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• **Supplementary File** •

# Global output feedback adaptive stabilization for systems with long uncertain input delay

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## Appendix A Some Definitions

$$
\begin{aligned}
V_{1s} &= \zeta^T P_s^{-1}\zeta + \bar{x}^T Q_s \bar{x},\\
V_{2s} &= \int_{-(\hat{d}_s+\Delta d_s)}^{-\hat{d}_s}\int_{t+\theta}^{t}\zeta^T(\tau)\left(A+e^{-A\hat{d}_s}BK_s\right)^T P_s^{-1}\left(A+e^{-A\hat{d}_s}BK_s\right)\zeta(\tau)d\tau d\theta\\
&\quad + \int_{-(\hat{d}_s+\Delta d_s)}^{-\hat{d}_s}\int_{t+\theta}^{t}\bar{x}^T(\tau)(L_sC)^T Q_s(L_sC)\bar{x}(\tau)d\tau d\theta,\\
V_{r,\zeta}(t) &= \zeta^T(t)P_r^{-1}\zeta(t) + \int_{-(\hat{d}_r+\Delta d_r)}^{-\hat{d}_r}\int_{t+\theta}^{t}\zeta^T(\tau)\left(A+e^{-A\hat{d}_r}BK_r\right)^T P_r^{-1}\left(A+e^{-A\hat{d}_r}BK_r\right)\zeta(\tau)d\tau d\theta,\\
V_{r,\bar{x}}(t) &= \bar{x}^T(t)Q_r\bar{x}(t) + \int_{-(\hat{d}_r+\Delta d_r)}^{-\hat{d}_r}\int_{t+\theta}^{t}\bar{x}^T(\tau)(L_rC)^T Q_r(L_rC)\bar{x}(\tau)d\tau d\theta,\\
\alpha_r &= \left\|e^{(A+L_rC)p_k}\right\|\theta_k + \sup_{\hat{d}_r\leqslant\delta\leqslant\hat{d}_r+\Delta d_r}\left\|\int_0^{p_k} e^{(A+L_rC)(p_k-s)}B\Big(u(s-\delta)-u(s-\hat{d}_r)\Big)ds\right\|,\\
\beta_r(\tau) &= \left\|L_rCe^{(A+L_rC)\tau}\right\|\theta_k + \sup_{\hat{d}_r\leqslant\delta\leqslant\hat{d}_r+\Delta d_r}\left\|\int_0^{p_k} L_rCe^{(A+L_rC)(\tau-s)}B\Big(u(s-\delta)-u(s-\hat{d}_r)\Big)ds\right\|.
\end{aligned}
$$

## Appendix B Proof of Lemma 1

*Proof.* The proof is divided into two parts. In the first part, it is showed that (8) holds if the inequalities

$$
\begin{aligned}
&P_s^{-1}\left(A+e^{-A\hat{d}_s}BK_s\right)+\left(A+e^{-A\hat{d}_s}BK_s\right)^T P_s^{-1}\\
&+(\Delta d_s)\left(A+e^{-A\hat{d}_s}BK_s\right)^T P_s^{-1}\left(A+e^{-A\hat{d}_s}BK_s\right)+\varepsilon_s P_s^{-1}Q_s^{-1}P_s^{-1}\leqslant -\varepsilon_s P_s^{-1}Q_s^{-1}P_s^{-1}
\end{aligned}
\tag{B1}
$$

and

$$
\begin{aligned}
&Q_s(A+L_sC)+(A+L_sC)^TQ_s+\left(\frac{1}{\varepsilon_s}+\Delta d_s\right)(L_sC)^TQ_s(L_sC)\\
&+(\Delta d_s)(Q_sBK_s)P_s(Q_sBK_s)^T+(\Delta d_s)(Q_sBK_s)Q_s^{-1}(Q_sBK_s)^T\leqslant 0
\end{aligned}
\tag{B2}
$$

have symmetric positive definite solutions $P_s$, $Q_s$.

***Part I:*** Denote $\delta d = d-\hat{d}_s$. The inequality $0\leqslant\hat{d}_s\leqslant d<\hat{d}_s+\Delta d_s$ implies $0\leqslant\delta d<\Delta d_s$. Taking the time derivative of $V_{1s}$ gives

$$
\begin{aligned}
\dot{V}_{1s} =&\zeta^T\left[P_s^{-1}\left(A+e^{-A\hat{d}_s}BK_s\right)+\left(A+e^{-A\hat{d}_s}BK_s\right)^T P_s^{-1}\right]\zeta-2\zeta^TP_s^{-1}L_sC\bar{x}\\
&+\bar{x}^T\left[Q_s(A+L_sC)+(A+L_sC)^TQ_s\right]\bar{x}+\int_{t-d}^{t-\hat{d}_s}2\bar{x}^T(\tau)Q_sBK_sL_sC\bar{x}(\tau)d\tau\\
&-\int_{t-d}^{t-\hat{d}_s}2\bar{x}^T(\tau)Q_sBK_s\left(A+e^{-A\hat{d}_s}BK_s\right)\zeta(\tau)d\tau,
\end{aligned}
\tag{B3}
$$

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resulting in

$$\begin{aligned}\dot{V}_{1s} \leqslant&\zeta^T \left[P_s^{-1}\left(A+e^{-A\hat{d}_s}BK_s\right)+\left(A+e^{-A\hat{d}_s}BK_s\right)^T P_s^{-1}+\varepsilon_s P_s^{-1}Q_s^{-1}P_s^{-1}\right]\zeta \\ &+\bar{x}^T\left[Q_s(A+L_sC)+(A+L_sC)^TQ_s+\frac{1}{\varepsilon_s}(L_sC)^TQ_s(L_sC)\right]\bar{x} \\ &+(\delta d)\bar{x}^T(Q_sBK_s)Q_s^{-1}(Q_sBK_s)^T\bar{x}+\int_{t-d}^{t-\hat{d}_s}\bar{x}^T(\tau)(L_sC)^TQ_s(L_sC)\bar{x}(\tau)d\tau \\ &+(\delta d)\bar{x}^T(Q_sBK_s)P_s(Q_sBK_s)^T\bar{x}+\int_{t-d}^{t-\hat{d}_s}\zeta^T(\tau)\left(A+e^{-A\hat{d}_s}BK_s\right)^TP_s^{-1}\left(A+e^{-A\hat{d}_s}BK_s\right)\zeta(\tau)d\tau.\end{aligned} \tag{B4}$$

Taking the time derivative of $V_{2s}$ gives

$$\begin{aligned}\dot{V}_{2s}=&(\Delta d_s)\bar{x}^T(L_sC)^TQ_s(L_sC)\bar{x}-\int_{t-(\hat{d}_s+\Delta d_s)}^{t-\hat{d}_s}\bar{x}^T(\tau)(L_sC)^TQ_s(L_sC)\bar{x}(\tau)d\tau \\ &+(\Delta d_s)\zeta^T\left(A+e^{-A\hat{d}_s}BK_s\right)^TP_s^{-1}\left(A+e^{-A\hat{d}_s}BK_s\right)\zeta \\ &-\int_{t-(\hat{d}_s+\Delta d_s)}^{t-\hat{d}_s}\zeta^T(\tau)\left(A+e^{-A\hat{d}_s}BK_s\right)^TP_s^{-1}\left(A+e^{-A\hat{d}_s}BK_s\right)\zeta(\tau)d\tau.\end{aligned} \tag{B5}$$

Note again that $\hat{d}_s \leqslant d < \hat{d}_s+\Delta d_s$, then we have

$$\begin{aligned}\dot{V}_s=&\dot{V}_{1s}+\dot{V}_{2s} \\ \leqslant&\zeta^T\Big[P_s^{-1}\left(A+e^{-A\hat{d}_s}BK_s\right)+\left(A+e^{-A\hat{d}_s}BK_s\right)^TP_s^{-1} \\ &+(\Delta d_s)\left(A+e^{-A\hat{d}_s}BK_s\right)^TP_s^{-1}\left(A+e^{-A\hat{d}_s}BK_s\right)+\varepsilon_sP_s^{-1}Q_s^{-1}P_s^{-1}\Big]\zeta \\ &+\bar{x}^T\Big[Q_s(A+L_sC)+(A+L_sC)^TQ_s+\left(\frac{1}{\varepsilon_s}+\Delta d_s\right)(L_sC)^TQ_s(L_sC) \\ &+(\Delta d_s)(Q_sBK_s)P_s(Q_sBK_s)^T+(\Delta d_s)(Q_sBK_s)Q_s^{-1}(Q_sBK_s)^T\Big]\bar{x}\end{aligned} \tag{B6}$$

for $p_s \leqslant t < t_{s+1}$. With reference to (B1) and (B2), the inequality (8) is proved.

In the second part, it is proved that (B1) and (B2) have solutions if the LMIs (9) have, which completes the proof.

***Part II:*** For ease of presentation, we divide (9) into the following inequalities:

$$\begin{bmatrix}-\left(AP_s+e^{-A\hat{d}_s}BR_s\right)-\left(AP_s+e^{-A\hat{d}_s}BR_s\right)^T-2\varepsilon_sP_s & (AP_s+e^{-A\hat{d}_s}BR_s)^T \\ (AP_s+e^{-A\hat{d}_s}BR_s) & \frac{1}{\Delta d_s}P_s\end{bmatrix}\geqslant 0, \tag{B7}$$

$$(Q_sA+S_sC)+(Q_sA+S_sC)^T+M_{1s}+2M_{2s}\leqslant 0, \tag{B8}$$

$$\begin{bmatrix}Q_s & I \\ I & P_s\end{bmatrix}\geqslant 0, \tag{B9}$$

$$\begin{bmatrix}M_{1s} & (S_sC)^T \\ S_sC & \left(\frac{1}{\varepsilon_s}+\Delta d_s\right)^{-1}Q_s\end{bmatrix}\geqslant 0, \tag{B10}$$

$$\begin{bmatrix}M_{2s} & Q_s \\ Q_s & \frac{1}{\varepsilon_s}I\end{bmatrix}\geqslant 0, \tag{B11}$$

$$\begin{bmatrix}\varepsilon_sI & BR_s \\ (BR_s)^T & \frac{1}{\Delta d_s}P_s\end{bmatrix}\geqslant 0. \tag{B12}$$

Then let $K_sP_s=R_s$, $Q_sL_s=S_s$. Multiplying the left and the right side of the inequality (B1) by $P_s$ from both left and right, the inequalities (B1) and (B2) are respectively transformed into

$$\begin{aligned}&\left(AP_s+e^{-A\hat{d}_s}BR_s\right)+\left(AP_s+e^{-A\hat{d}_s}BR_s\right)^T \\ &+(\Delta d_s)\left(AP_s+e^{-A\hat{d}_s}BR_s\right)^TP_s^{-1}\left(AP_s+e^{-A\hat{d}_s}BR_s\right)+2\varepsilon_sQ_s^{-1}\leqslant 0\end{aligned} \tag{B13}$$

and

$$\begin{aligned}&(Q_sA+S_sC)+(Q_sA+S_sC)^T+\left(\frac{1}{\varepsilon_s}+\Delta d_s\right)(S_sC)^TQ_s^{-1}(S_sC) \\ &+\Delta d_s(Q_sBR_s)P_s^{-1}(Q_sBR_s)^T+\Delta d_s(Q_sBR_sP_s^{-1})Q_s^{-1}(Q_sBR_sP_s^{-1})^T\leqslant 0.\end{aligned} \tag{B14}$$

In the following, we show (B7)-(B12) imply (B13)-(B14). According to Schur's complement, the LMIs (B7) and (B9) are equivalent to the following inequalities,

$$\begin{aligned}&\left(AP_s+e^{-A\hat{d}_s}BR_s\right)+\left(AP_s+e^{-A\hat{d}_s}BR_s\right)^T\\&+(\Delta d_s)\left(AP_s+e^{-A\hat{d}_s}BR_s\right)^TP_s^{-1}\left(AP_s+e^{-A\hat{d}_s}BR_s\right)+2\varepsilon_sP_s\leqslant 0;\\&P_s-Q_s^{-1}\geqslant 0.\end{aligned}\tag{B15}$$

Obviously when (B15) holds, (B13) must also hold. Furthermore, (B9) and (B11) are equivalent to $P_sQ_sP_s\geqslant P_s, Q_s^{-1}M_{2s}Q_s^{-1}\geqslant \varepsilon_sI$. According to the properties of positive definite matrices, for the inequality (B12), one has the result that

$$\begin{aligned}&\begin{bmatrix}\varepsilon_sI & BR_s\\ BR_s^T & \frac{1}{\Delta d_s}P_s\end{bmatrix}\geqslant 0\Rightarrow\begin{bmatrix}Q_s^{-1}M_{2s}Q_s^{-1} & BR_s\\ BR_s^T & \frac{1}{\Delta d_s}P_s\end{bmatrix}\geqslant 0\\&\Leftrightarrow\begin{bmatrix}Q_s^{-1} & O\\ O & I\end{bmatrix}\begin{bmatrix}M_{2s} & Q_sBR_s\\ Q_sBR_s^T & \frac{1}{\Delta d_s}P_s\end{bmatrix}\begin{bmatrix}Q_s^{-1} & O\\ O & I\end{bmatrix}\geqslant 0\Leftrightarrow\begin{bmatrix}M_{2s} & Q_sBR_s\\ (Q_sBR_s)^T & \frac{1}{\Delta d_s}P_s\end{bmatrix}\geqslant 0.\end{aligned}\tag{B16}$$

The similar procedure can be utilized again, then we have

$$\begin{bmatrix}\varepsilon_sI & BR_s\\ BR_s^T & \frac{1}{\Delta d_s}P_s\end{bmatrix}\geqslant 0\Rightarrow\begin{bmatrix}M_{2s} & Q_sBR_sP_s^{-1}\\ (Q_sBR_sP_s^{-1})^T & \frac{1}{\Delta d_s}Q_s\end{bmatrix}\geqslant 0.\tag{B17}$$

Combining (B12), (B16), (B17) and using Schur's complement, it's clear that the LMI (B14) has a solution if there are solutions to the inequalities (B8-B12).

## Appendix C Proof of Theorem 1

The proof is divided into two parts. In the first part, it is showed that, if there only exists a finite number of switching times, global asymptotic stability is reached. In the second one, it is showed that there exists a finite number of switching times, which ends the proof.

***Part I:*** Suppose $t_f$ is the final switching moment and $f=iN+q$ for some integer $i$ and $q\in\{0,1,2,...,N-1\}$. Since $t_f$ is the final switching moment, one has the following (otherwise, one would have another switching moment):

$$\int_{p_f}^{t}\zeta^T(\tau)(P_qQ_qP_q)^{-1}\zeta(\tau)d\tau\leqslant\varepsilon_q^{-1}\left[V_{q,\zeta}(p_f)-V_{q,\zeta}(t)+\lambda_{\max}(Q_q)\left(\alpha_q^2+\int_{-(\hat{d}_q+\Delta d_q)}^{-\hat{d}_q}\int_{p_f+\theta}^{p_f}\beta_q^2(\tau)d\tau d\theta\right)\right]$$

for all $t\geqslant p_f$. The right-hand side of the inequality is bounded from above by a constant number independent of $t$. Moreover, since $P_q$ and $Q_q$ are positive definite, one has that $\lim_{t\to\infty}\int_{p_f}^{t}\|\zeta(\tau)\|^2d\tau$ exists and is finite. Therefore, by Barbalat's lemma, $\zeta(t)$, and so is $u(t)$, is bounded on $[p_f,\infty)$ and we have $\zeta(t)\to 0$ as $t\to\infty$. From (5) and (6), we have

$$\|\hat{x}(t)\|\leqslant\|\zeta(t)\|+\hat{d}_r\left(\max\nolimits_{-\hat{h}_r\leqslant\theta\leqslant 0}\left\|e^{A\theta}\right\|\right)\|B\|\,\|K_r\|\,\|\zeta_t\|_c$$

for $t\in[p_f,\infty)$, where $\|\zeta_t\|_c$ denotes the supreme norm of continuous functions over $[t-d,t)$. Since $\hat{d}_r$ is bounded from above, $\hat{x}(t)$ is also bounded and we have $\hat{x}(t)\to 0$ as $t\to\infty$. From (4) one further has $\bar{x}(t)\to 0$ as $t\to\infty$, so we have the result that $x(t)\to 0$ as $t\to\infty$.

***Part II:*** Seeking a contradiction. Suppose on the contrary that there are infinite numbers of switching times. Then according to (2), there must exist a switching moment $t_k$ satisfying $\theta_k>\|\bar{x}(0)\|$, and $k=iN+s$ with the parameters $\hat{d}_s$, $\Delta d_s$, $\varepsilon_s$, $P_s$, $Q_s$, $K_s$, $L_s$,$V_s$, where delay estimate $\hat{d}_s$ satisfies $d\in[\hat{d}_s,\hat{d}_s+\Delta d_s)$. By Lemma 1, we have

$$\dot{V}_s\leqslant-\varepsilon_s\zeta^TP_s^{-1}Q_s^{-1}P_s^{-1}\zeta$$

for all $p_k\leqslant t<t_{k+1}$. Integrating both sides of the inequality, then we further get the following inequality:

$$V_{s,\zeta}(t)+V_{s,\bar{x}}(t)-V_{s,\zeta}(p_k)-V_{s,\bar{x}}(p_k)\leqslant-\varepsilon_s\int_{p_k}^{t}\zeta^T(\tau)(P_sQ_sP_s)^{-1}\zeta(\tau)d\tau.\tag{C1}$$

From the definition of $V_{s,\bar{x}}(t)$, we can obtain

$$V_{s,\bar{x}}(p_k)<\lambda_{\max}(Q_s)\left(\|\bar{x}(p_k)\|^2+\int_{-(\hat{d}_s+\Delta d_s)}^{-\hat{d}_s}\int_{p_k+\theta}^{p_k}\|L_sC\bar{x}(\tau)\|^2d\tau d\theta\right)$$

and by integrating the inequality (4) from $t=0$ to $t=p_k$, and from $t=0$ to $t=\tau$, respectively, we have $\|\bar{x}(p_k)\|\leqslant\alpha_s$ and $\|L_sC\bar{x}(\tau)\|\leqslant\beta_s(\tau)$. So one further has

$$-\lambda_{\max}(Q_s)\left(\alpha_s^2+\int_{-(\hat{d}_s+\Delta d_s)}^{-\hat{d}_s}\int_{p_k+\theta}^{p_k}\beta_s^2(\tau)d\tau d\theta\right)<-V_{s,\bar{x}}(p_k).\tag{C2}$$

As a result, substituting (C2) into (C1) and rearranging the terms, one will find that (10) cannot be satisfied for all $p_k\leqslant t<t_{k+1}$, i.e., $t_{k+1}=\infty$. A contradiction arises. To conclude, the total number of switching times is finite.

## Appendix D Numerical example

In this part, in order to validate the effectiveness of the theoretical results, some simulations to the problem of regulating a missile roll angle have been done. The plant is described by

$$\begin{aligned} \dot{x}_1 &= u(t-d), \\ \dot{x}_2 &= \varepsilon\alpha x_1 - \alpha x_2, \\ \dot{x}_3 &= x_2, \\ y &= x_3, \end{aligned} \tag{D1}$$

where $x_1 = \delta$ is the angle of tail mobile fins generating the control torque $u$, $x_2 = \omega$ is the roll angular velocity, $x_3 = \phi$ is the missile roll angle. $\varepsilon$ is the mobile fins efficiency, $\tau = \frac{1}{\alpha}$ is the roll time constant. According to [1], the following parameters values have been used: $\tau = 1s^{-1}$, $\varepsilon = 10s$, $d = 0.6$, $\underline{d} = 0$, $\overline{d} = 1.0$. Rewritting plant (D1), the corresponding matrices are given by

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 10 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}. \tag{D2}$$

In Subsection Appendix D.1, one compares the proposed logic switching-based delay estimator and the existing delay estimator mentioned in [2]. In Subsection Appendix D.2, one shows that the proposed scheme is robust to model uncertainties.

### Appendix D.1 Comparison with the delay estimator in [2]

Suppose that the system (D1) is subject to an unknown input delay $d = 0.6$ that is upper bounded by $\overline{d} = 1.0$. The initial state is given by $x(0) = [1; 0.2; 10]$ and $\hat{x}(0) = [1.2; 0.1; 0]$. Set $\hat{d}_0 = 0, \Delta d_r = 0.05, \varepsilon_r = 0.1, r = 0, 1, 2, \ldots, 20$. Then one can apply algorithm to find out $K_r, L_r, P_r, Q_r, V_r$ for every $r$. At the same time, the observer, controller and delay estimator in [2] are also applied into the same simulation. The results in Figure D1 and Figure D2 show the evolution of $x(t)$, $u(t)$ and the difference between the method in this note and that in [2]. It's clear that our adaptive controller is effective and the system is stabilised in both cases. Furthermore, Figure D3 shows that time delay estimates updated by two different strategies.

### Appendix D.2 Simulation with model uncertainties

In this subsection, one shows that the proposed control scheme is robust with respect to the model uncertainties. Considers the system (D1) with model uncertainties:

$$\begin{aligned} \dot{x}(t) &= (A + \Delta A)x(t) + (B + \Delta B)u(t-d), \\ y(t) &= (C + \Delta C)x(t), \end{aligned} \tag{D3}$$

with

$$\Delta A = \begin{bmatrix} -0.1 & 0 & 0 \\ 0 & 0 & 0.1 \\ 0 & 0 & -0.1 \end{bmatrix}, \Delta B = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}, \Delta C = \begin{bmatrix} 0.1 & 0.1 & 0 \end{bmatrix}.$$

where $\Delta A$, $\Delta B$ and $\Delta C$ are unknown. All of the parameters are the same as the ones given in subsection A and all the parameters are calculated by using $A$ and $B$.

The simulation results are presented in Figures D4–D6, the uncertain system (D3) is stabilised in spite of model uncertainties $\Delta A$, $\Delta B$ and $\Delta C$. This simulation highlights the robustness of the proposed method.

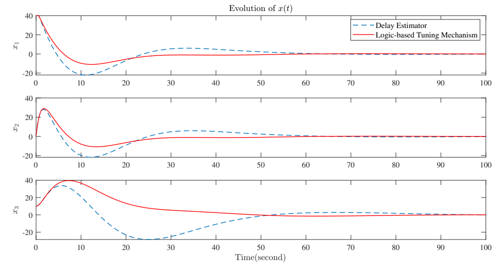


**Figure D1** Evolution of $x(t)$ for system (D1).

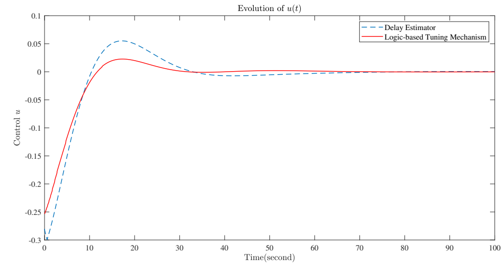


**Figure D2** Evolution of $u(t)$ for system (D1).

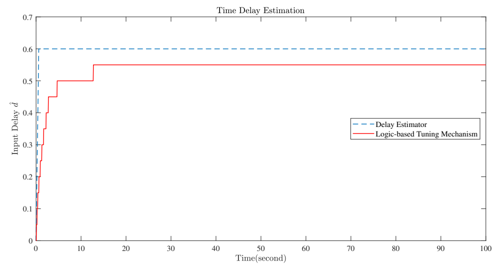


**Figure D3** Evolution of $\hat{d}(t)$ for system (D1).

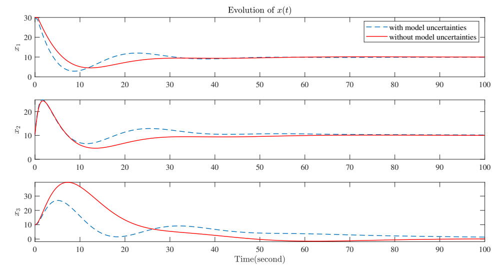


**Figure D4** Evolution of $x(t)$ for system (D3).

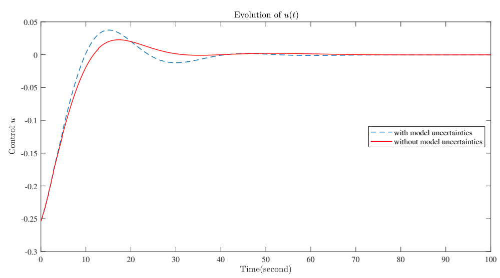


**Figure D5** Evolution of $u(t)$ for system (D3).

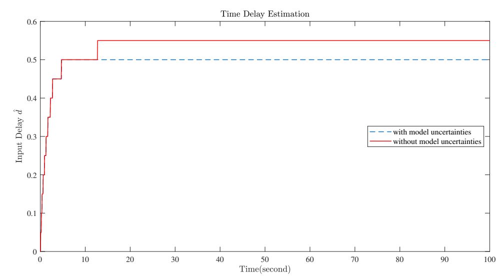


**Figure D6** Evolution of $\hat{d}(t)$ for system (D3).

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