\varphi(\mathbf{x})$, so that the samples mapped to this high-dimensional feature space are linearly separable. According to the representation theorem [7], the solution of $\mathbf{w}$ can be expressed as a linear combination of the samples:

$$\mathbf{w} = \sum_i \alpha_i \varphi(\mathbf{v}_i), \tag{A11}$$

and thus the variables under the optimization is the vector $\boldsymbol{\alpha}$ of coefficients α_i. The kernel function is defined as $\varphi^T(\mathbf{x})\varphi(\mathbf{x}') = \kappa(\mathbf{x}, \mathbf{x}')$. Thus for the $N \times N$ kernel matrix K, we have

$$K_{ij} = \kappa(\mathbf{v}_i, \mathbf{v}_j). \tag{A12}$$

Accordingly, Eq. (A8) becomes [6]:

$$\min_{\boldsymbol{\alpha}} \left\|K\boldsymbol{\alpha} - \mathbf{y}\right\|_2^2 + \delta \boldsymbol{\alpha}^T K \boldsymbol{\alpha}, \tag{A13}$$

and the solution is:

$$\boldsymbol{\alpha} = (K + \delta I)^{-1} \mathbf{y}. \tag{A14}$$

Theorem 1 [2]: Given the circulant data $C(\mathbf{x})$, the corresponding kernel matrix K is circulant if the kernel function satisfies $\kappa(\mathbf{x}, \mathbf{x}') = \kappa(P\mathbf{x}, P\mathbf{x}')$ for any permutation matrix P.

Similar to Eq. (A10), we can get

$$\hat{\boldsymbol{\alpha}} = \hat{\mathbf{y}} \ominus \left(\hat{\mathbf{k}}^{\mathbf{xx}} + \delta\right), \tag{A15}$$

where the kernel correlation $\mathbf{k}^{xx}$ is the first row of the kernel matrix $K = C(\mathbf{k}^{xx})$, with element

$$\kappa_i^{xx} = \kappa\left(\mathbf{x}, P^i\mathbf{x}\right) = \varphi^T(\mathbf{x})\varphi\left(P^i\mathbf{x}\right) = f\left(\mathbf{x}^T P^i \mathbf{x}\right), \tag{A16}$$

where $f()$ is the appropriate kernel function. A detailed derivation is in Ref. [1,2]. According to Ref. [1], in the 2D case, we have a kernel matrix $K = BCCB(\mathbf{k}^{xx})$, where the $M \times N$ matrix $\mathbf{k}^{xx}$ has the element $\mathbf{k}_{ij}^{xx} = \kappa\left(\mathbf{x}, P^i\mathbf{x}Q^j\right)$.

Theorem 2 [1]: The kernel matrix K with elements $K_{(ij),(i'j')} = \kappa\left(P^i\mathbf{x}Q^j, P^{i'}\mathbf{x}Q^{j'}\right)$ is a BCCB if $\kappa()$ is an unitarily invariant kernel.

Assume that $\kappa(\mathbf{x}, \mathbf{x}') = f\left(\mathbf{x}^T\mathbf{x}'\right)$ for some kernel function $f()$, and the kernel correlation can be expressed as:

$$\mathrm{vec}(\mathbf{k}^{xx}) = f(BCCB(\mathbf{x})\mathrm{vec}(\mathbf{x})) = f\begin{bmatrix} [\mathrm{vec}(\mathbf{v}_{00})]^T \mathrm{vec}(\mathbf{x}) \\ [\mathrm{vec}(\mathbf{v}_{01})]^T \mathrm{vec}(\mathbf{x}) \\ \vdots \\ \left[\mathrm{vec}\left(\mathbf{v}_{(M-1)(N-1)}\right)\right]^T \mathrm{vec}(\mathbf{x}) \end{bmatrix}. \tag{A17}$$

If the vectorization is undone, we have

$$\mathbf{k}^{xx} = f(V \star \mathbf{x}) = f\begin{bmatrix} \mathbf{v}_{00} \circ \mathbf{x} & \mathbf{v}_{01} \circ \mathbf{x} & \cdots & \mathbf{v}_{0(N-1)} \circ \mathbf{x} \\ \mathbf{v}_{10} \circ \mathbf{x} & \mathbf{v}_{11} \circ \mathbf{x} & \cdots & \mathbf{v}_{1(N-1)} \circ \mathbf{x} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{v}_{(M-1)0} \circ \mathbf{x} & \mathbf{v}_{(M-1)1} \circ \mathbf{x} & \cdots & \mathbf{v}_{(M-1)(N-1)} \circ \mathbf{x} \end{bmatrix}, \tag{A18}$$

where $\mathbf{v}_{ij} \circ \mathbf{x} = [\mathrm{vec}(\mathbf{v}_{ij})]^T \mathrm{vec}(\mathbf{x})$. Similarly, Eq. (A15) can be expressed as:

$$\mathrm{vec}(\hat{\boldsymbol{\alpha}}) = \mathrm{vec}(\hat{\mathbf{y}}) \ominus \left(\mathrm{vec}\left(\hat{\mathbf{k}}^{xx}\right) + \delta\right) \Rightarrow \hat{\boldsymbol{\alpha}} = \hat{\mathbf{y}} \ominus \left(\hat{\mathbf{k}}^{xx} + \delta\right), \tag{A19}$$

where the DFT operation involved is the 2D-DFT.

Appendix A.3 Response detecting

Given a testing base sample $\mathbf{z}$, the kernel correlation $\mathbf{k}^{zx}$ of z and $\mathbf{x}$ is needed, which is the first row of the kernel matrix K^z. The detection response can be formulated as [1,2]:

$$\mathbf{y}' = (K^z)^T \star \boldsymbol{\alpha} = \mathcal{F}^{-1}(\hat{\mathbf{k}}^{zx} \odot \hat{\boldsymbol{\alpha}}). \tag{A20}$$

Appendix B Quantum KCF algorithm

An unreasonable assumption is implicit in the quantum algorithm given by Yu et al. [8]: the virtual samples already exist. However, in the actual implementation of the classical KCF algorithm, only the original base samples $\mathbf{x}$ and $\mathbf{z}$ are required. The virtual samples obtained by a dense sampling operation on the original base samples are not physically present and they exist only in the theoretical derivation for the convenience of understanding the algorithm. The proposed quantum KCF algorithm in this paper inherits this characteristic. Besides, different from the classical KCF algorithm, in the proposed quantum KCF algorithm, the original base samples are stored in the quantum random access memory (QRAM) with a suitable data structure [9]. Based on this, the following unitary operations can be efficiently carried out in $\mathcal{O}(\mathrm{poly}\log(MN))$ time. Let's take the preparation of state $|\mathbf{x}\rangle$ as an example:

$$O_{\mathcal{M}} : |i\rangle|0\rangle \mapsto |i\rangle \frac{\sum_j x_{ij}|j\rangle}{\|\mathbf{x}_i\|}, \tag{B1}$$

$$O_{\mathcal{N}} : |0\rangle|j\rangle \mapsto \frac{\sum_i \|\mathbf{x}_i\| |i\rangle}{\|\mathbf{x}\|_F}|j\rangle, \tag{B2}$$

where $\|\mathbf{x}\|_F$ is the Frobenius norms of $\mathbf{x}$. All these operations can be used to generate $|\mathbf{x}\rangle$:

$$O_x : |00\rangle \mapsto O_{\mathcal{M}} O_{\mathcal{N}} |00\rangle = \frac{\sum_{ij} x_{ij}|ij\rangle}{\|\mathbf{x}\|_F} = |\mathbf{x}\rangle, \tag{B3}$$

which means that $|\mathbf{x}\rangle$ can be efficiently generated in time $\mathcal{O}(\mathrm{polylog}(MN))$ [10]. Similarly,

$$O_y : |00\rangle \mapsto |\mathbf{y}\rangle = \frac{\sum_{ij} y_{ij}|ij\rangle}{\|\mathbf{y}\|_F}, \tag{B4}$$

$$O_z : |00\rangle \mapsto |\mathbf{z}\rangle = \frac{\sum_{ij} z_{ij}|ij\rangle}{\|\mathbf{z}\|_F}, \tag{B5}$$

can also be prepared efficiently in time $\mathcal{O}(\mathrm{polylog}(MN))$.

The proposed quantum KCF algorithm will be formally described below. In Appendix B.1, an improved quantum Fourier transform in computational basis (QFTC) is presented which is used as a sub-algorithm of the proposed quantum KCF algorithm. Next, following the idea of Algorithm I, the quantum algorithm used to calculate the kernel correlation is given in Appendix B.2. Similarly, in Appendix B.3 and Appendix B.4, a quantum KCF algorithm is presented, where both training and detection stages are included. In the end, the amplitude of the quantum state we obtain will be encoded with the response score of the testing base sample.

Appendix B.1 Quantum Fourier transform in computational basis

This improved QFTC algorithm given here is a sub-algorithm which is used to encode the Fourier coefficients in the computational basis in the training and detection stages.

Note that conventions for the sign of the phase factor exponent vary. Here we use the convention that the quantum Fourier transform (QFT) has the same effect as the inverse discrete Fourier transform, and vice versa. Suppose we have an Oracle to achieve

$$O_p : |00\rangle \rightarrow \sum_{ij} p_{ij}|ij\rangle. \tag{B6}$$

In the two-dimensional situation, the inverse QFT realizes the DFT on the amplitude:

$$\sum_{ij} p_{ij}|ij\rangle \rightarrow \sum_{kl} \frac{1}{\sqrt{MN}} \sum_{ij} p_{ij} \mathrm{e}^{-2\pi\mathrm{i}\left(\frac{ik}{M}+\frac{jl}{N}\right)} |kl\rangle = \sum_{kl} q_{kl}|kl\rangle, \tag{B7}$$

and vice versa. And the quantum algorithm in this subsection enables the digital encoding of Fourier transform coefficients. Because

$$\frac{1}{\sqrt{MN}} \sum_{kl} |kl\rangle|0\rangle \rightarrow \frac{1}{\sqrt{MN}} \sum_{kl} |kl\rangle \left| \frac{1}{\sqrt{MN}} \sum_{ij} p_{ij} \mathrm{e}^{-2\pi\mathrm{i}\left(\frac{ik}{M}+\frac{jl}{N}\right)} \right\rangle \tag{B8}$$

needs to be realized in the second section, we will take it as an example to present the improved QFTC algorithm. Besides, since the operation for loading data in this example is O_p, we call it QFTC with O_p.

Appendix B.1.1 *Algorithm details*

(1) Initialize the quantum system in the state

$$\frac{1}{\sqrt{MN}}\sum_{kl}|kl\rangle\frac{1}{\sqrt{2}}\left(|0\rangle+|1\rangle\right)\left|0^{\log(MN)}\right\rangle. \tag{B9}$$

(2) Perform the controlled operation $\sum_{kl}|kl\rangle\langle kl|\otimes\left(|0\rangle\langle 0|\otimes(\mathrm{X})^{\otimes\log MN}+|1\rangle\langle 1|\otimes O_x\right)$ to obtain

$$\frac{1}{\sqrt{2MN}}\sum_{kl}|kl\rangle\left(|0\rangle|kl\rangle+|1\rangle\frac{1}{\|\mathbf{x}\|_F}\sum_{i_1j_1}x_{i_1j_1}|i_1j_1\rangle\right), \tag{B10}$$

where X represents the Pauli-X gate and $O_p=O_x$.

(3) Perform a controlled 2D QFT on the quantum state in the third register with the qubit of the second register being in the state $|0\rangle$. The state of the system is transformed into

$$\frac{1}{\sqrt{2MN}}\sum_{kl}|kl\rangle\left(|0\rangle\frac{1}{\sqrt{MN}}\sum_{i_2j_2}\mathrm{e}^{2\pi\mathrm{i}\left(\frac{i_2k}{M}+\frac{j_2l}{N}\right)}|ij\rangle+|1\rangle\frac{1}{\|\mathbf{x}\|_F}\sum_{i_1j_1}x_{i_1j_1}|i_1j_1\rangle\right). \tag{B11}$$

(4) Perform a Hadamard gate H on the second register and get

$$\begin{aligned}&\frac{1}{\sqrt{MN}}\sum_{kl}|kl\rangle\left(\sin\theta_{kl}|0\rangle|\psi_{kl}^0\rangle+\cos\theta_{kl}|1\rangle|\psi_{kl}^1\rangle\right)\\&=\frac{1}{\sqrt{MN}}\sum_{kl}|kl\rangle\frac{-\mathrm{i}}{\sqrt{2}}\left(\mathrm{e}^{\mathrm{i}\theta_{kl}}\left|\psi_{kl}^+\right\rangle-\mathrm{e}^{-\mathrm{i}\theta_{kl}}\left|\psi_{kl}^-\right\rangle\right)\end{aligned}, \tag{B12}$$

where

$$\begin{aligned}|\psi_{kl}^0\rangle&=\frac{\frac{1}{\sqrt{MN}}\sum_{i_2j_2}\mathrm{e}^{2\pi\mathrm{i}\left(\frac{i_2k}{M}+\frac{j_2l}{N}\right)}|i_2j_2\rangle+\frac{1}{\|\mathbf{x}\|_F}\sum_{i_1j_1}x_{i_1j_1}|i_1j_1\rangle}{2},\\|\psi_{kl}^1\rangle&=\frac{\frac{1}{\sqrt{MN}}\sum_{i_2j_2}\mathrm{e}^{2\pi\mathrm{i}\left(\frac{i_2k}{M}+\frac{j_2l}{N}\right)}|i_2j_2\rangle-\frac{1}{\|\mathbf{x}\|_F}\sum_{i_1j_1}x_{i_1j_1}|i_1j_1\rangle}{2},\\\left|\psi_{kl}^\pm\right\rangle&=\frac{|0\rangle|\psi_{kl}^0\rangle\pm\mathrm{i}|1\rangle|\psi_{kl}^1\rangle}{\sqrt{2}}.\end{aligned}$$

Besides,

$$\sin^2\theta_{kl}=\frac{1+\frac{1}{\|\mathbf{x}\|_F\sqrt{MN}}\sum_{ij}x_{ij}\mathrm{e}^{-2\pi\mathrm{i}\left(\frac{ik}{M}+\frac{jl}{N}\right)}}{2}, \tag{B13}$$

where $i_1=i_2=i$ and $j_1=j_2=j$.

(5) According to the quantum amplitude estimation (QAE) algorithm [11], we can get

$$\frac{1}{\sqrt{MN}}\sum_{kl}|kl\rangle\frac{-\mathrm{i}}{\sqrt{2}}\left(\mathrm{e}^{\mathrm{i}\theta_{kl}}\left|\psi_{kl}^+\right\rangle\left|\frac{\theta_{kl}}{\pi}\right\rangle-\mathrm{e}^{-\mathrm{i}\theta_{kl}}\left|\psi_{kl}^-\right\rangle\left|\frac{\pi-\theta_{kl}}{\pi}\right\rangle\right). \tag{B14}$$

(6) Add one additional register and perform a quantum arithmetic operation (QAO) [12] on it to obtain

$$\frac{1}{\sqrt{MN}}\sum_{kl}|kl\rangle\frac{-\mathrm{i}}{\sqrt{2}}\left(\mathrm{e}^{\mathrm{i}\theta_{kl}}\left|\psi_{kl}^+\right\rangle\left|\frac{\theta_{kl}}{\pi}\right\rangle-\mathrm{e}^{-\mathrm{i}\theta_{kl}}\left|\psi_{kl}^-\right\rangle\left|\frac{\pi-\theta_{kl}}{\pi}\right\rangle\right)\left|\frac{1}{\|\mathbf{x}\|_F\sqrt{MN}}\sum_{ij}x_{ij}\mathrm{e}^{-2\pi\mathrm{i}\left(\frac{ik}{M}+\frac{jl}{N}\right)}\right\rangle. \tag{B15}$$

(7) Uncompute the quantum registers except for the first and last registers, get

$$\frac{1}{\sqrt{MN}}\sum_{kl}|kl\rangle\left|\frac{1}{\|\mathbf{x}\|_F\sqrt{MN}}\sum_{ij}x_{ij}\mathrm{e}^{-2\pi\mathrm{i}\left(\frac{ik}{M}+\frac{jl}{N}\right)}\right\rangle. \tag{B16}$$

Appendix B.1.2 *Complexity analysis*

In this subsection, we presented an improved QFTC algorithm. In steps (1)-(3), we executed the Hadamard gates and O_x in time $\mathcal{O}(\mathrm{polylog}(MN))$. In step (5), we performed a QAE algorithm [11], which needs $\mathcal{O}\left(\frac{1}{\epsilon_x}\right)$ applications to achieve accuracy ϵ_x. This means that additional $\mathcal{O}\left(\log\frac{1}{\epsilon_x}\right)$ qubits are required to store θ_{ij}. Thus, the overall runtime is $\mathcal{O}\left(\frac{\mathrm{poly}\log(MN)}{\epsilon_x}\right)$. In step (6), the QAO has a complexity $\mathcal{O}\left(\mathrm{poly}\log\frac{1}{\epsilon_x}\right)$ within error $\mathcal{O}(\epsilon_x)$ (see Appendix E). Obviously, compared with step (5), the complexity of the QAO is negligible. In step (7), the time complexity of the undoing is just the same as that of prepare the quantum state in step (6). As a conclusion, the algorithm complexity in this subsection is $\mathcal{O}\left(\frac{\mathrm{poly}\log(MN)}{\epsilon_x}\right)$. The proposed algorithm has the same complexity as the QFTC algorithm given by Zhou et al. [12]. However, this improved QFTC has a more simplified process than the one [12] because in the latter the similar operation needs to be repeated twice thus leading to a larger space overhead.

Appendix B.2 Kernel correlation

In this subsection, the quantum algorithm for computing the kernel correlation is given, which will exist as a sub-algorithm used for the training stage and the detection stage.

Appendix B.2.1 *Algorithm details*

(1) Apply a 2D QFT on $|\mathbf{x}\rangle$ and assume that the output quantum state is expressed as

$$\sum_{kl} \hat{x}^*_{kl} \, |kl\rangle \, . \tag{B17}$$

(2) Add one register with $\mathcal{O}\left(\text{poly}\log\left(\frac{1}{\epsilon_x}\right)\right)$ qubits and implement the improved QFTC with $O_p = O_x$ given in the previous subsection,

$$\sum_{kl} \hat{x}^*_{kl} \, |kl\rangle \, |\hat{x}_{kl}\rangle \, , \tag{B18}$$

where

$$\hat{x}_{kl} = \frac{1}{\parallel \mathbf{x} \parallel_F \sqrt{MN}} \sum_{ij} x_{ij} \mathrm{e}^{-2\pi\mathrm{i}\left(\frac{ik}{M}+\frac{jl}{N}\right)}. \tag{B19}$$

(3) Add an ancillary qubit and perform a controlled rotation operation on it conditioned on the $|\hat{x}_{kl}\rangle$, obtain

$$\sum_{kl} \hat{x}^*_{kl} \, |kl\rangle \, |\hat{x}_{kl}\rangle \left(\Upsilon_{x1} \hat{x}_{kl} |0\rangle + \sqrt{1 - \Upsilon^2_{x1} \hat{x}^*_{kl} \hat{x}_{kl}} |1\rangle \right), \tag{B20}$$

where $\Upsilon_{x1} = \mathcal{O}\left(\frac{1}{\max_{kl} |\hat{x}_{kl}|}\right)$, $k = 0, 1, \cdots, M-1;\ l = 0, 1, \cdots, N-1$, and $|\ |$ represents the modulus of a complex number.

(4) Uncompute the second register and measure the auxiliary qubit to see the state $|0\rangle$, get

$$\frac{1}{\sqrt{\sum_{kl} \left(\hat{x}^*_{kl}\hat{x}_{kl}\right)^2}} \sum_{kl} \hat{x}^*_{kl} \hat{x}_{kl} \, |kl\rangle \, . \tag{B21}$$

In other words, steps (1)-(4) actually realize

$$O_{xx} : |00\rangle \rightarrow \frac{1}{\sqrt{\sum_{kl} \left(\hat{x}^*_{kl}\hat{x}_{kl}\right)^2}} \sum_{kl} \hat{x}^*_{kl} \hat{x}_{kl} \, |kl\rangle \, . \tag{B22}$$

(5) Perform the inverse QFTC with $O_p = O_{xx}$ to encode the inverse DFT coefficients into the computational basis similar to those described in the Appendix B.1:

$$\frac{1}{\sqrt{MN}} \sum_{ij} |ij\rangle \, |0\rangle \rightarrow \frac{1}{\sqrt{MN}} \sum_{ij} |ij\,\rangle \left| \frac{1}{\sqrt{MN \sum_{kl} \left(\hat{x}^*_{kl}\hat{x}_{kl}\right)^2}} \sum_{kl} \hat{x}^*_{kl} \hat{x}_{kl} \mathrm{e}^{2\pi\mathrm{i}\left(\frac{ik}{M}+\frac{jl}{N}\right)} \right\rangle. \tag{B23}$$

(6) Perform QAO to calculate kernel function, get

$$\frac{1}{\sqrt{MN}} \sum_{ij} |ij\,\rangle \, |\chi_{ij}\rangle \, , \tag{B24}$$

where $\chi_{ij} = f\left(\frac{1}{\sqrt{MN \sum_{kl} \left(\hat{x}^*_{kl}\hat{x}_{kl}\right)^2}} \sum_{kl} \hat{x}^*_{kl} \hat{x}_{kl} \mathrm{e}^{2\pi\mathrm{i}\left(\frac{ik}{M}+\frac{jl}{N}\right)} \right)$.

(7) Add an extra ancillary qubit, perform a controlled rotation operation again, we can obtain

$$\frac{1}{\sqrt{MN}} \sum_{ij} |ij\,\rangle \, |\chi_{ij}\rangle \left(\Upsilon_{x2} \chi_{ij} \, |0\rangle + \sqrt{1 - \Upsilon^2_{x2} \chi^*_{ij} \chi_{ij}} \, |1\rangle \right), \tag{B25}$$

where $\Upsilon_{x2} = \mathcal{O}\left(\frac{1}{\max\limits_{ij} |\,\chi_{ij}|}\right)$, $i = 0, 1, \cdots, M-1;\ j = 0, 1, \cdots, N-1$.

(8) Uncompute the second register, measure the auxiliary qubit to seeing $|0\rangle$, and transform the system into

$$\frac{1}{\sqrt{\sum_{ij} \chi^*_{ij} \chi_{ij}}} \sum_{ij} \chi_{ij} \, |ij\,\rangle. \tag{B26}$$

In short, the quantum algorithm in this subsection realizes

$$O_K : |00\rangle \rightarrow \frac{1}{\sqrt{\sum_{ij} \chi^*_{ij} \chi_{ij}}} \sum_{ij} \chi_{ij} \, |ij\,\rangle. \tag{B27}$$

Appendix B.2.2 *Complexity analysis*

In this subsection, based on the improved QFTC algorithm described in Appendix B.1, the complexity of step (2) is $\mathcal{O}\left(\frac{\text{poly}\log(MN)}{\epsilon_x}\right)$, where the time complexity of the 2D-QFT operation in step (1) is $\mathcal{O}\left(\text{polylog}\left(MN\right)\right)$. The complexity $\mathcal{O}\left(\text{poly}\log\frac{1}{\epsilon_x}\right)$ of the controlled rotation in step (3) is negligible [13,14]. The probability of getting the measurement outcome in step (4) is

$$P\left(0\right)=\sum_{kl}\left(\hat{x}_{kl}^{*}\right)^{2}\Upsilon_{x1}^{2}\hat{x}_{kl}^{2}=\Omega\left(\frac{1}{\varrho_{X}^{2}}\right), \tag{B28}$$

where ϱ_X is the condition number of the data matrix X. Thus, the quantum amplitude amplification (QAA) [11,15] algorithm need to execute $\mathcal{O}\left(\varrho_X\right)$ times to obtain the target quantum state with a measurement probability close to 1. Up to this point, the complexity $\mathcal{O}\left(\frac{\varrho_X\text{poly}\log(MN)}{\epsilon_x}\right)$. In step (5), we performed an inverse QFTC operation, and the complexity becomes $\mathcal{O}\left(\frac{\varrho_X\text{poly}\log(MN)}{\epsilon_x^2}\right)$. Similarly, the probability of measuring $|0\rangle$ in step (8) is

$$P\left(0\right)=\sum_{ij}\Upsilon_{x2}^{2}\chi_{ij}^{*}\chi_{ij}=\Omega\left(\frac{1}{\varrho_{0}^{2}}\right), \tag{B29}$$

where $\varrho_0=\frac{\max\limits_{ij}|\chi_{ij}|}{\min\limits_{ij}|\chi_{ij}|}$. Thus, the complexity of the quantum algorithm for computing the kernel correlation can be obtained synthetically as $\mathcal{O}\left(\frac{\varrho_0\varrho_X\text{poly}\log(MN)}{\epsilon_x^2}\right)$.

Appendix B.3 Training filtering parameters

The training phase of the proposed quantum algorithm is now formally described as follows.

Appendix B.3.1 *Algorithm details*

(1) Apply a 2D inverse QFT on $|\mathbf{y}\rangle$ and assume that the output quantum state is expressed as

$$\sum_{kl}\hat{y}_{kl}\left|kl\right\rangle. \tag{B30}$$

(2) Implement the improved QFTC with O_K to obtain

$$\sum_{kl}\hat{y}_{kl}\left|kl\right\rangle\left|\sigma_{kl}\right\rangle. \tag{B31}$$

where $\sigma_{kl}=\frac{1}{\sqrt{MN\sum_{ij}\chi_{ij}^{*}\chi_{ij}}}\sum_{ij}\chi_{ij}\mathrm{e}^{-2\pi\mathrm{i}\left(\frac{ik}{M}+\frac{jl}{N}\right)}$.

(3) Add one auxiliary qubit and rotate it by using a rotation operator $U\left(\sigma_{kl}\right)$, conditioned on $|\sigma_{kl}\rangle$, then the state becomes

$$\sum_{kl}\hat{y}_{kl}\left|kl\right\rangle\left|\sigma_{kl}\right\rangle\left[\Upsilon_{x3}g\left(\sigma_{kl}\right)|0\rangle+\sqrt{1-\Upsilon_{x3}^{2}g^{*}\left(\sigma_{kl}\right)g\left(\sigma_{kl}\right)}\,|1\rangle\right]. \tag{B32}$$

According to Appendix A, it holds:

$$g\left(\sigma_{kl}\right)=\frac{1}{\sigma_{kl}+\frac{\delta}{\sqrt{MN\sum_{ij}\chi_{ij}^{*}\chi_{ij}}}},$$

$$U\left(\sigma_{kl}\right)=\begin{bmatrix}\Upsilon_{x3}g\left(\sigma_{kl}\right) & \sqrt{1-\Upsilon_{x3}^{2}g^{*}\left(\sigma_{kl}\right)g\left(\sigma_{kl}\right)}\\ \sqrt{1-\Upsilon_{x3}^{2}g^{*}\left(\sigma_{kl}\right)g\left(\sigma_{kl}\right)} & -\Upsilon_{x3}g^{*}\left(\sigma_{kl}\right)\end{bmatrix},$$

$$\Upsilon_{x3}=\mathcal{O}\left(\frac{1}{\max_{kl}|g\left(\sigma_{kl}\right)|}\right),k=0,1,\cdots,M-1;\ j=0,1,\cdots,N-1.$$

The analysis of parameter Υ_{x3} can be found in Appendix F.

(4) Undo the improved QFTC operation, measure the auxiliary qubit to see the state $|0\rangle$, we can obtain

$$\frac{\sum_{kl}\hat{y}_{kl}g\left(\sigma_{kl}\right)|kl\rangle}{\sqrt{\sum_{kl}\hat{y}_{kl}^{*}\hat{y}_{kl}g^{*}\left(\sigma_{kl}\right)g\left(\sigma_{kl}\right)}}=\frac{\sum_{kl}\frac{\hat{y}_{kl}}{\sum_{ij}\chi_{ij}\mathrm{e}^{-2\pi\mathrm{i}\left(\frac{ik}{M}+\frac{jl}{N}\right)}+\delta}|kl\rangle}{\sqrt{\sum_{kl}\left(\frac{\hat{y}_{kl}}{\sum_{ij}\chi_{ij}\mathrm{e}^{-2\pi\mathrm{i}\left(\frac{ik}{M}+\frac{jl}{N}\right)}+\delta}\right)^{2}}}=\sum_{kl}\hat{\alpha}_{kl}\left|kl\right\rangle. \tag{B33}$$

Appendix B.3.2 *Complexity analysis*

In this subsection, the time complexity of the quantum Fourier transform in step (1) is $\mathcal{O}\left(\text{polylog}\left(MN\right)\right)$. The step (2) is based on the quantum sub-algorithm in Appendix B.1 and Appendix B.2 with time complexity $\mathcal{O}\left(\frac{\varrho_0\varrho_X\text{poly}\log(MN)}{\epsilon_x^3}\right)$. The probability of getting the measurement outcome $|0\rangle$ in step (4) is

$$P(0)=\sum_{kl}\hat{y}_{kl}\hat{y}_{kl}^{*}\Upsilon_{x3}^{2}g^{*}\left(\sigma_{kl}\right)g\left(\sigma_{kl}\right)=\Omega\left(\frac{1}{\varrho_{x}^{2}}\right), \tag{B34}$$

where ϱ_x is the condition number of the kernel matrix $BCCB\left(\mathbf{k}^{xx}\right)$. Here, the QAA algorithm needs to be executed $\mathcal{O}\left(\rho_x\right)$ times to obtain the target quantum state with a measurement probability close to . Thus, the complexity of the training stage is $\mathcal{O}\left(\frac{\varrho_x\varrho_0\varrho_X\text{poly}\log(MN)}{\epsilon_x^3}\right)$.

Appendix B.4 Response detection

Now we can calculate the response of the new testing base sample $\mathbf{z}$ according to the quantum KCF parameters obtained in Appendix B.2.

Appendix B.4.1 *Algorithm details*

(1) Calculate the QFTC of the kernel correlation $\mathbf{k}^{zx}$ and obtain

$$\sum_{kl} \hat{\alpha}_{kl} \left|kl\right\rangle \left|\lambda_{kl}\right\rangle , \tag{B35}$$

where

$$\lambda_{kl} = \frac{1}{\sqrt{MN\sum_{ij} \varkappa_{ij}^{*}\varkappa_{ij}}} \sum_{ij} \varkappa_{ij} \mathrm{e}^{-2\pi \mathrm{i}\left(\frac{ik}{M}+\frac{jl}{N}\right)},$$
$$\varkappa_{ij} = f\left(\frac{1}{\sqrt{MN\sum_{kl}\left(\hat{z}_{kl}^{*}\hat{x}_{kl}\right)^{2}}} \sum_{kl} \hat{z}_{kl}^{*}\hat{x}_{kl} \mathrm{e}^{2\pi \mathrm{i}\left(\frac{ik}{M}+\frac{jl}{N}\right)}\right).$$

(2) Add one auxiliary qubit, rotate it by using a rotation operator $U\left(\lambda_{kl}\right)$, obtain

$$\sum_{kl} \hat{\alpha}_{kl} \left|kl\right\rangle \left|\lambda_{kl}\right\rangle \left[\Upsilon_{z3} h\left(\lambda_{kl}\right)\left|0\right\rangle + \sqrt{1-\Upsilon_{z3}^{2} h^{*}\left(\lambda_{kl}\right) h\left(\lambda_{kl}\right)}\left|1\right\rangle\right], \tag{B36}$$

where

$$h\left(\lambda_{kl}\right) = \lambda_{kl},$$
$$U\left(\lambda_{kl}\right) = \begin{bmatrix} \Upsilon_{z3} h\left(\lambda_{kl}\right) & \sqrt{1-\Upsilon_{z3}^{2} h^{*}\left(\lambda_{kl}\right) h\left(\lambda_{kl}\right)} \\ \sqrt{1-\Upsilon_{z3}^{2} h^{*}\left(\lambda_{kl}\right) h\left(\lambda_{kl}\right)} & -\Upsilon_{z3} h^{*}\left(\lambda_{kl}\right) \end{bmatrix},$$
$$\Upsilon_{z2} = \mathcal{O}\left(\frac{1}{\max_{kl}\left|h\left(\lambda_{kl}\right)\right|}\right), k=0,1,\cdots,M-1;\ l=0,1,\cdots,N-1.$$

(3) Undo the step (1) and measure the auxiliary qubit to seeing $\left|0\right\rangle$, then the state becomes

$$\frac{\sum_{kl} \hat{\alpha}_{kl} h\left(\lambda_{kl}\right)\left|kl\right\rangle}{\sqrt{\sum_{kl} \hat{\alpha}_{kl}^{*}\hat{\alpha}_{kl} h^{*}\left(\lambda_{kl}\right) h\left(\lambda_{kl}\right)}} = \frac{\sum_{kl} \hat{\alpha}_{kl} \sum_{ij} \varkappa_{ij} \mathrm{e}^{-2\pi \mathrm{i}\left(\frac{ik}{M}+\frac{jl}{N}\right)}\left|kl\right\rangle}{\sqrt{\sum_{kl}\left(\hat{\alpha}_{kl}\sum_{ij} \varkappa_{ij} \mathrm{e}^{-2\pi \mathrm{i}\left(\frac{ik}{M}+\frac{jl}{N}\right)}\right)^{2}}} = \sum_{kl} \hat{y}_{kl}^{'} \left|kl\right\rangle . \tag{B37}$$

(4) Perform a 2D-QFT on the quantum state in Eq. (B37) and obtain the desirable state $\left|\mathbf{y}'\right\rangle$ whose amplitude is the superimposed form of the detection response.

Appendix B.4.2 *Complexity analysis*

Similar to Appendix B.3, we can obtain the time complexity of this subsequent response detection stage as $\mathcal{O}\left(\varrho_{z} \mathrm{poly}\log\left(MN\right)\left(\frac{\varrho_{x}\varrho_{0}\varrho_{X}}{\epsilon_{x}^{3}}+\frac{\varrho_{1}\varrho_{Z}}{\epsilon_{z}^{3}}\right)\right)$, where the definition of ϵ_{z} is similar to ϵ_{x}, ϱ_{z} represents the condition number of the kernel matrix $BCCB\left(\mathbf{k}^{zx}\right)$, ϱ_{Z} is the condition number of the data matrix Z and $\varrho_{1} = \frac{\max_{ij}\left|\varkappa_{ij}\right|}{\min_{ij}\left|\varkappa_{ij}\right|}$.

Appendix C Example of 2D generalization for dense sampling

An example of 2D generalization for dense sampling is given here. Suppose that there is a 2×3 training base sample

$$\mathbf{x} = \begin{bmatrix} x_{00} & x_{01} & x_{02} \\ x_{10} & x_{11} & x_{12} \end{bmatrix}. \tag{C1}$$

After the dense sampling, we can obtain

$$V = \left[\begin{array}{ccc|ccc|ccc} x_{00} & x_{01} & x_{02} & x_{02} & x_{00} & x_{01} & x_{01} & x_{02} & x_{00} \\ x_{10} & x_{11} & x_{12} & x_{12} & x_{10} & x_{11} & x_{11} & x_{12} & x_{10} \\ \hline x_{10} & x_{11} & x_{12} & x_{12} & x_{10} & x_{11} & x_{11} & x_{12} & x_{10} \\ x_{00} & x_{01} & x_{02} & x_{02} & x_{00} & x_{01} & x_{01} & x_{02} & x_{00} \end{array}\right] = \begin{bmatrix} P^{0}\mathbf{x}Q^{0} & P^{0}\mathbf{x}Q^{1} & P^{0}\mathbf{x}Q^{2} \\ P^{1}\mathbf{x}Q^{0} & P^{1}\mathbf{x}Q^{1} & P^{1}\mathbf{x}Q^{2} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{00} & \mathbf{v}_{01} & \mathbf{v}_{02} \\ \mathbf{v}_{10} & \mathbf{v}_{11} & \mathbf{v}_{12} \end{bmatrix}. \tag{C2}$$

Then we vectorize the virtual samples and concatenate all of them as follows,

$$X = BCCB(\mathbf{x}) = \begin{bmatrix} \left[\mathrm{vec}\left(\mathbf{v}_{00}\right)\right]^{T} \\ \left[\mathrm{vec}\left(\mathbf{v}_{01}\right)\right]^{T} \\ \left[\mathrm{vec}\left(\mathbf{v}_{02}\right)\right]^{T} \\ \left[\mathrm{vec}\left(\mathbf{v}_{10}\right)\right]^{T} \\ \left[\mathrm{vec}\left(\mathbf{v}_{11}\right)\right]^{T} \\ \left[\mathrm{vec}\left(\mathbf{v}_{12}\right)\right]^{T} \end{bmatrix} = \left[\begin{array}{ccc|ccc} x_{00} & x_{01} & x_{02} & x_{10} & x_{11} & x_{12} \\ x_{02} & x_{00} & x_{01} & x_{12} & x_{10} & x_{11} \\ x_{01} & x_{02} & x_{00} & x_{11} & x_{12} & x_{10} \\ \hline x_{10} & x_{11} & x_{12} & x_{00} & x_{01} & x_{02} \\ x_{12} & x_{10} & x_{11} & x_{02} & x_{00} & x_{01} \\ x_{11} & x_{12} & x_{10} & x_{01} & x_{02} & x_{00} \end{array}\right] = \begin{bmatrix} X_{1} & X_{2} \\ X_{2} & X_{1} \end{bmatrix}, \tag{C3}$$

where X_{1} and X_{2} are circulant matrix, and X is a BCCB matrix [1, 4, 5].

Appendix D Derivation and correction of linear ridge regression with circulant data

It needs to be pointed out that the derivation in Ref. [2] is wrong. Here, we give the correct derivation (take a one-dimensional scene as an example).

$$\begin{aligned} \mathbf{w} &= \frac{X^\dagger \mathbf{y}}{X^\dagger X + \delta I} \\ &= [F\mathrm{diag}\left(\hat{\mathbf{x}}^* \odot \hat{\mathbf{x}}\right) F^\dagger + \delta F I F^\dagger]^{-1} F\mathrm{diag}\left(\hat{\mathbf{x}}^*\right) F^\dagger \mathbf{y} \\ &= F\mathrm{diag}\left(\hat{\mathbf{x}}^* \odot \hat{\mathbf{x}} + \delta\right)^{-1} F^\dagger F\mathrm{diag}\left(\hat{\mathbf{x}}^*\right) F^\dagger \mathbf{y} \\ &= F[\hat{\mathbf{x}}^* \ominus \left(\hat{\mathbf{x}}^* \odot \hat{\mathbf{x}} + \delta\right)] F^\dagger \mathbf{y} \\ \Rightarrow F^\dagger \mathbf{w} &= \hat{\mathbf{w}}^* = [\hat{\mathbf{x}}^* \ominus \left(\hat{\mathbf{x}}^* \odot \hat{\mathbf{x}} + \delta\right)] \odot \hat{\mathbf{y}}^* \\ \Rightarrow \hat{\mathbf{w}} &= \left(\hat{\mathbf{x}} \odot \hat{\mathbf{y}}\right) \ominus \left(\hat{\mathbf{x}}^* \odot \hat{\mathbf{x}} + \delta\right) \end{aligned} . \tag{D1}$$

Appendix E Error analysis of QFTC

By defining that $\widetilde{\hat{x}_{kl}}$ represents the estimated value of $\hat{x}_{kl}$, we can get the absolute error

$$\begin{aligned} \left|\widetilde{\hat{x}_{kl}} - \hat{x}_{kl}\right| &= \left| \frac{1}{\parallel \mathbf{x} \parallel_F \sqrt{MN}} \widetilde{\sum_{ij} x_{ij}} \mathrm{e}^{-2\pi \mathrm{i}\left(\frac{ik}{M}+\frac{jl}{N}\right)} - \frac{1}{\parallel \mathbf{x} \parallel_F \sqrt{MN}} \sum_{ij} x_{ij} \mathrm{e}^{-2\pi \mathrm{i}\left(\frac{ik}{M}+\frac{jl}{N}\right)} \right| \\ &= 2\left|\sin^2 \widetilde{\theta_{ij}} - \sin^2 \theta_{ij}\right| = 2\left|\left(\sin\widetilde{\theta_{ij}} - \sin\theta_{ij}\right)\left(\sin\widetilde{\theta_{ij}} + \sin\theta_{ij}\right)\right| \\ &= 2\left|\left(2\sin\frac{\widetilde{\theta_{ij}} - \theta_{ij}}{2}\cos\frac{\widetilde{\theta_{ij}} + \theta_{ij}}{2}\right)\left(2\sin\frac{\widetilde{\theta_{ij}} + \theta_{ij}}{2}\cos\frac{\widetilde{\theta_{ij}} - \theta_{ij}}{2}\right)\right| \\ &= 2\left|\left(2\sin\frac{\widetilde{\theta_{ij}} - \theta_{ij}}{2}\cos\frac{\widetilde{\theta_{ij}} - \theta_{ij}}{2}\right)\left(2\sin\frac{\widetilde{\theta_{ij}} + \theta_{ij}}{2}\cos\frac{\widetilde{\theta_{ij}} + \theta_{ij}}{2}\right)\right| \\ &= 2\left|\sin\left(\widetilde{\theta_{ij}} - \theta_{ij}\right)\left(\sin\widetilde{\theta_{ij}} + \theta_{ij}\right)\right| \\ &\leqslant 2\left|\sin\left(\widetilde{\theta_{ij}} - \theta_{ij}\right)\right| \leqslant 2\left|\widetilde{\theta_{ij}} - \theta_{ij}\right| \leqslant 2\epsilon_x \end{aligned} . \tag{E1}$$

where the first equality comes from the mean value theorem [16].

Appendix F Parameter estimation analysis of Υ_{x3} and Υ_{z3}

In this appendix, we will give an analysis of Υ_{x3} and Υ_{z3}.

For Υ_{x3}, we already have

$$\frac{1}{\sqrt{MN\sum_{ij}\chi_{ij}^*\chi_{ij}}} \sum_{ij} \chi_{ij} \mathrm{e}^{-2\pi \mathrm{i}\left(\frac{ik}{M}+\frac{jl}{N}\right)} = \sigma_{kl}, \tag{F1}$$

and

$$g\left(\sigma_{kl}\right) = \frac{\sqrt{MN\sum_{ij}\chi_{ij}^*\chi_{ij}}}{\sum_{ij}\chi_{ij}\mathrm{e}^{-2\pi \mathrm{i}\left(\frac{ik}{M}+\frac{jl}{N}\right)} + \delta}, \tag{F2}$$

where $\chi_{ij} = f\left(\frac{1}{\sqrt{MN\sum_{kl}\left(\hat{x}_{kl}^*\hat{x}_{kl}\right)^2}} \sum_{kl} \hat{x}_{kl}^* \hat{x}_{kl} \mathrm{e}^{2\pi \mathrm{i}\left(\frac{ik}{M}+\frac{jl}{N}\right)}\right)$.

The modulus $|g\left(\sigma_{kl}\right)|$ can be expressed as

$$|g\left(\sigma_{kl}\right)| = \frac{\sqrt{MN\sum_{ij}\chi_{ij}^*\chi_{ij}}}{\sqrt{\left[\sum_{ij}\chi_{ij}\cos\left(2\pi\left(\frac{ik}{M}+\frac{jl}{N}\right)\right)+\delta\right]^2 + \left[\sum_{ij}\chi_{ij}\sin\left(2\pi\left(\frac{ik}{M}+\frac{jl}{N}\right)\right)\right]^2}}, \tag{F3}$$

and we assume $\sum_{ij}\chi_{ij}\cos\left(2\pi\left(\frac{ik}{M}+\frac{jl}{N}\right)\right) \geqslant 0$. Thus,

$$\begin{aligned} \frac{\max_{kl}|g\left(\sigma_{kl}\right)|}{\min_{kl}|g\left(\sigma_{kl}\right)|} &= \max_{(k_1l_1),(k_2l_2)} \frac{|g\left(\sigma_{k_1l_1}\right)|}{|g\left(\sigma_{k_2l_2}\right)|} \\ &= \max_{(k_1l_1),(k_2l_2)} \frac{\sqrt{\left[\sum_{ij}\chi_{ij}\cos\left(2\pi\left(\frac{ik_2}{M}+\frac{jl_2}{N}\right)\right)+\delta\right]^2 + \left[\sum_{ij}\chi_{ij}\sin\left(2\pi\left(\frac{ik_2}{M}+\frac{jl_2}{N}\right)\right)\right]^2}}{\sqrt{\left[\sum_{ij}\chi_{ij}\cos\left(2\pi\left(\frac{ik_1}{M}+\frac{jl_1}{N}\right)\right)+\delta\right]^2 + \left[\sum_{ij}\chi_{ij}\sin\left(2\pi\left(\frac{ik_1}{M}+\frac{jl_1}{N}\right)\right)\right]^2}} \\ &\leqslant \max_{(k_1l_1),(k_2l_2)} \frac{\sqrt{\left[\sum_{ij}\chi_{ij}\cos\left(2\pi\left(\frac{ik_2}{M}+\frac{jl_2}{N}\right)\right)\right]^2 + \left[\sum_{ij}\chi_{ij}\sin\left(2\pi\left(\frac{ik_2}{M}+\frac{jl_2}{N}\right)\right)\right]^2}}{\sqrt{\left[\sum_{ij}\chi_{ij}\cos\left(2\pi\left(\frac{ik_1}{M}+\frac{jl_1}{N}\right)\right)\right]^2 + \left[\sum_{ij}\chi_{ij}\sin\left(2\pi\left(\frac{ik_1}{M}+\frac{jl_1}{N}\right)\right)\right]^2}} \\ &= \max_{(k_1l_1),(k_2l_2)} \frac{|\sigma_{k_2l_2}|}{|\sigma_{k_1l_1}|} = \mathcal{O}\left(\varrho_x\right) \end{aligned} , \tag{F4}$$

where ϱ_x represents the condition number of $BCCB\left(\mathbf{k}^{xx}\right)$.

As for

$$\Delta = \left[\sum_{ij} \chi_{ij} \cos\left(2\pi\left(\frac{ik}{M} + \frac{jl}{N}\right)\right) + \delta\right]^2 + \left[\sum_{ij} \chi_{ij} \sin\left(2\pi\left(\frac{ik}{M} + \frac{jl}{N}\right)\right)\right]^2, \tag{F5}$$

we take the partial derivatives of k and l respectively to get

$$\begin{aligned}\Delta_k = 2&\left[\sum_{ij} \chi_{ij} \cos\left(2\pi\left(\frac{ik}{M} + \frac{jl}{N}\right)\right) + \delta\right]\left[\sum_{ij} \chi_{ij} \frac{-2\pi i}{M} \sin\left(2\pi\left(\frac{ik}{M} + \frac{jl}{N}\right)\right)\right] \\ &+2\left[\sum_{ij} \chi_{ij} \sin\left(2\pi\left(\frac{ik}{M} + \frac{jl}{N}\right)\right)\right]\left[\sum_{ij} \chi_{ij} \frac{2\pi i}{M} \cos\left(2\pi\left(\frac{ik}{M} + \frac{jl}{N}\right)\right)\right],\end{aligned} \tag{F6}$$

and

$$\begin{aligned}\Delta_l = 2&\left[\sum_{ij} \chi_{ij} \cos\left(2\pi\left(\frac{ik}{M} + \frac{jl}{N}\right)\right) + \delta\right]\left[\sum_{ij} \chi_{ij} \frac{-2\pi j}{N} \sin\left(2\pi\left(\frac{ik}{M} + \frac{jl}{N}\right)\right)\right] \\ &+2\left[\sum_{ij} \chi_{ij} \sin\left(2\pi\left(\frac{ik}{M} + \frac{jl}{N}\right)\right)\right]\left[\sum_{ij} \chi_{ij} \frac{2\pi j}{N} \cos\left(2\pi\left(\frac{ik}{M} + \frac{jl}{N}\right)\right)\right].\end{aligned} \tag{F7}$$

The value of the stagnation point is related to the value of χ_{ij}. One of the obvious stationary point coordinates is $(0, 0)$, and the corresponding extreme value is δ^2. In addition, there are three other boundary points $(0, N-1)$, $(M-1, 0)$ and $(M-1, N-1)$. Now suppose that we have selected the minimum value, and thus the maximum value of $|g(\sigma_{kl})|$ can be expressed as

$$\max_{kl} |g(\sigma_{kl})| = \frac{\sqrt{MN \sum_{ij} \chi_{ij}^* \chi_{ij}}}{\min_{kl} \sqrt{\left[\sum_{ij} \chi_{ij} \cos\left(2\pi\left(\frac{ik}{M} + \frac{jl}{N}\right)\right) + \delta\right]^2 + \left[\sum_{ij} \chi_{ij} \sin\left(2\pi\left(\frac{ik}{M} + \frac{jl}{N}\right)\right)\right]^2}}. \tag{F8}$$

In summary, we have

$$\Upsilon_{x3} |g(\sigma_{kl})| = \Omega\left(\frac{1}{\varrho_x}\right). \tag{F9}$$

As for Υ_{z3}, we know

$$h(\lambda_{kl}) = \lambda_{kl} = \frac{1}{\sqrt{MN \sum_{ij} \varkappa_{ij}^* \varkappa_{ij}}} \sum_{ij} \varkappa_{ij} \mathrm{e}^{-2\pi \mathrm{i}\left(\frac{ik}{M} + \frac{jl}{N}\right)}, \tag{F10}$$

where $\varkappa_{ij} = f\left(\frac{1}{\sqrt{MN \sum_{kl} \left(\hat{z}_{kl}^* \hat{x}_{kl}\right)^2}} \sum_{kl} \hat{z}_{kl}^* \hat{x}_{kl} \mathrm{e}^{2\pi \mathrm{i}\left(\frac{ik}{M} + \frac{jl}{N}\right)}\right)$, thus we can also obtain

$$\frac{\max_{kl} |h(\lambda_{kl})|}{\min_{kl} |h(\lambda_{kl})|} = \frac{\max_{kl} |(\lambda_{kl})|}{\min_{kl} |(\lambda_{kl})|} = \mathcal{O}(\varrho_z), \tag{F11}$$

where ϱ_z represents the condition number of $BCCB(\mathbf{k}^{zx})$. So we can obtain

$$\Upsilon_{z3} |h(\lambda_{kl})| = \Omega\left(\frac{1}{\varrho_z}\right). \tag{F12}$$

References

1 Henriques J F, Caseiro R, Martins P, et al. Exploiting the circulant structure of tracking-by-detection with kernels[C]. In: European conference on computer vision. Springer, Berlin, Heidelberg, 2012: 702-715.
2 Henriques J F, Caseiro R, Martins P, et al. High-speed tracking with kernelized correlation filters[J]. IEEE Transactions on Pattern Analysis and Machine Intelligence, 2014, 37(3): 583-596.
3 Gray R M. Toeplitz and circulant matrices: A review[J]. 2006.
4 Hansen P C, Nagy J G, O'leary D P. Deblurring images: matrices, spectra, and filtering[M]. Society for Industrial and Applied Mathematics, 2006.
5 Chaudhuri S, Velmurugan R, Rameshan R. Blind deconvolution methods: A review[J]. Blind Image Deconvolution, 2014: 37-60.
6 Rifkin R, Yeo G, Poggio T. Regularized least-squares classification[J]. Nato Science Series Sub Series III Computer and Systems Sciences, 2003, 190: 131-154.
7 Schölkopf B, Smola A J, Bach F. Learning with kernels: support vector machines, regularization, optimization, and beyond[M]. MIT press, 2002.
8 Yu C H, Gao F, Liu C, et al. Quantum algorithm for visual tracking[J]. Physical Review A, 2019, 99(2): 022301.
9 Giovannetti V, Lloyd S, Maccone L. Quantum random access memory[J]. Physical Review Letters, 2008, 100(16): 160501.
10 Kerenidis I, Prakash A. Quantum recommendation systems[J]. arXiv preprint arXiv:1603.08675, 2016.
11 Brassard G, Hoyer P, Mosca M, et al. Quantum amplitude amplification and estimation[J]. Contemporary Mathematics, 2002, 305: 53-74.
12 Zhou S S, Loke T, Izaac J A, et al. Quantum Fourier transform in computational basis[J]. Quantum Information Processing, 2017, 16(3): 82.
13 Wiebe N, Braun D, Lloyd S. Quantum algorithm for data fitting[J]. Physical Review Letters, 2012, 109(5): 050505.
14 Schuld M, Sinayskiy I, Petruccione F. Prediction by linear regression on a quantum computer[J]. Physical Review A, 2016, 94(2): 022342.
15 Nielsen M A, Chuang I L. Quantum computation and quantum information[J]. Cambridge: Cambridge University Press, 2000.
16 Sahoo P, Riedel T. Mean value theorems and functional equations[M]. World Scientific, 1998.