SCIENCE CHINA
Information Sciences

• **Supplementary File** •

Compensation of uncertain linear actuator dynamics for a class of cascaded PDE-ODE systems

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Appendix A Proof of Claim (9)

The proof is given by the following recursive steps:

Step 1: We choose $V_1 = \frac{1}{2}\left(X^{\mathrm{T}}PX + \rho\|\omega\|^2 + \zeta_1^2 + \tilde{\delta}_1^2\right)$ with $\tilde{\delta}_1 = \delta_1 - \hat{\delta}_1$ and $\delta_1 = \nu_1$. By (6) and (7), we have

$$\begin{aligned}\dot{\zeta}_1 &= \dot{Z}_1 - \dot{\chi} = Z_2 + Z_1^{\mathrm{T}}\nu_1 - \dot{\chi} = \zeta_2 + \tau_1 + Z_1^{\mathrm{T}}\nu_1 - \dot{\chi} \\ &= \zeta_2 - \rho\partial_x\omega(1) - \left(c_1 + \frac{3|PB|^2}{\underline{\lambda}_Q}\right)\zeta_1 + h_1^{\mathrm{T}}\tilde{\delta}_1. \end{aligned} \tag{A1}$$

Then, by (4), (8) and using the second equation of (5) with $k=1$, we compute $\dot{V}_1$, i.e.,

$$\dot{V}_1 = -\frac{1}{2}X^{\mathrm{T}}QX + X^{\mathrm{T}}PB\omega(0) + \rho\omega(1)\partial_x\omega(1) - \rho\|\partial_x\omega\|^2 + \zeta_1\left(\zeta_2 - \rho\partial_x\omega(1) - \left(c_1 + \frac{3|PB|^2}{\underline{\lambda}_Q}\right)\zeta_1\right).$$

By using Agmon's and then Poincaré's Inequalities (see Lemmas 1 and 2 in Appendix E below), we have

$$w^2(0) \leqslant w^2(1) + \|\omega\|^2 + \|\partial_x\omega\|^2 \leqslant 3w^2(1) + 5\|\partial_x\omega\|^2.$$

Then, by using Young's Inequality, we obtain

$$X^{\mathrm{T}}PB\omega(0) \leqslant \frac{\underline{\lambda}_Q}{4}|X|^2 + \frac{|PB|^2}{\underline{\lambda}_Q}\left(3w^2(1) + 5\|\partial_x\omega\|^2\right),$$

which results into that

$$\dot{V}_1 \leqslant -\frac{\underline{\lambda}_Q}{4}|X|^2 - \left(\rho - \frac{5|PB|^2}{\underline{\lambda}_Q}\right)\|\partial_x\omega\|^2 - c_1\zeta_1^2 + \zeta_1\zeta_2. \tag{A2}$$

Step 2: We choose $V_2 = V_1 + \frac{1}{2}\left(\zeta_2^2 + \tilde{\delta}_2^{\mathrm{T}}\tilde{\delta}_2\right)$ with $\tilde{\delta}_2 = \delta_2 - \hat{\delta}_2$ and $\delta_2 = \left(\nu_1^{\mathrm{T}}, \nu_2^{\mathrm{T}}\right)^{\mathrm{T}}$. Noting the first equation of (7), we have

$$\begin{aligned}\dot{\tau}_1 &= \frac{\partial\tau_1}{\partial Z_1}\dot{Z}_1 + \frac{\partial\tau_1}{\partial\hat{\delta}_1}\dot{\hat{\delta}}_1 + \sum_{i=0}^{1}\frac{\partial\tau_1}{\partial\chi^{(i)}}\chi^{(i+1)} + \frac{\partial\tau_1}{\partial\left(\partial_x\omega(1)\right)}\partial_x^3\omega(1) \\ &= \frac{\partial\tau_1}{\partial Z_1}Z_2 + \frac{\partial\tau_1}{\partial\hat{\delta}_1}\zeta_1 h_1 + \sum_{i=0}^{1}\frac{\partial\tau_1}{\partial\chi^{(i)}}\chi^{(i+1)} + \frac{\partial\tau_1}{\partial\left(\partial_x\omega(1)\right)}\partial_x^3\omega(1) + \frac{\partial\tau_1}{\partial Z_1}Z_{[1]}^{\mathrm{T}}\nu_1, \end{aligned} \tag{A3}$$

which, together with the choice of τ_2 (see the second equation of (7) with $k=2$), leads to

$$\dot{\zeta}_2 = \dot{Z}_2 - \dot{\tau}_1 = \zeta_3 + \tau_2 + Z_{[2]}^{\mathrm{T}}\nu_2 - \dot{\tau}_1 = \zeta_3 - c_2\zeta_2 - \zeta_1 + h_2^{\mathrm{T}}\tilde{\delta}_2, \tag{A4}$$

Then, by the second equation of (5) with $k=2$, we have

$$\begin{aligned}\dot{V}_2 &\leqslant -\frac{\underline{\lambda}_Q}{4}|X|^2 - \left(\rho - \frac{5|PB|^2}{\underline{\lambda}_Q}\right)\|\partial_x\omega\|^2 - c_1\zeta_1^2 + \zeta_1\zeta_2 + \zeta_2\dot{\zeta}_2 + \tilde{\delta}_2^{\mathrm{T}}\dot{\tilde{\delta}}_2 \\ &= -\frac{\underline{\lambda}_Q}{4}|X|^2 - \left(\rho - \frac{5|PB|^2}{\underline{\lambda}_Q}\right)\|\partial_x\omega\|^2 - c_1\zeta_1^2 - c_2\zeta_2^2 + \zeta_2\zeta_3. \end{aligned}$$

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Step k **(**$k = 3, \ldots, n-1$**):** We choose $V_k = V_{k-1} + \frac{1}{2}\left(\zeta_k^2 + \tilde{\delta}_k^{\mathrm{T}}\tilde{\delta}_k\right)$ with $\tilde{\delta}_k = \delta_k - \hat{\delta}_k$ and $\delta_k = \left(\nu_1^{\mathrm{T}}, \ldots, \nu_k^{\mathrm{T}}\right)^{\mathrm{T}}$. Noting that

$$\dot{\tau}_k = \sum_{i=1}^{k}\left(\frac{\partial \tau_k}{\partial Z_i} Z_{i+1} + \frac{\partial \tau_k}{\partial \hat{\delta}_i}\zeta_i h_i + \frac{\partial \tau_k}{\partial\left(\partial_x^{2i-1}\omega(1)\right)}\partial_x^{2i+1}\omega(1)\right) + \sum_{i=0}^{k}\frac{\partial \tau_k}{\partial \chi^{(i)}}\chi^{(i+1)} + \sum_{i=1}^{k}\frac{\partial \tau_k}{\partial Z_i} Z_{[i]}^{\mathrm{T}}\nu_i, \tag{A5}$$

the choice of τ_k (given by second equation of (7)) gives that

$$\dot{\zeta}_k = \dot{Z}_k - \dot{\tau}_{k-1} = \zeta_{k+1} + \tau_k + Z_{[k]}^{\mathrm{T}}\nu_k - \dot{\tau}_{k-1} = \zeta_{k+1} - c_k\zeta_k - \zeta_{k-1} + h_k^{\mathrm{T}}\tilde{\delta}_k, \tag{A6}$$

which, together with the second equation of (5), leads to

$$\begin{aligned}\dot{V}_k &\leqslant -\frac{\lambda_Q}{4}|X|^2 - \left(\rho - \frac{5|PB|^2}{\lambda_Q}\right)\|\partial_x\omega\|^2 - \sum_{i=1}^{k} c_i\zeta_i^2 + \zeta_k\zeta_{k+1} + \zeta_k\dot{\zeta}_k + \tilde{\delta}_k^{\mathrm{T}}\dot{\tilde{\delta}}_k \\ &= -\frac{\lambda_Q}{4}|X|^2 - \left(\rho - \frac{5|PB|^2}{\lambda_Q}\right)\|\partial_x\omega\|^2 - \sum_{i=1}^{k} c_i\zeta_i^2 + \zeta_k\zeta_{k+1}.\end{aligned}$$

Step n**:** We choose $V_n = V_{n-1} + \frac{1}{2}\left(\zeta_n^2 + \tilde{\delta}_n^{\mathrm{T}}\tilde{\delta}_n\right)$ with $\tilde{\delta}_k = \delta_k - \hat{\delta}_k$ and $\delta_n = \left(\nu_1^{\mathrm{T}}, \ldots, \nu_n^{\mathrm{T}}\right)^{\mathrm{T}}$. By the expression of $\dot{\tau}_{n-1}$ (see (A5) with $k = n-1$) and the choosing of U given by (5), we obtain that

$$\dot{\zeta}_n = \dot{Z}_n - \dot{\tau}_{n-1} = U + Z^{\mathrm{T}}\nu_n - \dot{\tau}_{n-1} = -c_n\zeta_n - \zeta_{n-1} + h_n^{\mathrm{T}}\tilde{\delta}_n. \tag{A7}$$

Then, the second equation of (5) with $k = n$ leads to that

$$\begin{aligned}\dot{V}_n &\leqslant -\frac{\lambda_Q}{4}|X|^2 - \left(\rho - \frac{5|PB|^2}{\lambda_Q}\right)\|\partial_x\omega\|^2 - \sum_{i=1}^{n-1} c_i\zeta_i^2 + \zeta_{n-1}\zeta_n + \zeta_n\dot{\zeta}_n + \tilde{\delta}_n^{\mathrm{T}}\dot{\tilde{\delta}}_n \\ &= -\frac{\lambda_Q}{4}|X|^2 - \left(\rho - \frac{5|PB|^2}{\lambda_Q}\right)\|\partial_x\omega\|^2 - \sum_{i=1}^{n} c_i\zeta_i^2.\end{aligned} \tag{A8}$$

Integrating both sides of (A8) over $[0,\, t]$ gives that $V_n(t) \leqslant V_n(0)$. Then, $V_n \in \mathcal{L}_\infty$, and hence the definition of V_n implies that $|X|$, $\|\omega\|$, ζ_k, $|\hat{\delta}_k| \in \mathcal{L}_\infty$. Moreover, integrating both sides of (A8) over $[0,\, +\infty)$ gives that

$$\int_0^{+\infty}\left(|X|^2 + \|\partial_x\omega\|^2 + \sum_{i=1}^{n}\zeta_i^2\right)\mathrm{d}t \leqslant M_1 V_n(0),$$

with M_1 being some positive constant, which shows that $|X|$, $\|\partial_x\omega\|$, $\zeta_k \in \mathcal{L}_2$. Then, by using Poincaré's Inequality, we obtain that $\|\omega\| \in \mathcal{L}_2$, and hence obtain the first line of (9). Moreover, by (A1) and using Young's Inequality, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x\omega\|^2 &= \partial_x\omega(1)\partial_t\omega(1) - \|\partial_x^2\omega\|^2 \\ &= \partial_x\omega(1)\left(\zeta_2 - \rho\partial_x\omega(1) - \left(c_1 + \frac{3|PB|^2}{\lambda_Q}\right)\zeta_1 + h_1^{\mathrm{T}}\tilde{\delta}_1\right) - \|\partial_x^2\omega\|^2 \\ &\leqslant -\frac{\rho}{4}\left(\partial_x\omega(1)\right)^2 + \frac{1}{\rho}\zeta_2^2 + \frac{1}{\rho}\left(c_1 + \frac{3|PB|^2}{\lambda_Q}\right)^2\zeta_1^2 + \frac{1}{\rho}(\zeta_1 + \chi)^2\tilde{\delta}_1^2 - \|\partial_x^2\omega\|^2.\end{aligned}$$

Noting that $|X|$, $\|\omega\| \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, the transformation (2) gives that $\|u\| \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, and hence $\chi \in \mathcal{L}_2 \cap \mathcal{L}_\infty$. This, together with the proven fact $\zeta_k \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, leads to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x\omega\|^2 \leqslant -\frac{\rho}{4}\left(\partial_x\omega(1)\right)^2 - \|\partial_x^2\omega\|^2 + \ell_1,$$

with ℓ_1 denoting the function which belongs to $\mathcal{L}_1$. Integrating above inequality over $[0, t]$ and $[0, +\infty)$ respectively gives that

$$\|\partial_x\omega\|^2 \leqslant \|\partial_x\omega(x,0)\|^2 + M_1, \quad \int_0^{+\infty}\left(\frac{\rho}{4}\left(\partial_x\omega(1)\right)^2 + \|\partial_x^2\omega\|^2\right)\mathrm{d}t \leqslant \|\partial_x\omega(x,0)\|^2 + M_1,$$

which show that $\|\partial_x\omega\| \in \mathcal{L}_\infty$ and $\partial_x\omega(1)$, $\|\partial_x^2\omega\| \in \mathcal{L}_2$. Then, Agmon's Inequality gives that $\omega(0)$, $\sup_{x\in[0,1]}|\omega(x,t)| \in \mathcal{L}_\infty$, and hence give the second line of (9).

To show the last line of (9), noting that similar to the derivation of (A2), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left(X^{\mathrm{T}}PX + \rho\|\omega\|^2 + \zeta_1^2\right) \leqslant -\frac{\lambda_Q}{4}|X|^2 - \left(\rho - \frac{5|PB|^2}{\lambda_Q}\right)\|\partial_x\omega\|^2 - c_1\zeta_1^2 + \zeta_1\zeta_2 + \zeta_1(\zeta_1 + \chi)\tilde{\delta}_1.$$

Since all the terms on the right-hand side of above inequality are bounded, there exists a positive constant M such that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(X^{\mathrm{T}}PX + \rho\|\omega\|^2 + \zeta_1^2\right) \leqslant M.$$

This, together with the proven fact that $|X|$, $\|\omega\|$, $\zeta_1 \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, leads to $\lim_{t\to\infty}|X| = 0$, $\lim_{t\to\infty}\zeta_1 = 0$ and $\lim_{t\to\infty}\|\omega\| = 0$ (by Lemma 3 in Appendix E). Then, the Agmon's Inequality implies that $\lim_{t\to+\infty}\sup_{x\in[0,1]}|\omega(x,t)| = 0$.

Appendix B Proof of Claim (10)

Before proceeding with the proof of claim (10), the following two propositions are first present to give some important properties of the closed-loop system. Specifically, Proposition 2 gives the equations that the PDE state w satisfies. Proposition 3 gives an equality and two inequalities that some closed-loop system signals satisfy. Both of the propositions are used to show the boundedness and integrability of the closed-loop system signals. Their proofs are postponed at Appendix E.

Proposition 2. The target system state w satisfies the following equations

$$\begin{cases} \partial_t^i\omega = \partial_x^{2i}\omega, \\ \partial_x^{2i+1}\omega(0) = 0, \\ \partial_t\omega(1) = -\rho\partial_x\omega(1) + f_1\left(\zeta_1,\zeta_2,\tilde h_1\right), \\ \partial_t^i\omega(1) = -\rho\partial_x^{2i-1}\omega(1) + f_i\Big(\zeta_1,\ldots,\zeta_{i+1},\tilde h_1^{(i-1)},\ldots,\tilde h_1,\tilde h_2^{(i-2)},\ldots,\tilde h_2,\ldots,\tilde h_i, \\ \qquad \partial_x^{2i-3}\omega(1),\partial_x^{2i-5}\omega(1),\ldots,\partial_x\omega(1)\Big), i=2,\ldots,n-1, \\ \partial_t^n\omega(1) = -\rho\partial_x^{2n-1}\omega(1) + f_n\Big(\zeta_1,\ldots,\zeta_n,\tilde h_1^{(n-1)},\ldots,\tilde h_1,\tilde h_2^{(n-2)},\ldots,\tilde h_2,\ldots,\tilde h_n, \\ \qquad \partial_x^{2n-3}\omega(1),\partial_x^{2n-5}\omega(1),\ldots,\partial_x\omega(1)\Big). \end{cases} \tag{B1}$$

with $\tilde h_i = h_i^{\mathrm T}\tilde\delta_i$ and f_i being a linear combination of its arguments therein.

Proposition 3. For any $k = 0, 1, \ldots$, there holds that

$$\chi^{(k)} = \int_0^1 \kappa(1,y)\partial_t^k u(y)\mathrm dy - \varphi(1)\left((A+BK)^k X + \sum_{i=0}^{k-1}(A+BK)^i B\partial_t^{k-i-1}\omega(0)\right), \tag{B2}$$

$$\left(\partial_t^k\omega(0)\right)^2 \leqslant \left(\partial_t^k\omega(1)\right)^2 + \|\partial_x^{2k}\omega\|\|\partial_x^{2k+1}\omega\|. \tag{B3}$$

Moreover, for any $k = 1, \ldots, n-1$, the following inequality holds:

$$\dot E_k \leqslant -\frac{\rho}{2}\left(\partial_x^{2k+1}\omega(1)\right)^2 - \|\partial_x^{2k+1}\omega\|^2 - \frac12\|\partial_x^{2k+2}\omega\|^2 + 2\left(\partial_t^k\omega(1)\right)^2 + \frac{1}{2\rho}f_{k+1}^2, \tag{B4}$$

with $E_k = \frac12\left(\|\partial_x^{2k}\omega\|^2 + \|\partial_x^{2k+1}\omega\|^2\right)$.

Claim (11) is proved by the following recursive steps.

Step 1: Noting the proven fact that $\|\omega\| \in \mathcal L_2\cap\mathcal L_\infty$, $\|\partial_x^2\omega\| \in \mathcal L_2$, transformation (2) implies that $\|u\| \in \mathcal L_2\cap\mathcal L_\infty$, $\|\partial_x^2 u\| \in \mathcal L_2$ (i.e., $\|\partial_t u\| \in \mathcal L_2$). Then, there holds that χ, $Z_1 \in \mathcal L_2\cap\mathcal L_\infty$, and hence h_1, $\tilde h_1$, $f_1 \in \mathcal L_2\cap\mathcal L_\infty$. This, together with the proven fact $\partial_x\omega(1) \in \mathcal L_2$, leads to $\dot\zeta_1$, $\partial_t\omega(1) \in \mathcal L_2$ (seeing from (A1) and the third equation of (B1), respectively). Moreover, the Agmon's Inequality implies that $\omega(0) \in \mathcal L_2\cap\mathcal L_\infty$, hence (B2) with $k=1$ gives that $\dot\chi \in \mathcal L_2$, and hence τ_1, Z_2, $\dot Z_1$, $\dot{\tilde h}_1$, h_2, $\tilde h_2$, $\dot\zeta_2$, $f_2 \in \mathcal L_2$.

Noting the proven fact $\partial_t\omega(1)$, $f_2 \in \mathcal L_2$, inequality (B4) with $k=1$ gives that

$$\dot E_1 \leqslant -\frac{\rho}{2}\left(\partial_x^3\omega(1)\right)^2 - \|\partial_x^3\omega\|^2 - \frac12\|\partial_x^4\omega\|^2 + \ell_1,$$

which implies that E_1, $\|\partial_x^2\omega\|$, $\|\partial_x^3\omega\| \in \mathcal L_\infty$ and $\partial_x^3\omega(1)$, $\|\partial_x^3\omega\|$, $\|\partial_x^4\omega\| \in \mathcal L_2$. Then, Agmon's Inequality results in $\partial_x\omega(1) \in \mathcal L_\infty$, (A1) leads to that $\dot\zeta_1 \in \mathcal L_2\cap\mathcal L_\infty$, the third and fourth equations of (B1) give that $\partial_t\omega(1) \in \mathcal L_2\cap\mathcal L_\infty$, $\partial_t^2\omega(1) \in \mathcal L_2$, (B3) with $k=1$ implies that $\partial_t\omega(0) \in \mathcal L_2\cap\mathcal L_\infty$. Moreover, transformation (2) implies that $\|\partial_x^2 u\| \in \mathcal L_\infty$, $\|\partial_x^4 u\| \in \mathcal L_2$, and hence $\|\partial_t u\| \in \mathcal L_\infty$, $\|\partial_t^2 u\| \in \mathcal L_2$. Hence, (B2) with $k=1,2$ gives that $\dot\chi \in \mathcal L_\infty$, $\ddot\chi \in \mathcal L_2$. Thus, there holds that τ_1, Z_2, $\dot Z_1$, $\dot{\tilde h}_1$, h_2, $\tilde h_2$, $\dot\zeta_2$, $f_2 \in \mathcal L_2\cap\mathcal L_\infty$. Also (7) and (A3) respectively give that τ_2, $\dot\tau_1 \in \mathcal L_2$, and hence Z_3, $\dot Z_2 \in \mathcal L_2$. Thus, there holds that $\ddot{\tilde h}_1$, $\dot{\tilde h}_2$, $\tilde h_3$, h_3, f_3, $\dot\zeta_3 \in \mathcal L_2$.

Step 2: By the proven fact that $\partial_t^2\omega(1)$, $f_3 \in \mathcal L_2$, (B4) with $k=2$ gives that

$$\dot E_2 \leqslant -\frac{\rho}{2}\left(\partial_x^5\omega(1)\right)^2 - \|\partial_x^5\omega\|^2 - \frac12\|\partial_x^6\omega\|^2 + \ell_1,$$

which implies that $\|\partial_x^4\omega\|$, $\|\partial_x^5\omega\| \in \mathcal L_\infty$ and $\partial_x^5\omega(1)$, $\|\partial_x^5\omega\|$, $\|\partial_x^6\omega\| \in \mathcal L_2$. Then, Agmon's Inequality results in $\partial_x^3\omega(1) \in \mathcal L_\infty$, the fourth equation of (B1) with $i=2,3$ gives that $\partial_t^2\omega(1) \in \mathcal L_2\cap\mathcal L_\infty$, $\partial_t^3\omega(1) \in \mathcal L_2$, (B3) with $k=2$ implies that $\partial_t^2\omega(0) \in \mathcal L_2\cap\mathcal L_\infty$. Moreover, transformation (2) implies that $\|\partial_x^4 u\| \in \mathcal L_\infty$ and $\|\partial_x^6 u\| \in \mathcal L_2$, and hence $\|\partial_t^2 u\| \in \mathcal L_\infty$, $\|\partial_t^3 u\| \in \mathcal L_2$. Thus, (B2) with $k=2$ and $k=3$ respectively give that $\ddot\chi \in \mathcal L_\infty$, $\chi^{(3)} \in \mathcal L_2$. Thus, (7) and (A3) gives that τ_2, $\dot\tau_1 \in \mathcal L_\infty$, $\tau_3 \in \mathcal L_2$, (7) with $k=3$ implies that $\dot\tau_2 \in \mathcal L_2$, and hence Z_3, $\dot Z_2$, $\ddot{\tilde h}_1$, $\dot{\tilde h}_2$, $\tilde h_3$, h_3, f_3, $\dot\zeta_3 \in \mathcal L_\infty$, and $\tilde h_1^{(3)}$, $\ddot{\tilde h}_2$, $\dot{\tilde h}_3$, $\tilde h_4$, h_4, f_4, $\dot\zeta_4 \in \mathcal L_2$.

Step k ($k = 3, \ldots, n-1$): Supposing that the recursive steps from 1 to $k-1$ have been completed, we have $\partial_t^k\omega(1)$, $f_{k+1} \in \mathcal L_2$. Then, (B4) leads to

$$\dot E_k \leqslant -\frac{\rho}{2}\left(\partial_x^{2k+1}\omega(1)\right)^2 - \|\partial_x^{2k+1}\omega\|^2 - \frac12\|\partial_x^{2k+2}\omega\|^2 + \ell_1,$$

which results in that

$$\begin{cases} \partial_t^k\omega(1), \partial_t^k\omega(0), \chi^{(k)}, \|\partial_x^{2k}\omega\|, \|\partial_x^{2k+1}\omega\|, \partial_x^{2k-1}\omega(1) \in \mathcal L_2\cap\mathcal L_\infty, \\ \partial_t^{k+1}\omega(1), \chi^{(k+1)}, \|\partial_x^{2k+1}\omega\|, \|\partial_x^{2k+2}\omega\|, \partial_x^{2k+1}\omega(1) \in \mathcal L_2, \\ Z_{k+1}, \dot Z_k, \dot\zeta_{k+1}, \tau_k, \dot\tau_{k-1}, \tilde h_1^{(k)}, \tilde h_2^{(k-1)}, \ldots, \tilde h_{k+1}, f_{k+1} \in \mathcal L_2\cap\mathcal L_\infty, \\ \dot\tau_k, \dot Z_{k+1}, \tilde h_1^{(k+1)}, \tilde h_2^{(k)}, \ldots, \tilde h_{k+2}, f_{k+2} \in \mathcal L_2. \end{cases} \tag{B5}$$

Step n: Let $E_n = \frac12\left(\|\partial_x^{2n}\omega\|^2 + \|\partial_x^{2n+1}\omega\|^2\right)$. Computing $\dot E_n$ along the solutions of (B1) while using integration by parts leads to

$$\dot E_n = \partial_x^{2n}\omega(1)\partial_x^{2n+1}\omega(1) + \partial_x^{2n+1}\omega(1)\partial_x^{2n+2}\omega(1) - \|\partial_x^{2n+1}\omega\|^2 - \|\partial_x^{2n+2}\omega\|^2. \tag{B6}$$

By (B5) with $k = n-1$,we obtain that $\partial_t^n \omega(1) \in \mathcal{L}_2$. Then, similar to the derivation of (E2), we have

$$\partial_x^{2n}\omega(1)\partial_x^{2n+1}\omega(1) \leqslant \frac{1}{2}\|\partial_x^{2n+2}\omega\|^2 + \ell_1. \tag{B7}$$

Moreover, by (B5) with $k = n-1$, the last equation of (B1) implies that

$$\begin{aligned}\partial_t^{n+1}\omega(1) &= -\rho\partial_x^{2n+1}\omega(1) + f_n\Big(\dot{\zeta}_1, \ldots, \dot{\zeta}_n, \tilde{h}_1^{(n)}, \ldots, \dot{\tilde{h}}_1, \tilde{h}_2^{(n-1)}, \ldots, \dot{\tilde{h}}_2, \ldots, \dot{\tilde{h}}_n,\\ &\qquad \partial_x^{2n-1}\omega(1), \partial_x^{2n-3}\omega(1), \ldots, \partial_x^3\omega(1)\Big)\\ &= -\rho\partial_x^{2n+1}\omega(1) + \ell_2.\end{aligned}$$

Then, there holds that

$$\begin{aligned}\partial_x^{2n+1}\omega(1)\partial_x^{2n+2}\omega(1) &= \partial_x^{2n+1}\omega(1)\partial_t^{n+1}\omega(1)\\ &= \partial_x^{2n+1}\omega(1)\left(-\rho\partial_x^{2n+1}\omega(1) + \ell_2\right)\\ &\leqslant -\frac{\rho}{2}\left(\partial_x^{2k+1}\omega(1)\right)^2 + \ell_1.\end{aligned} \tag{B8}$$

Substituting (B7),(B8) into (B6) arrives at

$$\dot{E}_n \leqslant -\frac{\rho}{2}\left(\partial_x^{2n+1}\omega(1)\right)^2 - \|\partial_x^{2n+1}\omega\|^2 - \frac{1}{2}\|\partial_x^{2n+2}\omega\|^2 + \ell_1,$$

which implies that $\|\partial_x^{2n}\omega\|$, $\|\partial_x^{2n+1}\omega\| \in \mathcal{L}_\infty$, $\partial_x^{2n+1}\omega(1)$, $\|\partial_x^{2n+1}\omega\|$, $\|\partial_x^{2n+2}\omega\| \in \mathcal{L}_2$. Then, Agmon's Inequality results in $\partial_x^{2n-1}\omega(1) \in \mathcal{L}_\infty$, transformation (2) implies that $\|\partial_x^{2n}u\|$, $\|\partial_t^n u\| \in \mathcal{L}_\infty$. This, together with $\partial_t^{n-1}\omega(0) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, gives that $\chi^{(n)} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$. Thus, (A5) and (5) respectively give that $\dot{\tau}_{n-1}$, $U \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, and hence $\dot{Z}_n \in \mathcal{L}_2 \cap \mathcal{L}_\infty$.

Appendix C Discuss on the implementation of the controller

Although the time-derivatives of χ are contained in (5), the designed controller U does not depend on the time-derivatives but only certain spatial partial derivatives of system states. In fact, see $\dot{\chi}$ and $\ddot{\chi}$ for example, by (1),(3) while using integration by parts, we have

$$\begin{cases}\dot{\chi} &= KBu(1) + \int_0^1 \partial_y^2\kappa(1,y)u(y)\mathrm{d}y + \varphi(1)AX,\\ \ddot{\chi} &= KB\partial_x^2 u(1) + KABu(1) + \int_0^1 \partial_y^4\kappa(1,y)u(y)\mathrm{d}y + \varphi(1)A^2X,\end{cases} \tag{C1}$$

which explicitly shows that no time-derivatives of system states are required.

Appendix D Simulation

In this section, we validate the effectiveness of the proposed controller for system (1) with $m = 1$, $n = 2$ and $A = 4$, $B = 2$, $\nu_1 = 5$, $\nu_2 = (1.2, 3)^{\mathrm{T}}$, the initial values being $X(0) = 1.5$, $u(x,0) = 2\sinh(\pi x) + 3\cos(0.5\pi x) + 2x^3$, $Z_1(0) = u(0,0)$ and $Z_2(0) = 2.5$.

By (5), the adaptive controller is explicitly constructed as follows:

$$\begin{cases}U &= -c_2\zeta_2 - \zeta_1 - h_2^{\mathrm{T}}\hat{\delta}_2 + \left(c_1 + \frac{2|PB|^2}{\lambda_Q}\right)\dot{\chi}\\ &\quad + \ddot{\chi} - \left(c_1 + \frac{3|PB|^2}{\lambda_Q}\right)Z_2 - Z_1\zeta_1 h_1 - \rho\partial_x^3\omega(1),\\ \dot{\hat{\delta}}_1 &= \zeta_1 h_1, \dot{\hat{\delta}}_2 = \zeta_2 h_2,\end{cases}$$

with ζ_1, ζ_2 defined before, $\dot{\chi}$ and $\ddot{\chi}$ being given by (9), $h_1 = Z_1$, $h_2^{\mathrm{T}} = \left(\left(c_1 + \frac{3|PB|^2}{\lambda_Q} + \hat{\delta}_1\right)Z_1, Z_1, Z_2\right)$, $c_1 = 85$, $c_2 = 8$. Choosing $K = -4$, $Q = 16$, the Lyapunov equation (8) gives that $P = 2$. Then, we choose $\rho = 10$. By the above controller with initial parameter estimation $\hat{\delta}_1(0) = 1.2$, $\hat{\delta}_2(0) = (1.5, 0.5, -1.5)^{\mathrm{T}}$, we implement the simulation in Matlab by explicit Euler method (see [1]) with 20-step discretization in space. Thus, four simulation figures are obtained and presented in Figs. 1–4 in the following. We observe from Figs. 1,2 that the original system states u, X and Z are all bounded and converge to zero ultimately. Moreover, Figs. 3,4 show that the dynamic estimations of unknown parameters are bounded and converge to different constants ultimately.

Appendix E Useful lemmas and proofs of Propositions 2 and 3

Proof of Proposition 2. The first two equations of (B1) can be directly obtained by the second and third ones of (4). Noting that $\zeta_1 = \omega(1)$, the expression of $\dot{\zeta}_1$ given by (A1) implies the third equation of (B1).

Suppose that the fourth equality of (B1) holds for any $2 \leqslant i < n-1$. Then, computing the time derivative of both side of the fourth equality of (B1) gives that

$$\begin{aligned}\partial_t^{i+1}\omega(1) &= -\rho\partial_x^{2i-1}\partial_t\omega(1) + f_i\Big(\dot{\zeta}_1, \ldots, \dot{\zeta}_{i+1}, \tilde{h}_1^{(i)}, \ldots, \dot{\tilde{h}}_1, \tilde{h}_2^{(i-1)}, \ldots, \dot{\tilde{h}}_2, \ldots, \dot{\tilde{h}}_i,\\ &\qquad \partial_x^{2i-3}\partial_t\omega(1), \partial_x^{2i-5}\partial_t\omega(1), \ldots, \partial_x\partial_t\omega(1)\Big).\end{aligned}$$

Substituting (A1),(A4) and (A6) into above equality and using the first equation of (B1) directly lead to

$$\partial_t^{i+1}\omega(1) = -\rho\partial_x^{2i+1}\omega(1) + f_{i+1}\Big(\zeta_1, \ldots, \zeta_{i+2}, \tilde{h}_1^{(i)}, \ldots, \tilde{h}_1, \tilde{h}_2^{(i-1)}, \ldots, \tilde{h}_2, \ldots, \tilde{h}_{i+1},$$

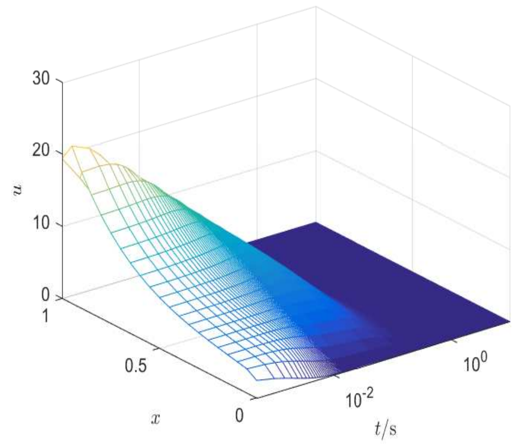

Fig. 1 Trajectories of u

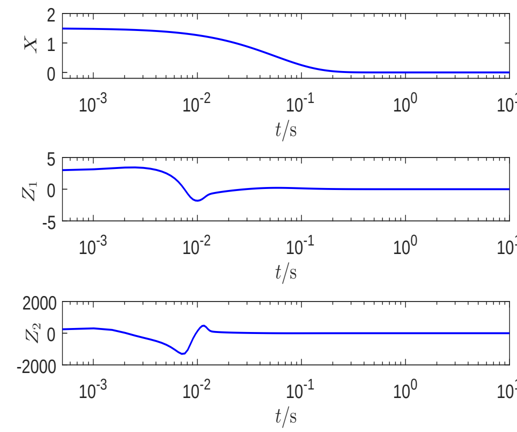

Fig. 2 Trajectories of X and Z

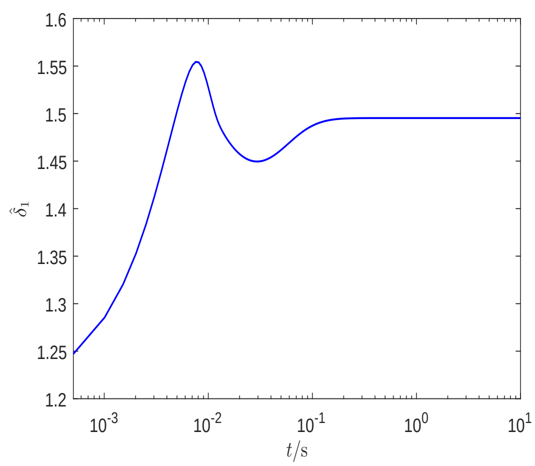

Fig. 3 Trajectories of $\hat{\delta}_1$

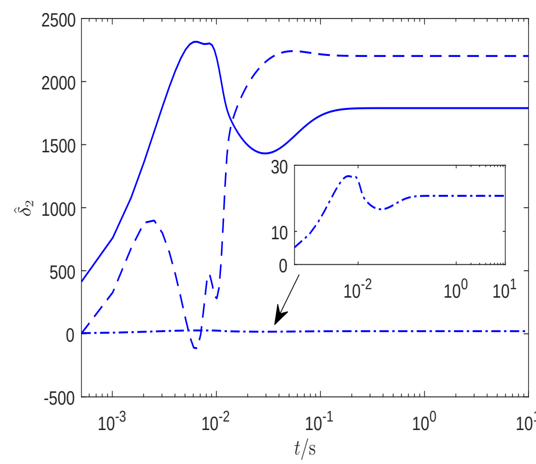

Fig. 4 Trajectories of $\hat{\delta}_2$

$$\partial_x^{2i-1}\omega(1), \partial_x^{2i-3}\omega(1), \ldots, \partial_x\omega(1)\Big).$$

Then, the fourth equality of (B1) holds for any $2 \leqslant i \leqslant n-1$. The last equation can be obtained similarly from the fourth one with $i = n-1$ by using (A7).

Proof of Proposition 3. In fact, (B2) can be directly verified by noting the definition of χ and using the first equation of (B1). Moreover, (B3) can be given by Agmon's Inequality while noting $\partial_t^k\omega(0) = \partial_x^{2k}\omega(0)$ and $\partial_t^k\omega(1) = \partial_x^{2k}\omega(1)$.

The following is devoted to the derivation of (B4). Computing $\dot{E}_k$ along the solutions of (B1) while using integration by parts lead to

$$\dot{E}_k = \partial_x^{2k}\omega(1)\partial_x^{2k+1}\omega(1) + \partial_x^{2k+1}\omega(1)\partial_x^{2k+2}\omega(1) - \|\partial_x^{2k+1}\omega\|^2 - \|\partial_x^{2k+2}\omega\|^2. \tag{E1}$$

By using Young's, Poincaré's and then Agmon's Inequalities, we have

$$\begin{aligned}\partial_x^{2k}\omega(1)\partial_x^{2k+1}\omega(1) &= \partial_t^k\omega(1)\partial_x^{2k+1}\omega(1)\\ &\leqslant \frac{1}{8}\left(\partial_x^{2k+1}\omega(1)\right)^2 + 2\left(\partial_t^k\omega(1)\right)^2\\ &\leqslant \frac{1}{2}\|\partial_x^{2k+2}\omega(1)\|^2 + 2\left(\partial_t^k\omega(1)\right)^2.\end{aligned} \tag{E2}$$

Moreover, by (B1) and using Young's Inequality, we directly have

$$\begin{aligned}\partial_x^{2k+1}\omega(1)\partial_x^{2k+2}\omega(1) &= \partial_x^{2k+1}\omega(1)\partial_t^{k+1}\omega(1)\\ &= \partial_x^{2k+1}\omega(1)\left(-\rho\partial_x^{2k+1}\omega(1) + f_{k+1}\right)\\ &\leqslant -\frac{\rho}{2}\left(\partial_x^{2k+1}\omega(1)\right)^2 + \frac{1}{2\rho}f_{k+1}^2.\end{aligned}$$

Substituting the above inequalities into the right-hand side of (E1) directly gives (B4).

Lemma 1. [2] (Agmon's Inequality) For any continuously differentiable function w defined on $[0,\ D]$, there hold

$$\begin{cases} \omega(x)^2 & \leqslant\ \omega(0)^2 + 2\|\omega\|\|\partial_x\omega\|, \\ \omega(x)^2 & \leqslant\ \omega(D)^2 + 2\|\omega\|\|\partial_x\omega\|. \end{cases}$$

Lemma 2. [2] (Poincaré's Inequality) For any continuously differentiable function w defined on $[0,\ D]$, there hold

$$\begin{cases} \|\omega\|^2 & \leqslant\ 2D\omega(0)^2 + 4D^2\|\partial_x\omega\|^2, \\ \|\omega\|^2 & \leqslant\ 2D\omega(D)^2 + 4D^2\|\partial_x\omega\|^2. \end{cases}$$

Lemma 3. [3] Suppose the function $f(t)$ defined on $[0, +\infty)$ satisfies:
1) $f(t) \geqslant 0$ for all $t \in [0, +\infty)$,
2) $f(t)$ is differentiable on $[0, +\infty)$ and there exists a constant M such that $\frac{\mathrm{d}f(t)}{\mathrm{d}t} \leqslant M$, $\forall t \geqslant 0$,
3) $\int_0^{+\infty} f(t)\ \mathrm{d}t < \infty$.
Then, there holds

$$\lim_{t\to+\infty} f(t) = 0.$$

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