SCIENCE CHINA
Information Sciences

• RESEARCH PAPER •

January 2023, Vol. 66 112203:1–112203:14
https://doi.org/10.1007/s11432-021-3426-1

Interval stability/stabilization of impulsive positive systems

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Received 9 September 2021/Revised 6 December 2021/Accepted 23 December 2021/Published online 26 December 2022

Abstract This paper investigates the criterion of interval stability of impulsive positive systems with time-varying delay using the idea of generalized pole placement and equivalent systems. Unlike previous stability conditions, the interval stability criterion can more accurately determine the stability of impulsive positive systems. In simple terms, the new criterion can determine the system's stability and convergence speed. Furthermore, a suitable sufficient condition of interval stabilization is obtained on the basis of the interval stability criterion, which can not only guarantee the stability of impulsive positive systems, but also modulate the state's convergence rate to the required level. A new algorithm is developed with the interval stabilization approach to accurately control the target system's convergence rate. Two instances are proposed to prove the superiority of the attainable interval stabilization criterion.

Keywords impulsive positive systems, interval stability/stabilization, copositive Lyapunov function, convergence rate

Citation Wang X Z, Zhang H S, Xia J W, et al. Interval stability/stabilization of impulsive positive systems. Sci China Inf Sci, 2023, 66(1): 112203, https://doi.org/10.1007/s11432-021-3426-1

1 Introduction

Impulsive positive systems (IPSs) [1–3] are specific hybrid systems [4] whose state variables and outputs are not only nonnegative as initial states and inputs but also arise instantaneous jumps or changes at some discrete moments. Such systems can describe a wide range of practical applications, such as portfolio management [2], epidemic dynamics [5], physiological pharmacokinetic model [6], and integrated pest management [7, 8]. Additionally, IPSs can represent the switched positive systems [9–14] and sampled-data positive systems [15]. The study on IPSs is both theoretically and practically significant.

Although it has many branches, stability is a fairly significant actuation for the advancement of IPSs. On account of a copositive Lyapunov function (CLF) [16], the adequate requirements for IPSs stability were first established in [1]. Refs. [3, 17] obtained several stability/stabilization requirements for linear IPSs through the concept of dwell-time and linear CLF. By constructing suitable CLF and using some analytical approaches, several sufficient conditions for finite-time stability are given in [18, 19]. Furthermore, Refs. [2, 20, 21] investigated IPSs' exponential stability.

However, no studies on the interval stability/stabilization of IPSs have been conducted. Particularly, interval stability/stabilization of linear stochastic systems was investigated in several studies [22, 23]. Refs. [24, 25] researched the Pth moment stability/stabilization of linear stochastic systems. Inspired by the above studies, the accurate control about the convergence speed for IPSs with time-varying delay is fulfilled through equivalent systems and elaborate definitions of interval stability.

Stimulated by the mentioned exposition, we analyze the interval stability for IPSs, focusing on uncovering the Giusto properties of interval stability and promoting the manifestation in the controller design. In this study, the conditions for interval stability and the design of an interval state feedback controller for such systems are presented, which is a significant breakthrough in the study of IPSs and precision control. This paper's main highlights and contributions are divided into three parts.

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(1) A new interval stability criterion of IPSs with time-varying delay is proposed, which can more accurately describe the performance of such systems. To achieve this, we convert the initial system into two equivalent systems and assign a CLF. The new stable criterion is considered an extension of the previous stable criterion for IPSs.

(2) The IPS interval stabilization problem is addressed by controlling the state matrix eigenvalues at a suitable interval, which ensures the system's stability and accurately controls the convergence speed of the states.

(3) An algorithm is developed as a supplement to interval stabilization to further control the convergence rate for IPSs. Combined with the interval stabilization approach, IPSs can be accurately controlled. There is no relevant conclusion that can accomplish this as far as we know.

The remaining paper is organized as follows. In Section 2, some preliminaries and definitions are presented in preparation for the main results. In Section 3, a standard is presented in relation to IPSs' interval stability. In Section 4, a sufficient condition is derived concerning interval stabilization of IPSs. Furthermore, the interval stabilization controller for IPSs is obtained. Two numerical examples are presented in Section 5. The conclusion is reached in Section 6.

Notation. $\mathcal{H}^{\mathrm{T}}$ is the transposition of matrix $\mathcal{H}$. $\mathbb{R}^n$ expresses the n-dimensional Euclidean space. $\mathbb{R}^n_+$ denotes nonnegative orthant of $\mathbb{R}^n$. $\mathbb{N}$ is the set of positive integer. $\mathrm{Re}(\xi)$ denotes real parts about matrix eigenvalues. The spectrum set of the matrix $\mathcal{H}$ is represented by $\sigma(\mathcal{H})$. $\mathcal{C}^{-q}_{-p}$ is the vertical strip region from $-p$ to $-q$ in the complex plane. The notation $\|\cdot\|$ denotes Euclidean norm. The 1-norm concerning $x \in \mathbb{R}^n$ is $\|x\|_1 = \sum_{i=1}^n \mid x_i \mid$. $\mathcal{H} \preceq 0$ ($\succeq 0, \prec 0, \succ 0$) means that all elements of matrix $\mathcal{H}$ are nonpositive (nonnegative, negative, and positive). $d \preceq 0$ ($\succeq 0, \prec 0, \succ 0$) means that all entries of the vector d are nonpositive (nonnegative, negative, and positive). For any given $\theta \in \mathbb{R}_+$, $g(\theta^-)$ means the limit of $g(t)$ as t goes to θ from the left. For $\tau > 0$, $\overline{\mathcal{PC}}([-\tau, 0], \mathbb{R}^n)$ denotes the space in relation to piecewise right continuous function $\bar{\phi} : [-\tau, 0] \to \mathbb{R}^n$ with $\|\bar{\phi}\|_\tau = \sup_{-\tau \leqslant \theta \leqslant 0} \|\bar{\phi}(\theta)\|_1$. $\min\{d\}$ denotes the minimum element of vector d. $\max\{d\}$ denotes the maximum element of vector d.

2 Preliminaries

Consider IPSs:

$$\begin{cases} \dot{x}(t) = \mathcal{A}x(t) + \mathcal{A}_d x(t - \tau(t)) + \mathcal{B}u(t),\ t \neq t_k,\ k \in \mathbb{N}, \\ x(t_k) = \mathcal{H}x(t_k^-),\ t = t_k,\ k \in \mathbb{N}, \\ x(t_0 + \theta) = \bar{\phi}(\theta),\ t_0 = 0,\ \theta \in [-\tau, t_0], \end{cases} \tag{1}$$

where $u(t) \in \mathbb{R}^r$ is the continuous control input, $x(t) \in \mathbb{R}^n$ is the state, $\tau(t)$ is the time-varying delay gratifying $0 < \tau(t) \leqslant \tau$, $\bar{\phi}(\theta) \in \overline{\mathcal{PC}}([-\tau, 0], \mathbb{R}^n)$ is the initial continuous function, $\mathcal{A}$, $\mathcal{A}_d$, $\mathcal{B}$, and $\mathcal{H}$ are system matrices with compatible dimensions.

The state of system (1) jumps at $t = t_k$, $0 = t_0 < t_1 < \cdots < t_k < \cdots$, and $\lim_{k\to\infty} t_k = +\infty$. Let $\mathcal{T}_k := t_{k+1} - t_k$, $k \in \mathbb{N}$ denote the dwell-time. The minimum dwell-time τ_0 and the maximum dwell-time τ_1 are defined as $\inf_k \{\mathcal{T}_k\} \geqslant \tau_0$ and $\sup_k \{\mathcal{T}_k\} \leqslant \tau_1$, respectively. The set of impulse time sequence satisfies $0 < \tau_0 \leqslant \mathcal{T}_k \leqslant \tau_1$.

Remark 1. In addition to the general characteristics of the general impulsive systems, IPSs also have the dynamic characteristics of the positive systems, that is, the state variables always remain in the non-negative quadrant.

Definition 1 ([2])**.** If the system path $x(t) \succeq 0$ ($t \geqslant t_0$) holds for all $\bar{\phi}(\cdot) \succeq 0$ and $u(t) \succeq 0$, then system (1) is positive.

Definition 2 ([26])**.** Real matrix $\mathcal{U} \triangleq (u_{ij})_{n\times n}$ is a Metzler matrix if $u_{ij} \geqslant 0$, $i \neq j$, as well as $\mathcal{M}$ is the set of Metzler matrices.

Definition 3 ([21])**.** If there exist constants $\bar{\lambda} > 0$ and $\bar{c} > 0$ satisfying that for any $\bar{\phi}(\cdot) \succeq 0$, $\|x(t)\| \leqslant \bar{c}\mathrm{e}^{-\bar{\lambda}(t-t_0)}\|\bar{\phi}\|_\tau$, $t \geqslant t_0$, then system (1) is exponentially stable.

Based on the idea of pole assignment and Definition 3, Definition 4 is gained.

Definition 4. System (1) is asymptotically interval $(-p, -q)$ stable with $0 \leqslant q < p$ if $\sigma(\mathcal{A}) \subset \mathcal{C}^{-q}_{-p} := \{\xi : -p < \mathrm{Re}(\xi) < -q\}$ and $\lim_{t\to\infty} \|x(t)\| = 0$.

Remark 2. p, q are non-negative constants and ξ is the eigenvalue of matrix $\mathcal{A}$, which can reflect some properties of system (1) such as its convergence rate. $\sigma(\mathcal{A}) \subset \mathcal{C}^{-q}_{-p} := \{\xi : -p < Re(\xi) < -q\}$ indicates that the real part of the eigenvalue of matrix $\mathcal{A}$ is located in the vertical strip region from $-p$ to $-q$ in the complex plane. When $q = 0$ and $p = \infty$, the interval stability becomes asymptotically stable.

Definition 5 ([27]). The function $\mathcal{V} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ pertains to class $\bar{v}_0$ if the following conditions hold:

(1) The function $\mathcal{V}$ is continuous for all sets $[t_k, t_{k+1}) \times \mathbb{R}^n$, as well as $\lim_{(t,y)\to(t_k^-,x)} \mathcal{V}(t, y) = \mathcal{V}(t_k^-, x)$ for each x, $y \in \mathbb{R}^n$, $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$.

(2) The function $\mathcal{V}(t, x(t))$ locally gratifies Lipschitz condition, and $\mathcal{V}(t, 0) \equiv 0$ holds for all $t \geqslant t_0$.

Definition 6 ([28]). Given a function $\mathcal{V} \in \bar{v}_0$, the upper right-hand derivative of $\mathcal{V}$ along the solution $x(t)$ of system (1) is defined as

$$D^+\mathcal{V}(t, x(t)) = \lim_{\varpi\to 0^+} \sup[\mathcal{V}(t + \varpi, x(t) + \varpi) - \mathcal{V}(t, x(t))]/\varpi.$$

Lemma 1 ([29]). Real square matrix $\mathcal{A} \in \mathcal{M}$ if and only if (iff) there exists a scalar ϑ such that $\mathcal{A} + \vartheta I \succeq 0$.

Lemma 2 ([1]). System (1) under $u(t) = 0$ is positive iff $\mathcal{A} \in \mathcal{M}$, $\mathcal{A}_d \succeq 0$, and $\mathcal{H} \succeq 0$.

Lemma 3. If there exist a costant $\mu > 0$ and a vector $v \in \mathbb{R}_n^+$ satisfying that

$$(\mathcal{H}^{\mathrm{T}} - \mu I)v \preceq 0, \tag{2}$$

then $\mathcal{V}(t_k) \leqslant \mu\mathcal{V}(t_k^-)$, $k \in \mathbb{N}$.

Proof. In light of system (1) and $\mathcal{V}(t) = x^{\mathrm{T}}(t)v$, one can deduce that

$$\mathcal{V}(t_k) - \mu\mathcal{V}(t_k^-) = x^{\mathrm{T}}(t_k^-)\mathcal{H}^{\mathrm{T}}v - \mu x^{\mathrm{T}}(t_k^-)v = x^{\mathrm{T}}(t_k^-)(\mathcal{H}^{\mathrm{T}} - \mu I)v.$$

Then, from (2), $\mathcal{V}(t_k) \leqslant \mu\mathcal{V}(t_k^-)$, $k \in \mathbb{N}$ is obtained.

Remark 3. When $\mu > 1$, the CLF may jump up at the impulsive time instants and the impulses may be viewed as disturbance impulses which are not conducive to the stability of the system (1). When $\mu < 1$, the CLF will jump down at the impulsive time instants and the impulses can be regarded as stabilizing impulses that are beneficial to the stability of the system. When $\mu = 1$, the impulses are 'neutral', indicating that they are neither beneficial nor harmful to the stability of system (1).

3 Interval stability

In this part, a sufficient criterion of interval $(-p, -q)$ stable for system (1) under $u(t) = 0$ is derived utilizing Lyapunov-Razumikhin techniques.

Construct a CLF:

$$\mathcal{V}(t) = x^{\mathrm{T}}(t)v, \tag{3}$$

where $v \succ 0$.

Remark 4. For IPSs, if the conventional quadratic Lyapunov function is selected to study the stability, the conventional quadratic Lyapunov function needs to realize the attenuation in the whole state space. In this case, there may be no solution for IPSs. If the CLF is used to study the stability of IPSs, it only needs that all non-negative states of the system meet the attenuation condition. This method is considered to be more effective in revealing the characteristics of IPSs and has been widely used in the study of IPSs.

Ground on copositive Lyapunov function (3) and Razumikhin method, the interval $(-p, -q)$ stable condition is deduced, which is presented as follows.

Theorem 1. For any $\mathcal{T}_k \in [\tau_0, \tau_1]$, system (1) under $u(t) = 0$ is interval $(-p, -q)$ stable, if there exist scalars $\mu > 0$, $\omega > 0$, p, q and appropriate dimensioned vectors $v \in \mathbb{R}_+^n$, such that when $\mu \in (0, 1)$, the following inequalities hold:

$$\left(-\mathcal{A}^{\mathrm{T}} + \left(\frac{\omega}{\mu} + \frac{\ln\mu}{\tau_1} - p\right) I\right) v \prec 0, \tag{4}$$

$$\left(\mathcal{A}^{\mathrm{T}} + \left(\frac{\omega}{\mu} + \frac{\ln\mu}{\tau_1} + q\right) I\right) v \prec 0, \tag{5}$$

$$(-\mathcal{A}_d^{\mathrm{T}} - \omega I)v \prec 0, \tag{6}$$
$$(\mathcal{A}_d^{\mathrm{T}} - \omega I)v \prec 0, \tag{7}$$
$$(\mathcal{H}^{\mathrm{T}} - \mu I)v \preceq 0, \tag{8}$$

or when $\mu \geqslant 1$, the following inequalities and inequalities (6)–(8) hold:

$$\left(-\mathcal{A}^{\mathrm{T}} + \left(\omega\mu + \frac{\ln\mu}{\tau_0} - p\right) I\right) v \prec 0, \tag{9}$$
$$\left(\mathcal{A}^{\mathrm{T}} + \left(\omega\mu + \frac{\ln\mu}{\tau_0} + q\right) I\right) v \prec 0. \tag{10}$$

Proof. By Definition 4, system (1) under $u(t) = 0$ is interval $(-p, -q)$ stable iff the system is asymptotically interval $(-\infty, -q)$ and $(-p, \infty)$ stable. Thus, only both following systems

$$\begin{cases} \dot{x}(t) = -(\mathcal{A} + pI)x(t) - \mathcal{A}_d x(t - \tau(t)),\ t \neq t_k,\ k \in \mathbb{N}, \\ x(t_k) = \mathcal{H}x(t_k^-),\ t = t_k,\ k \in \mathbb{N}, \\ x(t_0 + \theta) = \bar{\phi}(\theta),\ t_0 = 0,\ \theta \in [-\tau, t_0], \end{cases} \tag{11}$$

and

$$\begin{cases} \dot{x}(t) = (\mathcal{A} + qI)x(t) + \mathcal{A}_d x(t - \tau(t)),\ t \neq t_k,\ k \in \mathbb{N}, \\ x(t_k) = \mathcal{H}x(t_k^-),\ t = t_k,\ k \in \mathbb{N}, \\ x(t_0 + \theta) = \bar{\phi}(\theta),\ t_0 = 0,\ \theta \in [-\tau, t_0] \end{cases} \tag{12}$$

are asymptotically stable; then system (1) under $u(t) = 0$ is interval $(-p, -q)$ stable. About system (11), choose copositive Lyapunov function (3).

For sufficiently small scalar $\alpha > 0$, it follows (4), (6) and (9) that

$$\left(-\mathcal{A}^{\mathrm{T}} + \left(\frac{\omega}{\mu} + \frac{\ln\mu}{\tau_1} - p + \alpha\right) I\right) v \prec 0, \tag{13}$$
$$(-\mathcal{A}_d^{\mathrm{T}} - \omega \mathrm{e}^{-\alpha\tau} I)v \prec 0, \tag{14}$$
$$\left(-\mathcal{A}^{\mathrm{T}} + \left(\omega\mu + \frac{\ln\mu}{\tau_0} - p + \alpha\right) I\right) v \prec 0. \tag{15}$$

Let $\underline{v} = \min(v)$, $\overline{v} = \max(v)$, and

$$\mathcal{G}(t) = \mathrm{e}^{\alpha(t - t_0)}\mathcal{V}(t),$$

where sufficiently small scalars $\alpha > 0$. According to the different value range of μ, the following two cases will be discussed in $\mu \in (0, 1)$ and $\mu \geqslant 1$, respectively.

Case I. $\mu \in (0, 1)$.

The goal is to prove the asymptotical stability of the system (11). Setting $\delta = \frac{1}{\mu}\|\bar{\phi}(\theta)\|_\tau$, $\theta \in [-\tau, t_0]$, the following inequality:

$$\mathcal{G}(t) < \delta\overline{v}, \quad t \in [t_0 - \tau, \infty) \tag{16}$$

will be proved. On the basis of mathematical induction, the proof is implemented via three steps.

Step 1. In respect of time interval $t \in [t_0 - \tau, t_0]$.

$$\mathcal{G}(t) = \mathrm{e}^{\alpha(t - t_0)}\mathcal{V}(t) = \mathrm{e}^{\alpha\theta}\mathcal{V}(t_0 + \theta) \leqslant \mu\delta\overline{v} < \delta\overline{v}. \tag{17}$$

Therefore, when $t \in [t_0 - \tau, t_0]$, inequality (16) is satisfied.

Step 2. In respect of time interval $t \in (t_0, t_1)$.

In this step, inequality (16) will be proved via the contradiction method. Assume that there exists $t \in (t_0, t_1)$ satisfying $\mathcal{G}(t) \geqslant \delta\overline{v}$.

According to (17), it is shown that $\mathcal{G}(t_0) \leqslant \mu\delta\overline{v}$. Consequently, there exist $t^\star$ and $t^{\star\star}$ making that $t^{\star\star} = \inf\{t \in (t_0, t_1); \mathcal{G}(t) \geqslant \delta\overline{v}\}$, $t^\star = \sup\{t \in (t_0, t^{\star\star}); \mathcal{G}(t) \leqslant \mu\delta\overline{v}\}$. Considering that $\mathcal{G}(t)$ is continuous about $t \in (t_0, t_1)$, one has $\mathcal{G}(t^{\star\star}) = \delta\overline{v}$, $t^{\star\star} \in (t_0, t_1)$; $\mathcal{G}(t^{\star}) = \mu\delta\overline{v}$, $t^{\star} \in (t_0, t^{\star\star})$. In respect of $t \in [t^{\star}, t^{\star\star}]$, there holds:

$$\mu\mathcal{G}(t+\theta) \leqslant \mu\delta\overline{v} \leqslant \mathcal{G}(t), \quad \theta \in [-\tau, 0).$$

Then in relation to $t \in [t^{\star}, t^{\star\star}]$ and arbitrary given constant $\omega > 0$, it is able to acquire that

$$\begin{aligned} D^+\mathcal{G}(t) &= \mathrm{e}^{\alpha(t-t_0)}\big(\alpha\mathcal{V}(t) + D^+\mathcal{V}(t)\big) \\ &\leqslant -\frac{\ln\mu}{\tau_1}\mathcal{G}(t) + \frac{\ln\mu}{\tau_1}\mathcal{G}(t) + \mathrm{e}^{\alpha(t-t_0)}\big(\alpha\mathcal{V}(t) + D^+\mathcal{V}(t)\big) \\ &\quad + \omega\left(\frac{1}{\mu}\mathcal{G}(t) - \mathcal{G}\big(t-\tau(t)\big)\right). \end{aligned} \tag{18}$$

Moreover, calculating $D^+\mathcal{V}(t)$ along track of system (11), one can obtain that

$$D^+\mathcal{V}(t) = D^+x^{\mathrm{T}}(t)v = \Big(-(\mathcal{A}+pI)x(t) - \mathcal{A}_d x\big(t-\tau(t)\big)\Big)^{\mathrm{T}} v. \tag{19}$$

It can be verified that the following expressions hold:

$$\mathcal{G}(t) = \mathrm{e}^{\alpha(t-t_0)}x^{\mathrm{T}}(t)v, \tag{20}$$

$$-\mathcal{G}(t-\tau(t)) \leqslant -\,\mathrm{e}^{\alpha(t-t_0)}\mathrm{e}^{-\alpha\tau}x^{\mathrm{T}}\big(t-\tau(t)\big)v. \tag{21}$$

Substituting (19)–(21) into (18), it can be derived that

$$D^+\mathcal{G}(t) \leqslant -\frac{\ln\mu}{\tau_1}\mathcal{G}(t) + \mathrm{e}^{\alpha(t-t_0)}\left[\, x^{\mathrm{T}}(t)\ x^{\mathrm{T}}\big(t-\tau(t)\big) \,\right]\left[\begin{array}{c} \big(-\mathcal{A}^{\mathrm{T}} + (\frac{\omega}{\mu} + \frac{\ln\mu}{\tau_1} - p + \alpha)I\big)v \\ (-\mathcal{A}_d^{\mathrm{T}} - \omega\mathrm{e}^{-\alpha\tau}I)v \end{array}\right]. \tag{22}$$

Combining (13) and (14), one will get

$$D^+\mathcal{G}(t) < -\frac{\ln\mu}{\tau_1}\mathcal{G}(t), \quad t \in [t^{\star}, t^{\star\star}]. \tag{23}$$

Due to $-\frac{\ln\mu}{\tau_1}(t^{\star\star} - t^{\star}) \leqslant -\frac{\ln\mu}{\tau_1}\tau_1 = -\ln\mu$, $\mathcal{G}(t^{\star}) = \mu\delta\overline{v}$, it can be deduced from (23) that

$$\mathcal{G}(t^{\star\star}) < \mathrm{e}^{-\frac{\ln\mu}{\tau_1}(t^{\star\star}-t^{\star})}\mathcal{G}(t^{\star}) < \mathrm{e}^{-\ln\mu}\mu\delta\overline{v} = \frac{1}{\mu}\mu\delta\overline{v} = \delta\overline{v} = \mathcal{G}(t^{\star\star}), \tag{24}$$

which is a contradiction. As a result of the above discussion, $\mathcal{G}(t) < \delta\overline{v}$ holds for $t \in (t_0, t_1)$.

Step 3. In respect of time interval $t \in [t_k, t_{k+1})$.

Now presume that for $t \in [t_0 - \tau, t_k)$, $k \in \mathbb{N}$, the inequality $\mathcal{G}(t) < \delta\overline{v}$ holds. Coherently, for any $t \in [t_k, t_{k+1}), k \in \mathbb{N}$, the inequality $\mathcal{G}(t) < \delta\overline{v}$ will be proved. The proof is divided into two steps.

Step 3.1. When $t = t_k$, $k \in \mathbb{N}$, by (8), it can be acquired that

$$\mathcal{G}(t_k) = \mathrm{e}^{\alpha(t_k-t_0)}x^{\mathrm{T}}(t_k)v < \mu\delta\overline{v} < \delta\overline{v}.$$

Given that, $\mathcal{G}(t) < \delta\overline{v}$ holds in relation to $t = t_k$, $k \in \mathbb{N}$.

Step 3.2. When $t \in (t_k, t_{k+1})$, the inequality $\mathcal{G}(t) < \delta\overline{v}$ will be carried out via the contradiction method.

Naturally assume that $\mathcal{G}(t) < \delta\overline{v}$ does not hold for $t \in (t_k, t_{k+1})$. According to $\mathcal{G}(t_k) < \mu\delta\overline{v}$, there exist $t^{\star\star} = \inf\{t \in (t_k, t_{k+1}); \mathcal{G}(t) \geqslant \delta\overline{v}\}$ and $t^{\star} = \sup\{t \in (t_k, t^{\star\star}); \mathcal{G}(t) \leqslant \mu\delta\overline{v}\}$.

Given that $\mathcal{G}(t)$ is continuous for $t \in (t_k, t_{k+1})$, $\mathcal{G}(t^{\star\star}) = \delta\overline{v}$ and $\mathcal{G}(t^{\star}) = \mu\delta\overline{v}$ hold. When $t \in [t^{\star}, t^{\star\star}]$, the following inequality is true:

$$\mu\mathcal{G}(t+\theta) \leqslant \mu\delta\overline{v} \leqslant \mathcal{G}(t), \quad \theta \in [-\tau, 0).$$

With reference to $t \in [t^{\star}, t^{\star\star}]$ and any $\omega > 0$, one can gain that

$$\begin{aligned} D^+\mathcal{G}(t) &= \mathrm{e}^{\alpha(t-t_0)}\big(\alpha\mathcal{V}(t) + D^+\mathcal{V}(t)\big) \\ &\leqslant -\frac{\ln\mu}{\tau_1}\mathcal{G}(t) + \frac{\ln\mu}{\tau_1}\mathcal{G}(t) + \mathrm{e}^{\alpha(t-t_0)}\big(\alpha\mathcal{V}(t) + D^+\mathcal{V}(t)\big) \\ &\quad + \omega\left(\frac{1}{\mu}\mathcal{G}(t) - \mathcal{G}\big(t-\tau(t)\big)\right). \end{aligned}$$

Similar to the process of (18)–(24), it will get that $\mathcal{G}(t^{\star\star}) < \mathcal{G}(t^{\star\star})$. This is a contradiction, that is, the hypothesis is not valid. So, when $t \in (t_k, t_{k+1})$, $k \in \mathbb{N}$, inequality $\mathcal{G}(t) < \delta\overline{v}$ is established. According to steps 3.1 and 3.2, it can be seen that for any $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$, the inequality $\mathcal{G}(t) < \delta\overline{v}$ holds.

As described above, combining steps 1–3, utilizing mathematical induction, it is clear that in relation to any $t \in [t_0 - \tau, \infty)$, inequality $\mathcal{G}(t) < \delta\overline{v}$ holds. Considering δ, we have

$$\mathcal{G}(t) < \frac{\overline{v}}{\mu}\|\bar{\phi}(\theta)\|_\tau, \quad t \geqslant t_0 - \tau. \tag{25}$$

Since $\mathrm{e}^{\alpha(t-t_0)}\underline{v}\|x(t)\|_1 \leqslant \mathcal{G}(t)$, from (25), it is equal to showing that

$$\|x(t)\|_1 < \mathrm{e}^{-\alpha(t-t_0)}\frac{\overline{v}}{\mu\underline{v}}\|\bar{\phi}(\theta)\|_\tau, \quad t \geqslant t_0 - \tau.$$

Overall, system (11) is asymptotically stable as for $\mu \in (0, 1)$.

Case II. $\mu \geqslant 1$.

For the purpose of verifying that system (11) is asymptotically stable, via setting $\pounds = \mu\|\bar{\phi}(\theta)\|_\tau$, $\theta \in [-\tau, t_0]$, the following inequality

$$\mathcal{G}(t) < \pounds\overline{v}, \quad t \in [t_0 - \tau, \infty),$$

will be proved. We will take advantage of the method about mathematical induction to prove the above inequality in three steps.

Step 1. Concerning time interval $t \in [t_0 - \tau, t_0]$.

$$\mathcal{G}(t) = \mathrm{e}^{\alpha(t-t_0)}\mathcal{V}(t) < \pounds\overline{v}, \quad \theta \in [-\tau, 0].$$

Therefore, when $t \in [t_0 - \tau, t_0]$, $\mathcal{G}(t) < \pounds\overline{v}$ holds.

Step 2. Concerning time interval $t \in (t_0, t_1)$.

As $t \in (t_0, t_1)$, $\mathcal{G}(t) < \pounds\overline{v}$ will be testified via the method of reduction to absurdity.

Presume that $\mathcal{G}(t) < \pounds\overline{v}$ does not hold. There exists $t \in (t_0, t_1)$ satisfying that $\mathcal{G}(t) \geqslant \pounds\overline{v}$. Notice that $\mathcal{G}(t_0) < \frac{1}{\mu}\pounds\overline{v}$. There exist $t^{\star\star} = \inf\{t \in (t_0, t_1); \mathcal{G}(t) \geqslant \pounds\overline{v}\}$ and $t^\star = \sup\{t \in (t_0, t^{\star\star}); \mathcal{G}(t) \leqslant \frac{1}{\mu}\pounds\overline{v}\}$. As a result, $\mathcal{G}(t^{\star\star}) = \pounds\overline{v}$ and $\mathcal{G}(t^\star) = \frac{1}{\mu}\pounds\overline{v}$ hold. About any $t \in (t^\star, t^{\star\star})$, one can derive that

$$\mathcal{G}(t + \theta) \leqslant \pounds\overline{v} \leqslant \mu\mathcal{G}(t), \quad t \in [-\tau, 0]. \tag{26}$$

In relation to $t \in (t^\star, t^{\star\star})$ and any scalar $\omega > 0$, it can be deduced from (26) that

$$\begin{aligned} D^+\mathcal{G}(t) &= \mathrm{e}^{\alpha(t-t_0)}\big(\alpha\mathcal{V}(t) + D^+\mathcal{V}(t)\big) \\ &\leqslant -\frac{\ln\mu}{\tau_0}\mathcal{G}(t) + \frac{\ln\mu}{\tau_0}\mathcal{G}(t) + \mathrm{e}^{\alpha(t-t_0)}\big(\alpha\mathcal{V}(t) + D^+\mathcal{V}(t)\big) \\ &\quad + \omega\big(\mu\mathcal{G}(t) - \mathcal{G}\big(t - \tau(t)\big)\big). \end{aligned} \tag{27}$$

Given that $\mathcal{G}(t)$ and $\mathcal{V}(t)$ are continuous functions in the interval $[t_0, t_1)$, then substituting (19)–(21) into (27), one has

$$D^+\mathcal{G}(t) \leqslant -\frac{\ln\mu}{\tau_0}\mathcal{G}(t) + \mathrm{e}^{\alpha(t-t_0)}\begin{bmatrix} x^{\mathrm{T}}(t) & x^{\mathrm{T}}\big(t - \tau(t)\big)\end{bmatrix}\begin{bmatrix} \big(-\mathcal{A}^{\mathrm{T}} + (\omega\mu + \frac{\ln\mu}{\tau_0} - p + \alpha)I\big)v \\ (-\mathcal{A}_d^{\mathrm{T}} - \omega\mathrm{e}^{-\alpha\tau}I)v \end{bmatrix}. \tag{28}$$

Combining (14) and (15), one can get the following inequality:

$$D^+\mathcal{G}(t) < -\frac{\ln\mu}{\tau_0}\mathcal{G}(t), \quad t \in [t^\star, t^{\star\star}]. \tag{29}$$

Due to $-\frac{\ln\mu}{\tau_0} < 0$, from (29), it is clear that $D^+\mathcal{G}(t) < 0$, $t \in [t^\star, t^{\star\star}]$, which means that the inequality $\mathcal{G}(t^{\star\star}) < \mathcal{G}(t^\star)$ holds. But this contradicts $\mathcal{G}(t^{\star\star}) = \pounds\overline{v} > \frac{1}{\mu}\pounds\overline{v} = \mathcal{G}(t^\star)$. Therefore, the hypothesis is not valid, in other words, $\mathcal{G}(t) < \pounds\overline{v}$ holds for $t \in (t_0, t_1)$.

Step 3. Concerning time interval $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$.

Suppose that when $t \in [t_0 - \tau, t_k)$, $k \in \mathbb{N}$, the following inequality holds:

$$\mathcal{G}(t) < \pounds\overline{v}. \tag{30}$$

Then the inequality $\mathcal{G}(t) < \pounds\overline{v}$, $t \in [t_k, t_{k+1}), k \in \mathbb{N}$ will be proved. The proof steps are split into three steps.

Step 3.1. The estimation of $G(t)$ at t_k^-, $k \in \mathbb{N}$.

In this part, $\mathcal{G}(t_k^-) \leqslant \frac{1}{\mu}\pounds\overline{v}$, $k \in \mathbb{N}$ will be proved through the contradiction method.

As expected, we claim that the inequality $\mathcal{G}(t_k^-) \leqslant \frac{1}{\mu}\pounds\overline{v}$ holds. If not, suppose that $\mathcal{G}(t_k^-) > \frac{1}{\mu}\pounds\overline{v}$. The above hypothesis will be discussed in two cases as follows, namely, Case(a) and Case(b).

Case(a). $\mathcal{G}(t) > \frac{1}{\mu}\pounds\overline{v}$ holds for over $t \in [t_{k-1}, t_k)$, $k \in \mathbb{N}$.

It follows from (30) that

$$\mathcal{G}(t+\theta) < \pounds\overline{v} < \mu\mathcal{G}(t), \quad \theta \in [-\tau, 0],\ t \in [t_{k-1}, t_k). \tag{31}$$

In respect of $t \in [t_{k-1}, t_k)$ and any constant $\omega > 0$, it entails from (31) that

$$\begin{aligned} D^+\mathcal{G}(t) &= \mathrm{e}^{\alpha(t-t_0)}\big(\alpha\mathcal{V}(t) + D^+\mathcal{V}(t)\big) \\ &\leqslant -\frac{\ln\mu}{\tau_0}\mathcal{G}(t) + \frac{\ln\mu}{\tau_0}\mathcal{G}(t) + \mathrm{e}^{\alpha(t-t_0)}\big(\alpha\mathcal{V}(t) + D^+\mathcal{V}(t)\big) \\ &\quad + \omega\Big(\mu\mathcal{G}(t) - \mathcal{G}\big(t-\tau(t)\big)\Big). \end{aligned}$$

Similar to the process of (27)–(29), one can gain that

$$D^+\mathcal{G}(t) < -\frac{\ln\mu}{\tau_0}\mathcal{G}(t), \quad t \in [t_{k-1}, t_k). \tag{32}$$

Due to $-\frac{\ln\mu}{\tau_0}(t_k^- - t_{k-1}) < -\frac{\ln\mu}{\tau_0}\tau_0 = \ln\mu$ and $\mathcal{G}(t_{k-1}) < \pounds\overline{v}$, according to (32), one can check that

$$\mathcal{G}(t_k^-) < \mathrm{e}^{-\frac{\ln\mu}{\tau_0}(t_k^- - t_{k-1})}\mathcal{G}(t_{k-1}) < \frac{1}{\mu}\pounds\overline{v},$$

which is a contradiction. Thus, $\mathcal{G}(t_k^-) < \frac{1}{\mu}\pounds\overline{v}$ holds for Case(a).

Case(b). There exists $t \in [t_{k-1}, t_k)$ making that $\mathcal{G}(t) \leqslant \frac{1}{\mu}\pounds\overline{v}$.

Regulate $t^\star = \sup\{t \in [t_{k-1}, t_k); \mathcal{G}(t) \leqslant \frac{1}{\mu}\pounds\overline{v}\}$, and then $t^\star \in [t_{k-1}, t_k)$ and $\mathcal{G}(t^\star) = \frac{1}{\mu}\pounds\overline{v}$. Hence, for $t \in [t^\star, t_k)$, it can be deduced that

$$\begin{aligned} &\frac{1}{\mu}\pounds\overline{v} \leqslant \mathcal{G}(t) \leqslant \pounds\overline{v}, \quad t \in [t^\star, t_k), \\ &\mathcal{G}(t+\theta) < \pounds\overline{v} \leqslant \mu\mathcal{G}(t), \quad \theta \in [-\tau, 0],\ t \in [t^\star, t_k). \end{aligned} \tag{33}$$

About $t \in [t^\star, t_k)$ and any constant $\omega > 0$, from (33), it can be entailed that

$$\begin{aligned} D^+\mathcal{G}(t) &= \mathrm{e}^{\alpha(t-t_0)}\big(\alpha\mathcal{V}(t) + D^+\mathcal{V}(t)\big) \\ &\leqslant -\frac{\ln\mu}{\tau_0}\mathcal{G}(t) + \frac{\ln\mu}{\tau_0}\mathcal{G}(t) + \mathrm{e}^{\alpha(t-t_0)}\big(\alpha\mathcal{V}(t) + D^+\mathcal{V}(t)\big) \\ &\quad + \omega\Big(\mu\mathcal{G}(t) - \mathcal{G}\big(t-\tau(t)\big)\Big). \end{aligned}$$

Similar to the process of (27)–(29), it can acqurie that $D^+\mathcal{G}(t) < 0, t \in [t^\star, t_k)$. So $\mathcal{G}(t_k^-) < \mathcal{G}(t^\star) = \frac{1}{\mu}\pounds\overline{v}$. This is a contradiction. In a nutshell, $\mathcal{G}(t_k^-) \leqslant \frac{1}{\mu}\pounds\overline{v}$ is true as for Case(b).

Step 3.2. The estimation of $\mathcal{G}(t_k)$, $k \in \mathbb{N}$.

With the reference to $t = t_k, k \in \mathbb{N}$, by (8) and Lemma 3, considering $\mathcal{G}(t_k^-) \leqslant \frac{1}{\mu}\pounds\overline{v}$, one can get that

$$\mathcal{G}(t_k) \leqslant \mu\mathrm{e}^{\alpha(t_k-t_0)}x^{\mathrm{T}}(t_k^-)v < \pounds\overline{v}.$$

Step 3.3. For $t \in (t_k, t_{k+1})$, $k \in \mathbb{N}$, $\mathcal{G}(t) < \pounds\overline{v}$ is to be testified based on the method of reduction to absurdity.

As $t \in (t_k, t_{k+1})$, presume that $\mathcal{G}(t) < \pounds\overline{v}$ does not hold for $t \in (t_k, t_{k+1})$. So, there exists $t \in (t_k, t_{k+1})$ leading that $\mathcal{G}(t) \geqslant \pounds\overline{v}$. Let $t^{\star\star} = \inf\{t \in (t_k, t_{k+1}); \mathcal{G}(t) \geqslant \pounds\overline{v}\}, t^{\star\star} \in (t_k, t_{k+1})$. It can verify that $\mathcal{G}(t^{\star\star}) = \pounds\overline{v}$. If $\mathcal{G}(t) > \frac{1}{\mu}\pounds\overline{v}$, $t \in [t_k, t^{\star\star})$, set $t^\star = t_k$; if not, set $t^\star = \sup\{t \in (t_k, t^{\star\star}); \mathcal{G}(t) \leqslant \frac{1}{\mu}\pounds\overline{v}\}$. Undoubtedly, for $t \in [t^\star, t^{\star\star}]$, the following inequalities hold:

$$\begin{aligned}
&\frac{1}{\mu}\pounds\overline{v} \leqslant \mathcal{G}(t) \leqslant \pounds\overline{v}, \quad t \in [t^\star, t^{\star\star}),\\
&\mathcal{G}(t+\theta) \leqslant \pounds\overline{v} < \mu\mathcal{G}(t), \quad \theta \in [-\tau, 0],\ t \in [t^\star, t^{\star\star}).
\end{aligned} \tag{34}$$

About $t \in [t^\star, t^{\star\star})$ and any constant $\omega > 0$, it can be deduced from (34) that

$$\begin{aligned}
D^+\mathcal{G}(t) &= \mathrm{e}^{\alpha(t-t_0)}\big(\alpha\mathcal{V}(t) + D^+\mathcal{V}(t)\big)\\
&\leqslant -\frac{\ln\mu}{\tau_0}\mathcal{G}(t) + \frac{\ln\mu}{\tau_0}\mathcal{G}(t) + \mathrm{e}^{\alpha(t-t_0)}\big(\alpha\mathcal{V}(t) + D^+\mathcal{V}(t)\big)\\
&\quad + \omega\Big(\mu\mathcal{G}(t) - \mathcal{G}\big(t-\tau(t)\big)\Big).
\end{aligned}$$

Similar to the process of (27)–(29), one can derive that $D^+\mathcal{G}(t) < 0$, $t \in [t^\star, t^{\star\star})$. So $\pounds\overline{v} = \mathcal{G}(t^{\star\star}) < \mathcal{G}(t^\star) = \frac{1}{\mu}\pounds\overline{v}$. This leads to a contradiction. Consequently, $\mathcal{G}(t) < \pounds\overline{v}$ holds with reference to $t \in (t_k, t_{k+1})$.

By synthesizing steps 1–3, and utilizing mathematical induction, we acquire that $\mathcal{G}(t) < \pounds\overline{v}$ holds for $t \in [t_0 - \tau, \infty)$. Analogous to the proof about (26), the following results can be obtained:

$$\|x(t)\|_1 < \mathrm{e}^{-\alpha(t-t_0)}\frac{\mu\overline{v}}{\underline{v}}\|\bar{\phi}(\theta)\|_\tau, \quad t \geqslant t_0 - \tau.$$

Thus, the system (11) is asymptotically stable. In other words, the system (1) under $u(t) = 0$ is interval $(-p, \infty)$ stable. Simultaneously, the stable criterion (5), (7), (8) and (10) of system (12) can be inferred by means of the similar process, which can make the system (1) under $u(t) = 0$ interval $(-\infty, -q)$ stable.

Remark 5. When the impulses are disturbance impulses, Theorem 1 requires that there exists a minimum dwell-time τ_0 such that the impulse time sequence satisfies $\inf_k\{\mathcal{T}_k\} \geqslant \tau_0$, which guarantees that the impulses will not occur too frequently. When the impulses are stabilizing impulses, it is required that there exists a maximum dwell-time τ_1 such that the impulse time sequence satisfies $\sup_k\{\mathcal{T}_k\} \leqslant \tau_1$. This ensures the impulses occur frequently enough for the stabilizing effect.

Remark 6. As is known to all, most of the existing theories about the stability of IPSs, such as asymptotic stability, can only judge the steady-state performance of such systems, but cannot judge some dynamic performance, such as the convergence rate of the system. To tackle this limitation, we make a breakthrough improvement on these theories and gain a more precise stability criterion of IPSs with time-varying delay, that is, interval stability, which can accurately judge the convergence speed of the system.

On the basis of Theorem 1, the common stability conditions of IPSs can be acquired as follows.

Corollary 1. For any $\mathcal{T}_k \in [\tau_0, \tau_1]$, system (1) under $u(t) = 0$ is asymptotically stable, if there exist scalars $\mu > 0$, $\omega > 0$ as well as appropriate dimensioned vectors $v \in \mathbb{R}^n_+$, such that when $\mu \in (0, 1)$, the following inequalities hold:

$$\left(\mathcal{A}^{\mathrm{T}} + \left(\frac{\omega}{\mu} + \frac{\ln\mu}{\tau_1}\right)I\right)v \prec 0,$$

$$(\mathcal{A}_d^{\mathrm{T}} - \omega I)v \prec 0, \tag{35}$$

$$(\mathcal{H}^{\mathrm{T}} - \mu I)v \preceq 0, \tag{36}$$

or when $\mu \geqslant 1$, the following inequality and (35), (36) hold:

$$\left(\mathcal{A}^{\mathrm{T}} + \left(\omega\mu + \frac{\ln\mu}{\tau_0}\right)I\right)v \prec 0.$$

4 Interval stabilization

Based on the previous results, this part works out the interval stabilization problem of the system (1) via state-feedback.

The main purpose of this section is to hatch up a state-feedback controller for the system (1):

$$u(t) = \mathcal{K}x(t),$$

where $\mathcal{K} \in \mathbb{R}^{r\times n}$ is a gain matrix.

Consider

$$\begin{cases} \dot{x}(t) = (\mathcal{A} + \mathcal{B}\mathcal{K})x(t) + \mathcal{A}_d x(t-\tau(t)),\ t \neq t_k,\ k \in \mathbb{N}, \\ x(t_k) = \mathcal{H}x(t_k^-),\ t = t_k,\ k \in \mathbb{N}, \\ x(t_0+\theta) = \bar{\phi}(\theta),\ t_0 = 0,\ \theta \in [-\tau, t_0], \end{cases} \tag{37}$$

where $\mathcal{A} \in \mathcal{M}$, $\mathcal{A}_d \succeq 0$, $\mathcal{B} \in \mathbb{R}^{n\times r}$, $\mathcal{B} \succeq 0$, and $\mathcal{H} \succeq 0$. $\tau(t)$, $x(t)$, $x(t_k^-)$, and $\bar{\phi}(\theta)$ are defined as in (1).

Lemma 4 ([20])**.** System (37) is positive iff $\mathcal{A} + \mathcal{B}\mathcal{K} \in \mathcal{M}$, $A_d \succeq 0$, and $H \succeq 0$.

Theorem 2. Considering the unstable system (1), for given scalars $\tau_1 \geqslant \tau_0 > 0$ and a stipulated vector $\tilde{v} \in \mathbb{R}^n_+$, if there exist scalars $\mu > 0$, $\omega > 0$, p, q, ℓ, vectors $v \in \mathbb{R}^n_+$, and $z \in \mathbb{R}^n$, such that when $\mu \in (0,1)$, the following inequalities hold:

$$\tilde{v}^{\mathrm{T}}\mathcal{B}^{\mathrm{T}}v\mathcal{A} + \mathcal{B}\tilde{v}z^{\mathrm{T}} + \ell I \succeq 0, \tag{38}$$

$$\left(-\mathcal{A}^{\mathrm{T}} + \left(\frac{\omega}{\mu} + \frac{\ln\mu}{\tau_1} - p\right) I\right) v - z \prec 0, \tag{39}$$

$$\left(\mathcal{A}^{\mathrm{T}} + \left(\frac{\omega}{\mu} + \frac{\ln\mu}{\tau_1} + q\right) I\right) v + z \prec 0, \tag{40}$$

$$(-\mathcal{A}_d^{\mathrm{T}} - \omega I)v \prec 0, \tag{41}$$

$$(\mathcal{A}_d^{\mathrm{T}} - \omega I)v \prec 0, \tag{42}$$

$$(\mathcal{H}^{\mathrm{T}} - \mu I)v \preceq 0, \tag{43}$$

or when $\mu \geqslant 1$, the following inequalities and (38), (41)–(43) hold:

$$\left(-\mathcal{A}^{\mathrm{T}} + \left(\omega\mu + \frac{\ln\mu}{\tau_0} - p\right) I\right) v - z \prec 0, \tag{44}$$

$$\left(\mathcal{A}^{\mathrm{T}} + \left(\omega\mu + \frac{\ln\mu}{\tau_0} + q\right) I\right) v + z \prec 0, \tag{45}$$

where

$$u(t) = \mathcal{K}x(t) = \frac{1}{\tilde{v}^{\mathrm{T}}\mathcal{B}^{\mathrm{T}}v}\tilde{v}z^{\mathrm{T}}x(t), \tag{46}$$

the system (37) is interval $(-p,-q)$ stable.

Proof. The proof is split into two steps.

(1) The positivity about system (37). Owing to $\tilde{v} \succ 0$, $\tilde{v} \in \mathbb{R}^r$, $\mathcal{B} \succeq 0$, $\mathcal{B} \in \mathbb{R}^{n\times r}$, and $v \in \mathbb{R}^n$, then it is acquired that $\tilde{v}^{\mathrm{T}}\mathcal{B}^{\mathrm{T}}v$ is a positive constant. It can be deduced from (38) that

$$\mathcal{A} + \frac{1}{\tilde{v}^{\mathrm{T}}\mathcal{B}^{\mathrm{T}}v}\mathcal{B}\tilde{v}z^{\mathrm{T}} + \frac{\ell}{\tilde{v}^{\mathrm{T}}\mathcal{B}^{\mathrm{T}}v}I \succeq 0. \tag{47}$$

In the light of Lemma 4, the system (37) is positive.

(2) The interval $(-p,-q)$ stability of system (37). Through utilizing the same demonstration as those in the stability of Theorem 1, the sufficient conditions of the interval stability criterion of system (37) are expressed in the following form, that is, when $\mu \in (0,1)$, the following inequalities hold:

$$\left(-\mathcal{A}^{\mathrm{T}} + \left(\frac{\omega}{\mu} + \frac{\ln\mu}{\tau_1} - p\right) I\right) v - \mathcal{K}^{\mathrm{T}}\mathcal{B}^{\mathrm{T}}v \prec 0, \tag{48}$$

$$\left(\mathcal{A}^{\mathrm{T}}+\left(\frac{\omega}{\mu}+\frac{\ln\mu}{\tau_1}+q\right)I\right)v+\mathcal{K}^{\mathrm{T}}\mathcal{B}^{\mathrm{T}}v\prec 0, \tag{49}$$

$$(-\mathcal{A}_d^{\mathrm{T}}-\omega I)v\prec 0, \tag{50}$$

$$(\mathcal{A}_d^{\mathrm{T}}-\omega I)v\prec 0, \tag{51}$$

$$(\mathcal{H}^{\mathrm{T}}-\mu I)v\preceq 0, \tag{52}$$

or when $\mu\geqslant 1$, the following inequalities and (50)–(52) hold:

$$\left(-\mathcal{A}^{\mathrm{T}}+\left(\omega\mu+\frac{\ln\mu}{\tau_0}-p\right)I\right)v-\mathcal{K}^{\mathrm{T}}\mathcal{B}^{\mathrm{T}}v\prec 0, \tag{53}$$

$$\left(\mathcal{A}^{\mathrm{T}}+\left(\omega\mu+\frac{\ln\mu}{\tau_0}+q\right)I\right)v+\mathcal{K}^{\mathrm{T}}\mathcal{B}^{\mathrm{T}}v\prec 0. \tag{54}$$

Considering the form of expression about $\mathcal{K}$ in (46), we can derive that inequalities (48)–(54) are equivalent to inequalities (39)–(45), which demonstrates that the system (37) is interval $(-p,-q)$ stable in case of the conditions of Theorem 2. The proof is completed.

Remark 7. Arbitrary $0\leqslant q<p$ cannot guarantee the existence of the solution of Theorem 1 or Theorem 2, but Theorem 1 or Theorem 2 can be solved by increasing the value of p or decreasing the value of q. Therefore, in practice, appropriate p and q can always be found to control the convergence rate of the system.

Remark 8. In this paper, sufficient conditions for Theorems 1 and 2 are given. In Theorems 1 and 2, the sufficient conditions are divided into two cases: $\mu\in(0,1)$ or $\mu\geqslant 1$. As long as the corresponding sufficient conditions in the cases of $\mu\in(0,1)$ or $\mu\geqslant 1$ are solved, the convergence rate of the system state can be controlled within a certain range.

Remark 9. An interval stabilization method is provided in Theorem 2, which can modulate the convergence speed of the system states more precisely than the existing stabilization methods. The primary thought of this method is analogous to the pole assignment method, which has not been extended to IPSs with time-varying delay as far as we are concerned.

Corollary 2. Considering the unstable system (1), for a vector $\tilde{v}\in\mathbb{R}_+^n$ and scalars $\tau_1\geqslant\tau_0>0$, if there exist scalars $\mu>0$, $\omega>0$ and ℓ, vectors $v\in\mathbb{R}_+^n$ and $z\in\mathbb{R}^n$, such that when $\mu\in(0,1)$, the following inequalities hold:

$$\tilde{v}^{\mathrm{T}}\mathcal{B}^{\mathrm{T}}v\mathcal{A}+\mathcal{B}\tilde{v}z^{\mathrm{T}}+\ell I\succeq 0, \tag{55}$$

$$\left(\mathcal{A}^{\mathrm{T}}+\left(\frac{\omega}{\mu}+\frac{\ln\mu}{\tau_1}\right)I\right)v+z\prec 0,$$

$$(\mathcal{A}_d^{\mathrm{T}}-\omega I)v\prec 0, \tag{56}$$

$$(\mathcal{H}^{\mathrm{T}}-\mu I)v\preceq 0, \tag{57}$$

or when $\mu\geqslant 1$, the following inequality and (55)–(57) hold:

$$\left(\mathcal{A}^{\mathrm{T}}+\left(\omega\mu+\frac{\ln\mu}{\tau_0}\right)I\right)v+z\prec 0,$$

where

$$u(t)=\mathcal{K}x(t)=\frac{1}{\tilde{v}^{\mathrm{T}}\mathcal{B}^{\mathrm{T}}v}\tilde{v}z^{\mathrm{T}}x(t),$$

the system (37) is asymptotically stable.

Theorem 2 can control the real parts of the eigenvalues for IPSs in a suitable interval location, so as to achieve the goal of controlling the system convergence rate. Consequently, the relevant algorithm is designed as Algorithm 1.

5 Illustrative examples

In this section, two examples will be put forward to examine the validity of the above conclusion.

Algorithm 1 System eigenvalues adjustment algorithm

Step 1. For the interval $M = (-p, -q)$, the corresponding controller can be solved according to Theorem 2, which can ensure that the real parts of eigenvalues about the system matrix for the closed-loop system are located in M. If the dynamic performance of the system meets the requirements, the algorithm is terminated. Otherwise, execute Step 2.
Step 2. If the system converges too slowly, increase the values of p and q until they satisfy the requirement. Otherwise, execute Step 3.
Step 3. If the system converges too rapidly, decrease the values of p and q until they satisfy the requirement. The algorithm is terminated.

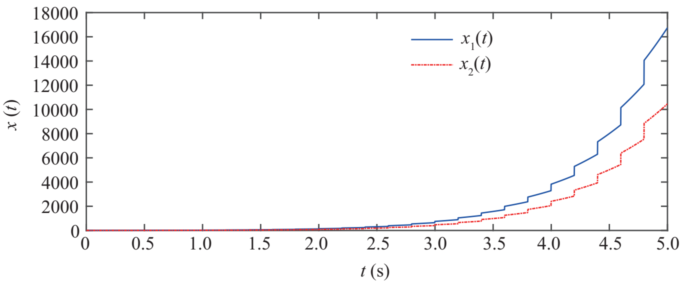

Figure 1 (Color online) The open-loop system state response.

5.1 Simulation for the convergence rate of IPSs under disturbance impulses

System (1) with $\mu = 1.2$ and $u(t) = \mathcal{K}x(t)$ is noted as below:

$$\mathcal{A} = \begin{bmatrix} 0.2 & 1 \\ 1 & -1 \end{bmatrix}, \quad \mathcal{H} = \begin{bmatrix} 1.1 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}, \quad \mathcal{A}_d = \begin{bmatrix} 0.01 & 0.09 \\ 0.2 & 0.01 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 1.6 \\ 0.8 \end{bmatrix},$$

$\tau_0 = 0.2$, $\tau(t) = 0.2 + 0.13\sin(t)$. Let the initial states be $x_i(t_0) = 5$ $(i = 1, 2)$. The open-loop system state response can be displayed in Figure 1.

As a common method of stabilization, the system gain matrix can be solved via Corollary 2 as follows:

$$\mathcal{K}_1 = \begin{bmatrix} -1.0302 & -0.5595 \end{bmatrix}.$$

The paths of the system states with state feedback control are shown in Figure 2(a). When $t > 8.7$ s, $x_i(t) < 0.01$ $(i = 1, 2)$.

Remark 10. Up to now, the usual stabilization methods can only ensure the asymptotic convergence of IPSs, not to exert influence on the convergence speed. Whereas, Theorem 2 can regulate the time-delay system convergence speed through regulating the interval $(-p, -q)$, as stated below.

According to Theorem 2, the system convergence rate can be regulated by conducting its generalized eigenvalues in a suitable interval location. Performing Yalmip toolbox when $\tilde{v} = 1$ and $\omega = 0.2$, one can set $(-p, -q) = (-5, -4)$. One can derive a feasible solution according to the inequalities (44)–(46) in Theorem 2, so that the corresponding gain matrix is

$$\mathcal{K}_2 = \begin{bmatrix} -1.3427 & -1.1845 \end{bmatrix},$$

which can ensure the faster convergence speed than above.

Remark 11. In this paper, based on the copositive Lyapunov function method, the sufficient condition of interval stability is expressed in the form of linear programming (LP). LP condition can be easily solved by using the Yalmip toolbox, which has the characteristics of small calculation, convenience, and good applicability.

The paths of the system states with state feedback control are plotted in Figure 2(b). When $t > 4.3$ s, $x_i(t) < 0.01$.

Without doubt, one can arrange $(-p, -q) = (-10, -9)$, so that the corresponding gain matrix is

$$\mathcal{K}_3 = \begin{bmatrix} -2.7802 & -2.0671 \end{bmatrix}.$$

The states of the system will converge faster. As is presented in Figure 2(c), when $t > 1.7$ s, $x_i(t) < 0.01$ $(i = 1, 2)$. Comparing Figures 2(a)–(c), we can make the system convergence speed even faster than above via picking the befitting interval $(-p, -q)$ and obtaining the relevant gain matrix.

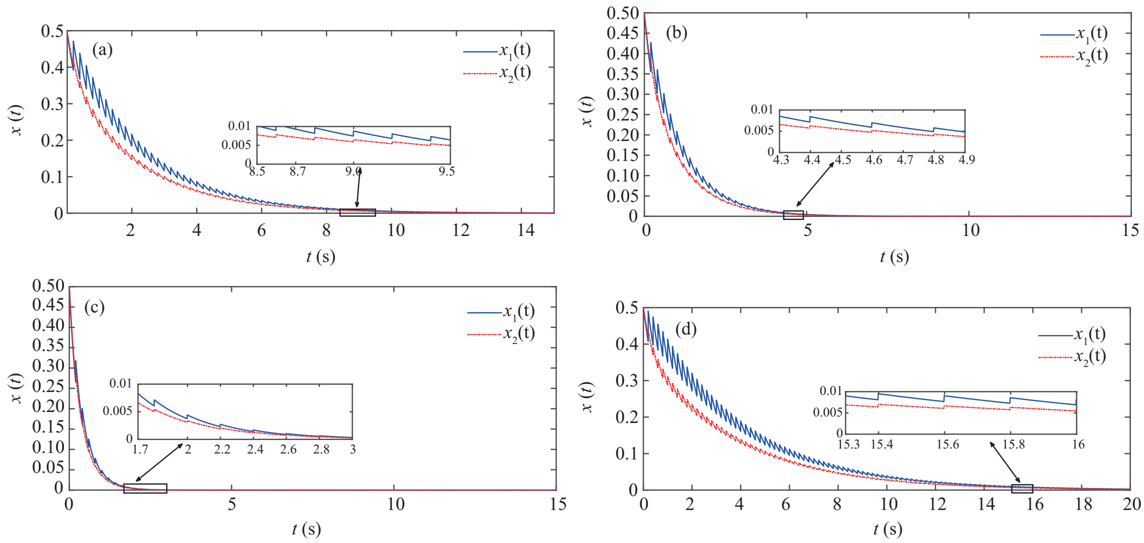

Figure 2 (Color online) The paths of the system states with state feedback control. (a) The system state's convergence rate trajectory under Corollary 2; (b) the trajectory in the interval $(-5,-4)$; (c) the trajectory in the interval $(-10,-9)$; (d) the trajectory in the interval $(-1.7,-1.3)$.

Meanwhile, one can reduce the convergence rates of the system by setting the interval $(-p,-q) = (-1.7,-1.3)$. Based on Theorem 2, the gain matrix can be derived as below:

$$\mathcal{K}_4 = \begin{bmatrix} -0.6053 & -0.3095 \end{bmatrix}.$$

Apparently, the system will converge more slowly as displayed in Figure 2(d), where $x_i(t) < 0.01$ $(i = 1,2)$ when $t > 15.3$ s.

Remark 12. From the above example, it can be concluded that, the method in this paper can ensure the stability of IPSs with time-varying delay and control the convergence speed of such systems via regulating the suitable interval $(-p,-q)$. Nevertheless, the common stabilization cannot do this [1–3, 20, 21]. In simple terms, the usual approaches for stabilization of IPSs with time-varying delay only satisfy $x_i(t) < 0.01$ $(i = 1,2)$ at $t = 8.7$ s (see Figure 2(a)), but the stabilization method in this paper can adjust the convergence rate of the system (shown in Figures 2(b)–(d)).

5.2 Simulation for the convergence rate of IPSs under stabilizing impulses

System (1) with $\mu = 0.5$ and $u(t) = \mathcal{K}x(t)$ is displayed as

$$\mathcal{A} = \begin{bmatrix} 1 & 2.4 \\ 2 & 1 \end{bmatrix}, \quad \mathcal{H} = \begin{bmatrix} 0.15 & 0.12 \\ 0.5 & 0.3 \end{bmatrix}, \quad \mathcal{A}_d = \begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0.1 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 1.2 \\ 1.6 \end{bmatrix},$$

$\tau_1 = 0.6$, $\tau(t) = 0.2 + 0.1\sin(t)$. Let the initial states be $x_i(t_0) = 5$ $(i = 1,2)$. The open-loop system state response can be delineated in Figure 3.

As a usual stabilization approach, the system gain matrix can be addressed through Corollary 2 as below:

$$\mathcal{K}_1 = \begin{bmatrix} -1.2071 & -1.3724 \end{bmatrix}.$$

The paths of the system states with state feedback control can be presented in Figure 4(a). When $t > 1.8$ s, $x_i(t) < 0.01$ $(i = 1,2)$.

Applying the Yalmip toolbox when $\tilde{v} = 1$ and $\omega = 0.4$, let $(-p,-q) = (-9,-8)$. We can get the corresponding gain matrix is

$$\mathcal{K}_2 = \begin{bmatrix} -7.9052 & -8.6194 \end{bmatrix},$$

which can make the system convergence rate faster than above.

The paths of the system states with state feedback control are exhibited in Figure 4(b). When $t > 0.6$ s, $x_i(t) < 0.01 (i = 1,2)$.

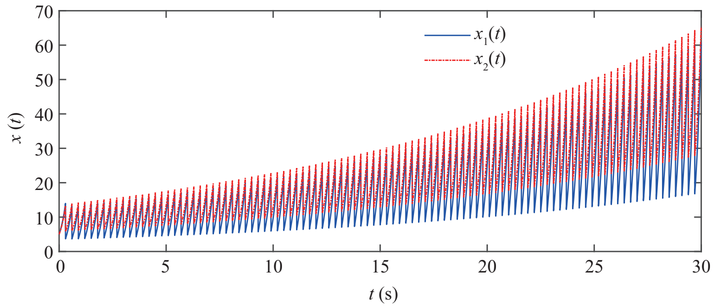

Figure 3 (Color online) The open-loop system state response.

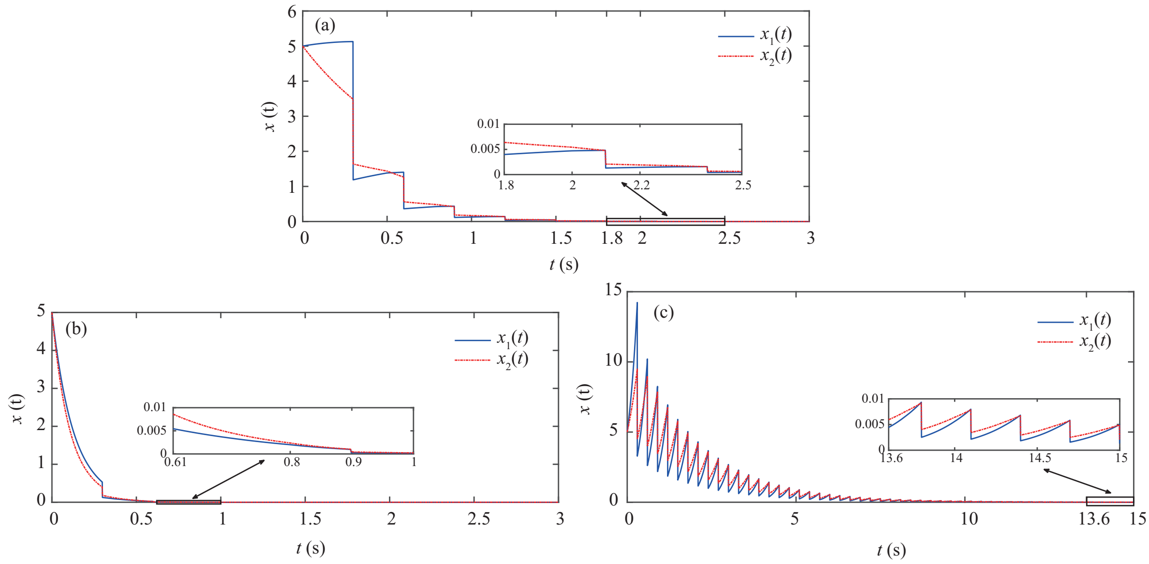

Figure 4 (Color online) The paths of the system states with state feedback control. (a) The system state's convergence rate trajectory under Corollary 2; (b) the trajectory in the interval $(-9,-8)$; (c) the trajectory in the interval $(-0.5,-0.3)$.

Meanwhile, we can reduce the convergence rates of the system by regulating the interval $(-p,-q) = (-0.5,-0.3)$. Based on Theorem 2, the related gain matrix can be obtained as follows:

$$\mathcal{K}_3 = \begin{bmatrix} -0.1849 & -0.2151 \end{bmatrix}.$$

The system states will converge slower as drawn in Figure 4(c), where $x_i(t) < 0.01$ $(i = 1, 2)$ when $t > 13.6$ s.

6 Conclusion

This paper establishes a new interval stability criterion for IPSs with time-varying delay that has not been investigated so far. The interval stabilization controller is designed to regulate the system's convergence speed and ensure stability. We discovered that the system stability improvement made the new criterion outperform the current approaches in actual system control. Combined with the interval stabilization approach, the system eigenvalues adjustment algorithm is designed to accurately control the target system's convergence rate. Finally, two numerical examples are presented to demonstrate the theoretical analysis validity.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant Nos. 61973148, 62173174), Natural Science Foundation of Shandong Province for Key Projects (Grant No. ZR2020KA010), Support Plan for Outstanding Youth Innovation Team in Shandong Higher Education Institutions (Grant No. 2019KJI010), and Graduate Education High-Quality Curriculum Construction Project for Shandong Province (Grant No. SDYKC20185).

References

1 Zhang J S, Wang Y W, Xiao J W, et al. Stability analysis of impulsive positive systems. IFAC Proc Vol, 2014, 47: 5987–5991

2 Wang Y W, Zhang J S, Liu M. Exponential stability of impulsive positive systems with mixed time-varying delays. IET Control Theor Appl, 2014, 8: 1537–1542
3 Briat C. Dwell-time stability and stabilization conditions for linear positive impulsive and switched systems. Nonlinear Anal-Hybrid Syst, 2017, 24: 198–226
4 Li S, Xiang Z R, Zhang J F. Exponential stability analysis for singular switched positive systems under dwell-time constraints. J Franklin Inst, 2020, 357: 13834–13871
5 Chang Z B, Xing X Y, Liu S Y, et al. Spatiotemporal dynamics for an impulsive eco-epidemiological system driven by canine distemper virus. Appl Math Comput, 2021, 402: 126135
6 Rivadeneira P S, Ferramosca A, Gonzalez A H. Control strategies for nonzero set-point regulation of linear impulsive systems. IEEE Trans Automat Contr, 2018, 63: 2994–3001
7 Liu B, Zhang Y, Chen L. The dynamical behaviors of a Lotka-Volterra predator-prey model concerning integrated pest management. Nonlinear Anal-Real World Appl, 2005, 6: 227–243
8 Tang S, Cheke R A. State-dependent impulsive models of integrated pest management (IPM) strategies and their dynamic consequences. J Math Biol, 2005, 50: 257–292
9 Zhang J F, Liu L Y, Li S, et al. Event-triggered L1-gain control of nonlinear positive switched systems. J Syst Sci Complex, 2021, 34: 873–898
10 Li S, Xiang Z R. Positivity, exponential stability and disturbance attenuation performance for singular switched positive systems with time-varying distributed delays. Appl Math Comput, 2020, 372: 124981
11 Lin M L, Sun Z D. Performance analysis of switched linear systems under arbitrary switching via generalized coordinate transformations. Sci China Inf Sci, 2019, 62: 012203
12 Zhang J F, Han Z Z, Zhu F, et al. Stability and stabilization of positive switched systems with mode-dependent average dwell time. Nonlinear Anal-Hybrid Syst, 2013, 9: 42–55
13 Li S, Xiang Z R, Zhang J F. Dwell-time conditions for exponential stability and standard L1-gain performance of discrete-time singular switched positive systems with time-varying delays. Nonlinear Anal-Hybrid Syst, 2020, 38: 100939
14 Zhang J F, Han Z Z, Wu H. Robust finite-time stability and stabilisation of switched positive systems. IET Control Theor Appl, 2014, 8: 67–75
15 Gao S S, You X, Jia X C, et al. A new stabilizing method for linear aperiodic sampled-data systems with time delay inputs and uncertainties. Sci China Inf Sci, 2020, 63: 149203
16 Li S, Xiang Z R, Zhang J F. Robust stability and stabilization conditions for uncertain switched positive systems under mode-dependent dwell-time constraints. Int J Robust Nonlinear Control, 2021, 31: 8569–8604
17 Zhu B L, Zhang J, Suo M L, et al. Robust stability analysis and controller synthesis for uncertain impulsive positive systems under L1-gain performance. ISA Trans, 2019, 93: 55–69
18 Hu M J, Wang Y W, Xiao J W. On finite-time stability and stabilization of positive systems with impulses. Nonlinear Anal-Hybrid Syst, 2019, 31: 275–291
19 Park J H. Recent Advances in Control Problems of Dynamical Systems and Networks. Cham: Springer, 2021. 37–60
20 Hu M J, Xiao J W, Xiao R B, et al. Impulsive effects on the stability and stabilization of positive systems with delays. J Franklin Inst, 2017, 354: 4034–4054
21 Zhu B L, Ma J, Zhang Z P, et al. Robust stability analysis and stabilisation of uncertain impulsive positive systems with time delay. Int J Syst Sci, 2018, 49: 2940–2956
22 Zhang W H, Xie L H. Interval stability and stabilization of linear stochastic systems. IEEE Trans Automat Contr, 2009, 54: 810–815
23 Zhang H S, Xia J W, Park J H, et al. Interval stability and interval stabilization of linear stochastic systems with time-varying delay. Int J Robust Nonlinear Control, 2021, 31: 2334–2347
24 Zhang H S, Xia J W, Shen H, et al. Pth moment regional stability/stabilization and generalized pole assignment of linear stochastic systems: based on the generalized $\mathcal{H}$-representation method. Int J Robust Nonlinear Control, 2020, 30: 3234–3249
25 Zhang H S, Xia J W, Zhang Y N, et al. pth moment $\mathcal{D}$-stability/stabilization of linear discrete-time stochastic systems. Sci China Inf Sci, 2020, 65: 139202
26 Kaczorek T. Positive 1D and 2D System. London: Springer, 2002
27 Zhou J, Wu Q J. Exponential stability of impulsive delayed linear differential equations. IEEE Trans Circ Syst II, 2009, 56: 744–748
28 Liu Y B, Feng W Z. Razumikhin-Lyapunov functional method for the stability of impulsive switched systems with time delay. Math Comput Model, 2009, 49: 249–264
29 Horn R A, Johnson C R. Matrix Analysis. Cambridge: Cambridge University Press, 2012. 505–507