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# Fully actuated system approaches for continuous-time delay systems: part 1. Systems with state delays only

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**Abstract** In this paper, the fully actuated system (FAS) approaches for continuous-time systems with time-varying state delays are proposed. Two types of continuous-time high-order FAS models with time delays: single-order time-delay FAS models and multi-order time-delay FASs, are proposed. Particularly, the type of sub-FASs that do not completely but partially satisfy the full actuation is investigated, and the sets of feasible points are defined. When the system states are constrained to the feasible set, a controller can be easily constructed for the sub-FAS such that the closed-loop system is a constant linear system with an arbitrarily assignable eigenstructure. In addition, it is demonstrated that the feasibility constraint can be transformed into a constraint on the initial values of the system, which vanishes when the system is a (global) FAS. Based on the unique control characteristic of the type of FAS models, the concepts of controllability and stabilizability of general dynamical time-delay systems are also proposed. The effect of the proposed theories is illustrated with examples.

**Keywords** time-delay systems, fully actuated systems, controller designs, controllability, stabilizability

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## 1 Introduction

We begin with a brief overview of the development of time-delay systems.

### 1.1 Time-delay systems

Time-delay systems have attracted much attention in the control community and have been developing extremely fast over the past several decades since they are ubiquitous in many physical systems and practical applications, such as chemical process control [1], combustion engines [2], road traffic systems [3], communication networks [4], and fluid dynamics [5].

The history of time-delay systems control can be traced back to the 18th century when Euler and Bernoulli contributed some pioneering work [6,7] to its development. Even though time-delay systems are very complex and difficult, significant advances have been made in the stability analysis and stabilization of delay systems. It is recalled that a major burst of activities occurred throughout the 1970s which was devoted to systems over rings, finite spectrum assignment, and early efforts in control of distributed parameter systems [8]. In addition, the introduction of linear matrix inequalities has led to another resurgence in the study on time-delay systems [6]. Recently, both stability analysis and the design of stabilizing control laws for time-delay systems have become highly active research areas. As for the stability analysis, more relaxed negativity conditions of the time derivative of Razumikhin functions and Krasovskii functionals were derived [9–11]. As for the design of stabilizing control laws, various controllers have been designed to deal with different types of systems, such as feed forward systems [12–18], multiple integrator type systems [19–23], and systems in a lower-triangular form [24–28].

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Even though a burst of research activities has occurred and fruitful results have been obtained, many fundamental problems in the control of time-delay systems remain unsolved. The challenge of the time-delay systems comes across one or all of the following several aspects. For example, time-delay systems are infinite-dimensional, the characteristic equation is not polynomial, stability analysis requires Krasovskii functionals rather than Lyapunov functions, etc. [8, 29]. Generally, stability analysis methods can be divided into two categories: the characteristic equation-based method and the Lyapunov-Krasovskii functional approach. However, the characteristic equation-based method suffers from a difficulty in computing exact zeros of the characteristic equations, and only sufficient conditions can be obtained for the Lyapunov-Krasovskii functional approach in most cases [30]. Similarly, the controller design can also be roughly categorized as finite-dimensional or predictor-based. Typically, the former controller is incapable of stabilizing systems with arbitrarily large delays, and the latter suffers some implementation problems due to its infinite-dimensional feedback.

Regarding research on time-delay systems, the following two aspects can be observed:

(1) Most control design approaches rely on the stability analysis results. Since most of the stability results available today are only sufficient ones, many of these design approaches are severely constrained by the conservatism of the employed stability results [31–34]. This has been a major challenge with the state-space approach ever since its development.

(2) Time-varying delays prove to be a very tough and thorny problem. Up to date, discussions on time-varying delays are mostly restricted in the case of constant linear time-delay systems [35–37]. Even today, the analysis and control of a time-varying linear system with time-varying delays remains an unresolved problem, let alone that of a nonlinear system with time-varying delays, which faces a series of insurmountable challenges in the state-space context.

### 1.2 Fully actuated system approaches

In contrast to the above mentioned state-space approaches, this paper introduces the fully actuated system approaches initially proposed by two series of studies (see [38–40] and [41–50]) for continuous-time systems with state delays.

This physical world is primarily governed by a set of physical laws, such as Newton's Law, theorems of linear and angular momentum, Lagrangian equation, and Kirchhoff's law of voltage or current. When modeling with such laws, numerous basic second-order systems are established, which are often fully actuated in the sense that a control actuator is fixed at each degree of freedom. For such a system, a controller can be easily designed to cancel the nonlinearities in the system, resulting in a constant linear closed-loop system [38, 41, 47]. Motivated by such an observation, a fully actuated system approach is recently proposed for nonlinear control system analysis and design. Specifically, it was discovered that the physical concept of full actuation can be mathematically generalized to describe a variety of control systems and that most under actuated systems can be converted to fully actuated ones. Particularly, it has been demonstrated that linear controllable systems [39, 50], feedback linearizable systems [38, 50], strict-feedback systems [38, 42, 50], and a more general type of nonlinear systems [41] can all be converted equivalently into high-order fully actuated systems (FASs). Once a dynamical system is expressed in an FAS model, the various design problems can be easily solved [43–49].

This paper aims to generalize the FAS approach to continuous-time systems with time delays.

For physical systems with certain delay mechanisms, modeling using physical laws usually yields second-order FASs with time delays. Many time-delay systems in a state-space form can be converted equivalently into FASs with time delays (see [38, 42, 50]). In the discrete-time case, it has been shown in [51, 52] that a controllable discrete-time linear system with time delays can always be converted equivalently into a discrete-time FAS with time delays, and that a discrete-time strict-feedback system with time delays can also be converted into a discrete-time FAS with time delays. Now in this paper, it is also demonstrated that a continuous-time strict-feedback system with time delays can be converted into a continuous-time FAS with time delays. With such a motivation, two general continuous-time nonlinear FASs with time delays are proposed: one for single-order affine and non-affine FASs with time-varying state delays and the other for multi-order affine and non-affine FASs with time-varying state delays.

The (global) FASs with time delays have the feature that the full-actuation condition is met in the whole state space. Furthermore, sub-FASs, or systems that do not satisfy the full-actuation condition in the complete state space but only partially in a feasible set of the whole state space, are also particularly studied. Herein, two types of continuous-time sub-FAS models with time-varying delays are proposed:

single-order affine and non-affine sub-FASs and multi-order affine and non-affine sub-FASs. Eventually, the concepts of the sets of feasible points and the sets of singular points for these sub-FASs are proposed. As in the global FAS case, controllers are also designed with sub-full-actuation features that convert the closed-loop systems into constant linear ones with arbitrarily assignable eigenstructures. However, different from the global FAS case, the derived closed-loop linear systems are subject to some state constraints. In other words, the linear closed-loop systems are restricted to work only on the sets of feasible points. The state constraint clearly vanishes when the system is globally fully actuated, and it is further shown to be a constraint on only the system's initial values.

The control feature of FASs, in essence, fully represents the meaning of complete controllability. As a result of this observation, complete controllability of a general nonlinear system is proposed in [39, 47]. In this paper, the concepts of complete controllability and stabilizability are further modified to suit the proposed more general FASs with time delays. Corresponding to the proposed sub-FASs case, sub-controllability and sub-stabilizability of general dynamical systems with time delays are defined.

We point out that a continuous-time delay system is essentially different from a discrete-time delay system. Taking the linear systems with constant time delays as an example, a discrete-time one of a finite dimension is after all a finite dimensional problem, while a continuous-time one of a finite dimension is no longer a finite dimensional but an infinite dimensional problem because there are an infinite number of system eigenvalues. Hence, in general, the continuous-time delay systems are more challenging.

In the paper, $I_n$ denotes the identity matrix, $\emptyset$ denotes the null set, and $\Omega\backslash\Theta$ represents the complement of the set $\Theta$ in set $\Omega$. Furthermore, $\det(A)$ and $\operatorname{adj}(A)$ denote the determinant and the adjoint matrix of a matrix $A$, respectively. For $x, x_i \in \mathbb{R}^m$, $A_i \in \mathbb{R}^{m\times m}$, $n_0, n_i \in \mathbb{N}$, $n_0 < n_i$, $i = 1, 2, \ldots, n$, as in [43], the following symbols are used in the paper:

$$x^{(n_1\sim n_2)} = \begin{bmatrix} x^{(n_1)} \\ x^{(n_1+1)} \\ \vdots \\ x^{(n_2)} \end{bmatrix}, \ n_1 \leqslant n_2,$$

$$x_{i\sim j}^{(n_1\sim n_2)} = \begin{bmatrix} x_i^{(n_1\sim n_2)} \\ x_{i+1}^{(n_1\sim n_2)} \\ \vdots \\ x_j^{(n_1\sim n_2)} \end{bmatrix}, \ i \leqslant j, \ n_1 \leqslant n_2,$$

$$x_k^{(n_k)}|_{k=i\sim j} = \begin{bmatrix} x_i^{(n_i)} \\ x_{i+1}^{(n_{i+1})} \\ \vdots \\ x_j^{(n_j)} \end{bmatrix}, \ i \leqslant j,$$

$$x_k^{(n_0\sim n_k)}|_{k=i\sim j} = \begin{bmatrix} x_i^{(n_0\sim n_i)} \\ x_{i+1}^{(n_0\sim n_{i+1})} \\ \vdots \\ x_j^{(n_0\sim n_j)} \end{bmatrix}, \ i \leqslant j,$$

and

$$A_{0\sim n-1} = \begin{bmatrix} A_0 & A_1 & \cdots & A_{n-1} \end{bmatrix}.$$

The paper is organized as follows. In Section 2, the time-delay strict-feedback system is introduced as a motivation. In Sections 3 and 4, the continuous-time delay FASs and sub-FASs are respectively proposed. Controllers for the proposed time-delay sub-FASs are provided in Section 5. A state constrained control problem arising from the control of sub-FASs is further solved in Section 6, and a further generalization of FAS model is introduced in Section 7, followed by a brief concluding remark in Section 8. Appendix A presents a proof of a preliminary result.

## 2 Time-delay strict-feedback systems

As a motivation, let us consider the following two-stage strict-feedback system with time delays:

$$\begin{cases} \dot{x}_1(t) = \alpha(x_1(t-\tau_j(t))|_{j=1\sim\zeta}, t) + B_1(x_1(t-\sigma_j(t))|_{j=1\sim\gamma}, t)x_2(t), \\ \dot{x}_2(t) = \beta(x_k(t-\tau_j(t))|_{j=1\sim\zeta,k=1,2}, t) + B_2(x_k(t-\sigma_j(t))|_{j=1\sim\gamma,k=1,2}, t)u(t), \end{cases} \tag{1}$$

where $x_i \in \mathbb{R}^r, i = 1, 2$, are the state vectors, $u \in \mathbb{R}^r$ is the control vector, $\zeta$ and $\gamma$ are two nonnegative intergers. By our notations,

$$x_i(t-\tau_j(t))|_{j=1\sim\zeta} = \begin{bmatrix} x_i(t-\tau_1(t)) \\ x_i(t-\tau_2(t)) \\ \vdots \\ x_i(t-\tau_\zeta(t)) \end{bmatrix}, \ i = 1, 2, \tag{2}$$

$$x_i(t-\sigma_j(t))|_{j=1\sim\gamma} = \begin{bmatrix} x_i(t-\sigma_1(t)) \\ x_i(t-\sigma_2(t)) \\ \vdots \\ x_i(t-\sigma_\gamma(t)) \end{bmatrix}, \ i = 1, 2, \tag{3}$$

and

$$x_k(t-\tau_j(t))|_{j=1\sim\zeta,k=1,2} = \begin{bmatrix} x_1(t-\tau_j(t))|_{j=1\sim\zeta} \\ x_2(t-\tau_j(t))|_{j=1\sim\zeta} \end{bmatrix},$$

$$x_k(t-\sigma_j(t))|_{j=1\sim\gamma,k=1,2} = \begin{bmatrix} x_1(t-\sigma_j(t))|_{j=1\sim\gamma} \\ x_2(t-\sigma_j(t))|_{j=1\sim\gamma} \end{bmatrix},$$

where $\tau_j(t), \ j = 1, 2, \ldots, \zeta$ and $\sigma_j(t), \ j = 1, 2, \ldots, \gamma$ are two sets of non-negative scalar functions. Furthermore, $\alpha(\cdot, t), \beta(\cdot, t) \in \mathbb{R}^r$ are two nonlinear functions, and $B_1(\cdot, t) \in \mathbb{R}^{r\times r}$ and $B_2(\cdot, t) \in \mathbb{R}^{r\times r}$ are two matrix functions satisfying the following full-actuation assumptions.

**Assumption A.** There exists a non-empty set $\Omega_1 \subset \mathbb{R}^{\gamma r}$ such that $\det B_1(X, t) \neq 0$ or $\infty$, for all $X \in \Omega_1$ and $t \geqslant 0$.

**Assumption B.** There exists a non-empty set $\Omega_2 \subset \mathbb{R}^{2\gamma r}$ such that $\det B_2(Y, t) \neq 0$ or $\infty$, for all $Y \in \Omega_2$ and $t \geqslant 0$.

**Proposition 1.** Let Assumptions A and B hold, then the state-space system (1) is equivalent to the following second-order system:

$$\ddot{x}_1(t) = f(t) + B\left(x_k(t-\sigma_j(t))|_{j=1\sim\gamma,k=1,2}, t\right)u(t), \tag{4}$$

where

$$\begin{aligned} &B\left(x_k(t-\sigma_j(t))|_{j=1\sim\gamma,k=1,2}, t\right) \\ &\quad = B_1\left(x_1(t-\sigma_j(t))|_{j=1\sim\gamma}, t\right) B_2\left(x_k(t-\sigma_j(t))|_{j=1\sim\gamma,k=1,2}, t\right), \end{aligned} \tag{5}$$

and

$$\begin{aligned} f(t) \triangleq\ & \dot{\alpha}\left(x_1(t-\tau_j(t))|_{j=1\sim\zeta}, t\right) + \dot{B}_1\left(x_1(t-\sigma_j(t))|_{j=1\sim\gamma}, t\right)x_2(t) \\ & + B_1\left(x_1(t-\sigma_j(t))|_{j=1\sim\gamma}, t\right)\beta\left(x_k(t-\tau_j(t))|_{j=1\sim\zeta,k=1,2}, t\right), \end{aligned} \tag{6}$$

with the $x_2$ related terms determined by the following relation:

$$x_2(t) = B_1^{-1}\left(x_1(t-\sigma_j(t))|_{j=1\sim\gamma}, t\right)\left[\dot{x}_1(t) - \alpha\left(x_1(t-\tau_j(t))|_{j=1\sim\zeta}, t\right)\right]. \tag{7}$$

*Proof.* Due to Assumption A, we can obtain (7) from the first equation in (1). Further, taking the derivatives of both sides of the first equation in (1), we have

$$\begin{aligned}\ddot{x}_1(t) = {} & \dot{\alpha}\left(x_1(t-\tau_j(t))|_{j=1\sim\zeta}, t\right) + \dot{B}_1\left(x_1(t-\sigma_j(t))|_{j=1\sim\gamma}, t\right)x_2(t) \\ & + B_1\left(x_1(t-\sigma_j(t))|_{j=1\sim\gamma}, t\right)\dot{x}_2(t).\end{aligned} \tag{8}$$

It follows from the second one in (1) that

$$\begin{aligned}\dot{x}_2(t) = {} & \beta\left(x_k(t-\tau_j(t))|_{j=1\sim\zeta,k=1,2}, t\right) \\ & + B_2\left(x_k(t-\sigma_j(t))|_{j=1\sim\gamma,k=1,2}, t\right)u(t).\end{aligned} \tag{9}$$

Substituting the above equation into (8) yields

$$\begin{aligned}\ddot{x}_1(t) = {} & \dot{\alpha}\left(x_1(t-\tau_j(t))|_{j=1\sim\zeta}, t\right) + \dot{B}_1\left(x_1(t-\sigma_j(t))|_{j=1\sim\gamma}, t\right)x_2(t) \\ & + B_1\left(x_1(t-\sigma_j(t))|_{j=1\sim\gamma}, t\right)\beta\left(x_k(t-\tau_j(t))|_{j=1\sim\zeta,k=1,2}, t\right) \\ & + B_1\left(x_1(t-\sigma_j(t))|_{j=1\sim\gamma}, t\right)B_2\left(x_k(t-\sigma_j(t))|_{j=1\sim\gamma,k=1,2}, t\right)u(t).\end{aligned} \tag{10}$$

This can clearly be written into the form of (4)–(6), which turns out to be a system with only the variable $x_1$ when the $x_2$ related terms therein are converted into $x_1$ related ones by using (7).

Let

$$\mathbb{F} = (\Omega_1 \times \mathbb{R}^{\gamma r}) \cap \Omega_2 \subset \mathbb{R}^{2\gamma r},$$

and then we clearly have

$$\det B(Y,t) \neq 0 \text{ or } \infty \text{ for all } Y \in \mathbb{F} \text{ and } t \geqslant 0. \tag{11}$$

When this set $\mathbb{F}$ is not empty, the above relation (11) represents an incomplete full-actuation condition, and in this case we call the system (4)–(6) a sub-FAS.

In the special case of

$$\Omega_1 = \mathbb{R}^{\gamma r} \text{ and } \Omega_2 = \mathbb{R}^{2\gamma r},$$

we have $\mathbb{F} = \mathbb{R}^{2\gamma r}$, and the above relation (11) turns into the following complete full-actuation condition:

$$\det B(Y,t) \neq 0 \text{ or } \infty \text{ for all } Y \in \mathbb{R}^{2\gamma r} \text{ and } t \geqslant 0. \tag{12}$$

In this case we call the system (4)–(6) a (global) FAS.

In fact, based on the form of a usual continuous-time strict-feedback system ([42]), we can further propose the following strict-feedback system with time delays:

$$\begin{cases}\dot{x}_1(t) = \alpha_1\left(x_1(t-\tau_{1j}(t))|_{j=1\sim\zeta_1}, t\right) + B_1\left(x_1(t-\sigma_{1j}(t))|_{j=1\sim\gamma_1}, t\right)x_2(t), \\ \dot{x}_2(t) = \alpha_2\left(x_{1\sim 2}(t-\tau_{2j}(t))|_{j=1\sim\zeta_2}, t\right) + B_2\left(x_{1\sim 2}(t-\sigma_{2j}(t))|_{j=1\sim\gamma_2}, t\right)x_3(t), \\ \quad\vdots \\ \dot{x}_{n-1}(t) = \alpha_{n-1}\left(x_{1\sim n-1}(t-\tau_{n-1,j}(t))|_{j=1\sim\zeta_{n-1}}, t\right) \\ \qquad\qquad + B_{n-1}\left(x_{1\sim n-1}(t-\sigma_{n-1,j}(t))|_{j=1\sim\gamma_{n-1}}, t\right)x_n(t), \\ \dot{x}_n(t) = \alpha_n\left(x_{1\sim n}(t-\tau_{nj}(t))|_{j=1\sim\zeta_n}, t\right) + B_n\left(x_{1\sim n}(t-\sigma_{nj}(t))|_{j=1\sim\gamma_n}, t\right)u(t),\end{cases} \tag{13}$$

where $x_i \in \mathbb{R}^r, i = 1,2,\ldots,n$ are the state vectors, $u \in \mathbb{R}^r$ is the control vector, $\tau_{ij}(t), j = 1,2,\ldots,\zeta_i$, and $\sigma_{ij}(t), j = 1,2,\ldots,\gamma_i$, $i = 1,2,\ldots,n$ are all nonnegative scalar functions representing the time delays in the system. Furthermore, $\alpha_i(\cdot,t), i = 1,2,\ldots,n$ are a set of nonlinear vector functions, and $B_i(\cdot,t), i = 1,2,\ldots,n$ are a set of matrix functions.

If the following full-actuation condition

$$\det B_i\left(x_{1\sim i}(t-\sigma_{ij}(t))|_{j=1\sim\gamma_i}, t\right) \neq 0 \text{ or } \infty,\ i = 1,2,\ldots,n \tag{14}$$

is met for all possible values of $x_{1\sim i}\left(t-\sigma_{ij}\left(t\right)\right)|_{j=1\sim\gamma_i}$, $i=1,2,\ldots,n$, and $t\geqslant 0$, then the general time-delay strict-feedback system can be converted into a high-order FAS. When the above condition (14) is not met for all, but part of, the possible values of $x_{1\sim i}\left(t-\sigma_{ij}\left(t\right)\right)|_{j=1\sim\gamma_i}$, $i=1,2,\ldots,n$, and $t\geqslant 0$, the general time-delay strict-feedback system can be still converted into a high-order sub-FAS. A detailed deduction will be given elsewhere.

In order to demonstrate the multi-layer case, let us give an example.

**Example 1.** Consider a strict feedback system in the form of (13) with state delays as follows:

$$\begin{cases} \dot{x}_1\left(t\right)=f\left(x_1\left(t-\tau_1\left(t\right)\right),t\right)+x_2\left(t\right), \\ \dot{x}_2\left(t\right)=x_1\left(t-\tau_2\left(t\right)\right)x_2\left(t-\tau_3\left(t\right)\right)+g\left(x_1\left(t-\sigma\left(t\right)\right),t\right)x_3\left(t\right), \\ \dot{x}_3\left(t\right)=u\left(t\right), \end{cases} \tag{15}$$

where $x_i, i=1,2,3$ are the state variables, and $u$ is the control variable, $\tau_i\left(t\right), i=1,2,3$ and $\sigma\left(t\right)$ are non-negative scalar functions representing the time delays in the system, $f\left(x_1\left(t-\tau_1\left(t\right)\right),t\right)$ and $g\left(x_1\left(t-\sigma\left(t\right)\right),t\right)$ are two scalar functions, and

$$g\left(x,t\right)\neq 0,\ \forall x\in\mathbb{R} \text{ and } t\geqslant 0. \tag{16}$$

Taking derivative of the first equation in (15) and substituting the second one into the result, we have

$$\ddot{x}_1\left(t\right)=\dot{f}\left(x_1\left(t-\tau_1\left(t\right)\right),t\right)+x_1\left(t-\tau_2\left(t\right)\right)x_2\left(t-\tau_3\left(t\right)\right)+g\left(x_1\left(t-\sigma\left(t\right)\right),t\right)x_3\left(t\right). \tag{17}$$

From the first equation in (15) we can obtain

$$x_2\left(t\right)=\dot{x}_1\left(t\right)-f\left(x_1\left(t-\tau_1\left(t\right)\right),t\right), \tag{18}$$

and substituting this into (17), we have

$$\ddot{x}_1\left(t\right)=f_1\left(x_1^{\lceil\tau_{1\sim 2}(t)\rceil},\dot{x}_1^{\lceil\tau_3(t)\rceil},t\right)+g\left(x_1\left(t-\sigma\left(t\right)\right),t\right)x_3\left(t\right), \tag{19}$$

where $x^{\lceil\tau\rceil}\triangleq x(t-\tau)$ and

$$\begin{aligned} f_1\left(x_1^{\lceil\tau_{1\sim 2}(t)\rceil},\dot{x}_1^{\lceil\tau_3(t)\rceil},t\right) =& \dot{f}\left(x_1\left(t-\tau_1\left(t\right)\right),t\right) \\ &+x_1\left(t-\tau_2\left(t\right)\right)\left[\dot{x}_1\left(t-\tau_3\left(t\right)\right)-f\left(x_1\left(t-\tau_1\left(t\right)-\tau_3\left(t\right)\right),t-\tau_3\left(t\right)\right)\right]. \end{aligned} \tag{20}$$

Next, taking derivative of (19) and substituting the third equation in (15) into the result yield

$$\dddot{x}_1\left(t\right)=\dot{f}_1\left(x_1^{\lceil\tau_{1\sim 2}(t)\rceil},\dot{x}_1^{\lceil\tau_3(t)\rceil},t\right)+\dot{g}\left(x_1\left(t-\sigma\left(t\right)\right),t\right)x_3\left(t\right)+g\left(x_1\left(t-\sigma\left(t\right)\right),t\right)u\left(t\right). \tag{21}$$

On the other side, under the condition that $g\left(x_1\left(t-\sigma\left(t\right)\right),t\right)$ is different from zero for all $t\geqslant 0$, we can obtain from (19)

$$x_3\left(t\right)=\frac{\ddot{x}_1\left(t\right)-f_1(x_1^{\lceil\tau_{1\sim 2}(t)\rceil},\dot{x}_1^{\lceil\tau_3(t)\rceil},t)}{g\left(x_1\left(t-\sigma\left(t\right)\right),t\right)}. \tag{22}$$

Substituting this into (21), produce the following FAS model for the system (15):

$$\dddot{x}_1\left(t\right)=\frac{f_2(x_1^{\lceil\tau_{1\sim 2}(t)\rceil},x_1^{\lceil\sigma(t)\rceil},\dot{x}_1^{\lceil\tau_3(t)\rceil},\ddot{x}_1\left(t\right),t)}{g\left(x_1\left(t-\sigma\left(t\right)\right),t\right)}+g\left(x_1\left(t-\sigma\left(t\right)\right),t\right)u\left(t\right), \tag{23}$$

where

$$\begin{aligned} f_2\left(x_1^{\lceil\tau_{1\sim 2}(t)\rceil},x_1^{\lceil\sigma(t)\rceil},\dot{x}_1^{\lceil\tau_3(t)\rceil},\ddot{x}_1\left(t\right),t\right) =& g\left(x_1\left(t-\sigma\left(t\right)\right),t\right)\dot{f}_1\left(x_1^{\lceil\tau_{1\sim 2}(t)\rceil},\dot{x}_1^{\lceil\tau_3(t)\rceil},t\right) \\ &+\dot{g}\left(x_1\left(t-\sigma\left(t\right)\right),t\right)\left[\ddot{x}_1\left(t\right)-f_1\left(x_1^{\lceil\tau_{1\sim 2}(t)\rceil},\dot{x}_1^{\lceil\tau_3(t)\rceil},t\right)\right]. \end{aligned} \tag{24}$$

Under the condition (16), the obtained system (23)-(24) is a third-order (global) FAS. When this condition is replaced by

$$g\left(x,t\right)\neq 0,\ \forall x\in\Omega \text{ and } t\geqslant 0, \tag{25}$$

where $\Omega$ is some non-empty interval, the obtained system (23)-(24) turns to be a third-order sub-FAS.

# 3 FAS models

## 3.1 Single-order FASs with state delays

Let us first start with the single-order affine FAS models with time delays.

### 3.1.1 *Single-order affine FAS models*

Motivated by the fact presented in Section 2, we can give the following single-order affine FAS with state delays:

$$x^{(n)}(t) = f\left(x^{(0\sim n-1)}(t-\tau_j(t))\,|_{j=1\sim\zeta}, t\right) + B\left(x^{(0\sim n-1)}(t-\sigma_j(t))\,|_{j=1\sim\gamma}, t\right)u(t), \tag{26}$$

where $x, u \in \mathbb{R}^r$ are the state vector and the control vector, respectively. Again, by convention,

$$x^{(0\sim n-1)}(t-\tau_j(t))\,|_{j=1\sim\zeta} = \begin{bmatrix} x^{(0\sim n-1)}(t-\tau_1(t)) \\ x^{(0\sim n-1)}(t-\tau_2(t)) \\ \vdots \\ x^{(0\sim n-1)}(t-\tau_\zeta(t)) \end{bmatrix}, \tag{27}$$

$$x^{(0\sim n-1)}(t-\sigma_j(t))\,|_{j=1\sim\gamma} = \begin{bmatrix} x^{(0\sim n-1)}(t-\sigma_1(t)) \\ x^{(0\sim n-1)}(t-\sigma_2(t)) \\ \vdots \\ x^{(0\sim n-1)}(t-\sigma_\gamma(t)) \end{bmatrix}, \tag{28}$$

with $\tau_j(t), j = 1, 2, \ldots, \zeta$ and $\sigma_j(t), j = 1, 2, \ldots, \gamma$ being two sets of non-negative scalar functions representing the delays in the system. Furthermore, $f(\cdot, t) \in \mathbb{R}^r$ and $B(\cdot, t) \in \mathbb{R}^{r\times r}$ are nonlinear functions. Noting that

$$x^{(0\sim n-1)}(t-\sigma_j(t))\,|_{j=1\sim\gamma} \in \mathbb{R}^{\gamma nr},$$

we require the function $B(\cdot, t)$ to satisfy the following full-actuation assumption.

**Assumption A1.** $\det B(X, t) \neq 0$ or $\infty$, $\forall X \in \mathbb{R}^{\gamma nr}$ and $t \geqslant 0$.

Recall that the system (23)-(24) obtained in Example 1 serves as an example for a single-order FAS when the condition (16) is met.

Under the above assumption, we can define a new control

$$\tilde{u}(t) = B\left(x^{(0\sim n-1)}(t-\sigma_j(t))\,|_{j=1\sim\gamma}, t\right)u(t), \tag{29}$$

and write the FAS (26) into the following standard form:

$$x^{(n)}(t) = f\left(x^{(0\sim n-1)}(t-\tau_j(t))\,|_{j=1\sim\zeta}, t\right) + \tilde{u}(t). \tag{30}$$

Particularly, a constant linear single-order FAS with state time delays is of the following form:

$$x^{(n)}(t) = \sum_{i=0}^{m} C_i x^{(0\sim n-1)}(t-\tau_i(t)) + Bu(t), \tag{31}$$

where $B$ is a constant square nonsingular matrix, and $C_i \in \mathbb{R}^{r\times nr}, i = 0, 1, \ldots, m$ are the coefficient matrices.

**Remark 1.** It is clearly seen from the above discussion that, in order to control the FAS (26), it suffices only to control the standard FAS (30). However, such a fact fails for a sub-FAS, in which the full actuation assumption, that is, Assumption A1, is not valid. In order to maintain the consistency in the treatment of both FASs and sub-FASs, in the following we still treat the original FAS (26) in Section 5 for controller design.

#### 3.1.2 *Single-order non-affine FAS models*

Parallel to the above single-order affine FAS (26), we can also define the following non-affine one (also, refer to Definition 2 in [39]):

$$x^{(n)}(t) = f\left(x^{(0\sim n-1)}(t-\tau_j(t))\,|_{j=1\sim\zeta}, t\right) + g\left(x^{(0\sim n-1)}(t-\sigma_j(t))\,|_{j=1\sim\gamma}, t, u(t)\right), \tag{32}$$

where $g(\cdot) \in \mathbb{R}^r$ is a nonlinear function, and the other variables are as stated before. Particularly, we call the system (32) a single-order non-affine (global) FAS if the function $g(\cdot)$ satisfies the following assumption.

**Assumption A2.** The following mapping

$$\tilde{u}(t) = g(X, t, u(t)) \tag{33}$$

forms a differential homeomorphism from $u$ to $\tilde{u}$ for all $X \in \mathbb{R}^{\gamma nr}$ and $t \geqslant 0$.

Due to Assumption A2, we can introduce the control vector transformation

$$\tilde{u}(t) = g\left(x^{(0\sim n-1)}(t-\sigma_j(t))\,|_{j=1\sim\gamma}, t, u(t)\right), \tag{34}$$

under which the single-order non-affine (global) FAS (32) can be equivalently represented into the standard form of (30).

### 3.2 Multi-order FASs with state delays

In this subsection, let us generalize the single-order time-delay FAS proposed in Section 2 into the case of multi-order ones.

#### 3.2.1 *Multi-order affine FAS models*

Stimulated by the multi-order FAS without delays ( [47–49]), we can give the following general form of a multi-order affine FAS model with time delays:

$$\begin{bmatrix} x_1^{(\mu_1)}(t) \\ x_2^{(\mu_2)}(t) \\ \vdots \\ x_\eta^{(\mu_\eta)}(t) \end{bmatrix} = \begin{bmatrix} f_1\left(x_k^{(0\sim\mu_k-1)}(t-\tau_j(t))\,|_{k=1\sim\eta,\ j=1\sim\zeta}, t\right) \\ f_2\left(x_k^{(0\sim\mu_k-1)}(t-\tau_j(t))\,|_{k=1\sim\eta,\ j=1\sim\zeta}, t\right) \\ \vdots \\ f_\eta\left(x_k^{(0\sim\mu_k-1)}(t-\tau_j(t))\,|_{k=1\sim\eta,\ j=1\sim\zeta}, t\right) \end{bmatrix} + B\left(x_k^{(0\sim\mu_k-1)}(t-\sigma_j(t))\,|_{k=1\sim\eta,\ j=1\sim\gamma}, t\right)u(t), \tag{35}$$

where $\mu_k, k = 1, 2, \ldots, \eta$ are a set of integers, $\tau_j(t), j = 1, 2, \ldots, \zeta$ and $\sigma_j(t), j = 1, 2, \ldots, \gamma$ are two sets of positive bounded scalar functions representing the time delays in the system; and $x_k \in \mathbb{R}^{r_k}, k = 1, 2, \ldots, \eta$ are a set of vectors of proper dimensions, with $r_k, k = 1, 2, \ldots, \eta$ being a set of distinct integers satisfying

$$r_1 + r_2 + \cdots + r_\eta = r. \tag{36}$$

Furthermore, $f_k(\cdot) \in \mathbb{R}^{r_k}, k = 1, 2, \ldots, \eta$ are a set of nonlinear vector functions, $B(\cdot) \in \mathbb{R}^{r\times r}$ is a matrix function, and by our notations,

$$x_k^{(0\sim\mu_k-1)}(t-\tau_j(t))\,|_{k=1\sim\eta} = \begin{bmatrix} x_1^{(0\sim\mu_1-1)}(t-\tau_j(t)) \\ x_2^{(0\sim\mu_2-1)}(t-\tau_j(t)) \\ \vdots \\ x_\eta^{(0\sim\mu_\eta-1)}(t-\tau_j(t)) \end{bmatrix},\ j = 1, 2, \ldots, \zeta, \tag{37}$$

and

$$x_k^{(0\sim\mu_k-1)}(t-\tau_j(t))\,|_{k=1\sim\eta,\ j=1\sim\zeta} = \begin{bmatrix} x_k^{(0\sim\mu_k-1)}(t-\tau_1(t))\,|_{k=1\sim\eta} \\ x_k^{(0\sim\mu_k-1)}(t-\tau_2(t))\,|_{k=1\sim\eta} \\ \vdots \\ x_k^{(0\sim\mu_k-1)}(t-\tau_\zeta(t))\,|_{k=1\sim\eta} \end{bmatrix}. \tag{38}$$

Let

$$\varkappa_0 = \sum_{k=1}^{\eta} \mu_k r_k. \tag{39}$$

Then, clearly,

$$x_k^{(0\sim\mu_k-1)}(t)\,|_{k=1\sim\eta} \in \mathbb{R}^{\varkappa_0},\ x_k^{(0\sim\mu_k-1)}(t-\tau_j(t))\,|_{k=1\sim\eta,\ j=1\sim\gamma} \in \mathbb{R}^{\gamma\varkappa_0}. \tag{40}$$

The above system (35) is called a (global) FAS with delays if the following assumption is met.
**Assumption A3.** $\det B(X,t) \neq 0$ or $\infty$, $\forall X \in \mathbb{R}^{\gamma\varkappa_0}$ and $t \geqslant 0$.

Let

$$f\left(x_k^{(0\sim\mu_k-1)}(t-\tau_j(t))\,|_{k=1\sim\eta,\ j=1\sim\zeta}, t\right) = \begin{bmatrix} f_1\left(x_k^{(0\sim\mu_k-1)}(t-\tau_j(t))\,|_{k=1\sim\eta,\ j=1\sim\zeta}, t\right) \\ f_2\left(x_k^{(0\sim\mu_k-1)}(t-\tau_j(t))\,|_{k=1\sim\eta,\ j=1\sim\zeta}, t\right) \\ \vdots \\ f_\eta\left(x_k^{(0\sim\mu_k-1)}(t-\tau_j(t))\,|_{k=1\sim\eta,\ j=1\sim\zeta}, t\right) \end{bmatrix}. \tag{41}$$

Then the FAS (35) can be compactly written as

$$\begin{aligned} x_k^{(\mu_k)}|_{k=1\sim\eta}(t) = {} & f\left(x_k^{(0\sim\mu_k-1)}(t-\tau_j(t))\,|_{k=1\sim\eta,\ j=1\sim\zeta}, t\right) \\ & + B\left(x_k^{(0\sim\mu_k-1)}(t-\sigma_j(t))\,|_{k=1\sim\eta,\ j=1\sim\gamma}, t\right) u(t). \end{aligned} \tag{42}$$

Obviously, in the case of $\eta = 1$, the above FAS model (35), or (42), reduces to the form of the single-order FAS (26).

Under Assumption A3, we can in fact introduce a new control vector

$$\tilde{u}(t) = B\left(x_k^{(0\sim\mu_k-1)}(t-\sigma_j(t))\,|_{k=1\sim\eta,\ j=1\sim\gamma}, t\right) u(t), \tag{43}$$

and now the (global) FAS (42) can be written in the following standard form:

$$x_k^{(\mu_k)}|_{k=1\sim\eta}(t) = f\left(x_k^{(0\sim\mu_k-1)}(t-\tau_j(t))\,|_{k=1\sim\eta,\ j=1\sim\zeta}, t\right) + \tilde{u}(t), \tag{44}$$

or

$$x_i^{(\mu_i)}(t) = f_i\left(x_k^{(0\sim\mu_k-1)}(t-\tau_j(t))\,|_{k=1\sim\eta,\ j=1\sim\zeta}, t\right) + \tilde{u}_i(t),\ i = 1, 2, \ldots, \eta, \tag{45}$$

where $\tilde{u}_i, i = 1, 2, \ldots, \eta$ are defined by

$$\tilde{u} = \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \vdots \\ \tilde{u}_\eta \end{bmatrix}, \quad \tilde{u}_i \in \mathbb{R}^{r_i}. \tag{46}$$

As a special case of the multi-order nonlinear time-delay FAS (42), a constant linear multi-order FAS with state delays is in the form of

$$x_k^{(\mu_k)}|_{k=1\sim\eta}(t) = \sum_{i=0}^{m} D_i x_k^{(0\sim\mu_k-1)}(t-\tau_i(t))\,|_{k=1\sim\eta} + Bu(t), \tag{47}$$

where $B$ is a constant square nonsingular matrix, and $D_i \in \mathbb{R}^{r\times\gamma\varkappa_0}, i = 0, 1, \ldots, m$ are the coefficient matrices.

For convenience, in the following of this paper we also use the following notation introduced in [51]:

$$x^{\lceil\tau_{1\sim\zeta}(t)\rceil} \triangleq x(t-\tau_j(t))\big|_{j=1\sim\zeta} = \begin{bmatrix} x(t-\tau_1(t)) \\ x(t-\tau_2(t)) \\ \vdots \\ x(t-\tau_\zeta(t)) \end{bmatrix}. \tag{48}$$

### 3.2.2 *Multi-order non-affine FAS models*

Parallel to the above multi-order affine FAS (35), we can also define the following multi-order non-affine one (refer to Definition 2 in [39]):

$$x_k^{(\mu_k)}|_{k=1\sim\eta}(t) = f\left(x_k^{(0\sim\mu_k-1)}(t-\tau_j(t))\,|_{k=1\sim\eta,\ j=1\sim\zeta}, t\right) + g\left(x_k^{(0\sim\mu_k-1)}(t-\sigma_j(t))\,|_{k=1\sim\eta,\ j=1\sim\gamma}, t, u(t)\right), \tag{49}$$

where $g(\cdot) \in \mathbb{R}^r$ is a nonlinear vector-valued function, and the other variables are as stated before. We call the system (49) a non-affine (global) FAS if the function $g(\cdot)$ satisfies the following assumption.
**Assumption A4.** The following mapping

$$\tilde{u}(t) = g(X, t, u(t)) \tag{50}$$

forms a differential homeomorphism from $u$ to $\tilde{u}$ for all $X \in \mathbb{R}^{\gamma\varkappa_0}$ and $t \geqslant 0$.

Due to the above Assumption A4, we can similarly introduce the following control vector transformation:

$$\tilde{u}(t) = g\left(x_k^{(0\sim\mu_k-1)}(t-\sigma_j(t))\,|_{k=1\sim\eta,\ j=1\sim\gamma}, t, u(t)\right), \tag{51}$$

and the multi-order non-affine (global) FAS (49) can also be represented into the standard form of (44).
**Example 2.** Consider the following system studied in [53]:

$$\begin{cases} \dot{x}_1 = -0.1x_1 + x_2 + u_1, \\ \dot{x}_2 = -x_1 - 0.01x_2 - 0.5x_2^3 + x_3 + u_2, \\ \dot{x}_3 = -x_2 - 0.3x_3 + 0.1\tanh x_3, \end{cases} \tag{52}$$

where $x_i, i = 1,2,3$ are the state variables, and $u_i, i = 1,2$ are the control variables. By adding some time delays in the first two equations and also adding a control element $u_2$ in the first equation, we propose an example system as follows:

$$\begin{cases} \dot{x}_1(t) = -0.1x_1(t-\tau_1(t)) + x_2(t-\tau_2(t)) + u_1(t) + u_2(t), \\ \dot{x}_2(t) = -x_1(t-\tau_3(t)) - 0.01x_2(t) - 0.5x_2^3(t) + x_3(t-\tau_4(t)) + u_2(t), \\ \dot{x}_3(t) = -x_2(t) - 0.3x_3(t) + 0.1\tanh x_3(t), \end{cases} \tag{53}$$

where $\tau_i(t), i = 1,2,3,4$ are the time delays.

From the third equation in (53), we obtain

$$x_2 = -0.3x_3 + 0.1\tanh x_3 - \dot{x}_3. \tag{54}$$

Hence,

$$\dot{x}_2 = -\left(0.2 + 0.1\tanh^2 x_3\right)\dot{x}_3 - \ddot{x}_3. \tag{55}$$

Substituting (54) into the first equation in (53) yields

$$\dot{x}_1(t) = f_1\left(x_1^{\lceil\tau_1(t)\rceil}, x_3^{(0\sim1)}(t-\tau_2(t))\right) + u_1(t) + u_2(t), \tag{56}$$

where

$$f_1\left(x_1^{\lceil\tau_1(t)\rceil}, x_3^{(0\sim1)}(t-\tau_2(t))\right) = -0.1x_1(t-\tau_1(t)) - 0.3x_3(t-\tau_2(t)) + 0.1\tanh x_3(t-\tau_2(t)) - \dot{x}_3(t-\tau_2(t)). \tag{57}$$

Substituting both (54) and (55) into the second equation in (53) yields

$$\begin{aligned} &-\left(0.2 + 0.1\tanh^2 x_3(t)\right)\dot{x}_3(t) - \ddot{x}_3(t) \\ &\quad= -x_1(t-\tau_3(t)) - 0.01(-0.3x_3(t) + 0.1\tanh x_3(t) - \dot{x}_3(t)) \\ &\qquad -0.5(-0.3x_3(t) + 0.1\tanh x_3(t) - \dot{x}_3(t))^3 + x_3(t-\tau_4(t)) + u_2(t), \end{aligned}$$

which gives

$$\ddot{x}_3(t) = f_2\left(x_1^{\lceil\tau_3(t)\rceil}, x_3^{(0\sim1)}, x_3^{\lceil\tau_4(t)\rceil}\right) - u_2(t), \tag{58}$$

where

$$\begin{aligned} & f_2\left(x_1^{\lceil\tau_3(t)\rceil}, x_3^{(0\sim1)}, x_3^{\lceil\tau_4(t)\rceil}\right) \\ & \quad = -\left(0.2 + 0.1\tanh^2 x_3(t)\right)\dot{x}_3(t) + 0.01\left(-0.3x_3(t) + 0.1\tanh x_3(t) - \dot{x}_3(t)\right) \\ & \qquad +0.5\left(-0.3x_3(t) + 0.1\tanh x_3(t) - \dot{x}_3(t)\right)^3 + x_1(t-\tau_3(t)) - x_3(t-\tau_4(t)). \end{aligned} \tag{59}$$

Combining (56) and (58), we have the following multi-order FAS:

$$\begin{bmatrix} \dot{x}_1(t) \\ \ddot{x}_3(t) \end{bmatrix} = \begin{bmatrix} f_1\left(x_1^{\lceil\tau_1(t)\rceil}, x_3^{(0\sim1)}(t-\tau_2(t))\right) \\ f_2\left(x_1^{\lceil\tau_3(t)\rceil}, x_3^{(0\sim1)}(t), x_3^{\lceil\tau_4(t)\rceil}\right) \end{bmatrix} + Bu(t), \tag{60}$$

where

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}. \tag{61}$$

Therefore, the time-delay system (53) in the state-space form is equivalent to the FAS (60).

## 4 Sub-FAS models

In this section, we look into the case that the full-actuation assumptions, namely, Assumptions A1–A4 are satisfied not completely, but conditionally.

### 4.1 Single-order sub-FASs

Let us start with the single-order affine sub-FAS models.

#### 4.1.1 *Affine sub-FAS models*

By adding a structural matrix $T$ in the single-order system (26), we immediately have the following single-order affine high-order system with state delays:

$$x^{(n)}(t) = f\left(x^{(0\sim n-1)}(t-\tau_j(t))\,|_{j=1\sim\zeta}, t\right) + B\left(Tx^{(0\sim n-1)}(t-\sigma_j(t))\,|_{j=1\sim\gamma}, t\right)u(t), \tag{62}$$

where $T \in \mathbb{R}^{m\times\gamma nr}, m \leqslant \gamma nr$ is a constant matrix, while all the other variables are as stated before.

Regarding the above system (62), we can introduce the following definitions.

**Definition 1.** Let $\mathbb{F} \subset \mathbb{R}^m, m \leqslant \gamma nr$ be the largest set such that the following full-actuation condition is met:

$$\det B(X,t) \neq 0 \text{ or } \infty, \ \forall X \in \mathbb{F} \text{ and } t \geqslant 0.$$

Then, we have the following:

(1) the set $\mathbb{F} \subset \mathbb{R}^m$ is called the set of feasible points of the system (62), and any element $Tx^{(0\sim n-1)}(t-\sigma_j(t))\,|_{j=1\sim\gamma}$ in $\mathbb{F}$ is called a feasible point of system (62);

(2) the following set

$$\mathbb{S} = \mathbb{R}^m \backslash \mathbb{F}$$

is called the set of singular points, and any element $Tx^{(0\sim n-1)}(t-\sigma_j(t))\,|_{j=1\sim\gamma}$ in $\mathbb{S}$ is called a singular point of system (62).

**Definition 2.** Let $\mathbb{F} \subset \mathbb{R}^m, m \leqslant \gamma nr$ be the set of feasible points of the system (62). Then the system (62) is called a sub-FAS if $\mathbb{F}$ is not empty. Particularly, the single-order affine system (62) with time delays is called

(1) a (global) FAS if $\mathbb{F} = \mathbb{R}^m$;

(2) an almost FAS if $\mathbb{S}$ is a set of isolated points; and

(3) a basic FAS if $\mathbb{S}$ is a set of a finite number of points.

#### 4.1.2 *Non-affine sub-FAS models*

Parallel to the above single-order affine high-order system (62), we can also define the following non-affine one:

$$x^{(n)}(t) = f\left(x^{(0\sim n-1)}(t-\tau_j(t))\,|_{j=1\sim\zeta}, t\right) + g\left(Tx^{(0\sim n-1)}(t-\sigma_j(t))\,|_{j=1\sim\gamma}, t, u(t-h)\right), \tag{63}$$

where $g(\cdot) \in \mathbb{R}^r$ is a nonlinear function, and the other variables are as stated before.

Parallel to the above Definition 1, we now have the following one.

**Definition 3.** Let $\mathbb{F} \subset \mathbb{R}^m, m \leqslant \gamma nr$ be the largest set such that the mapping

$$\tilde{u} = g(X, t, u)$$

forms a homeomorphism between $u$ and $\tilde{u}$ for all $X \in \mathbb{F}$ and $t \geqslant 0$. Then, we have the following:

(1) the set $\mathbb{F} \subset \mathbb{R}^m$ is called the set of feasible points of the sub-FAS (63), and any element $Tx^{(0\sim n-1)}(t-\sigma_j(t))\,|_{j=1\sim\gamma}$ in $\mathbb{F}$ is called a feasible point of sub-FAS (63);

(2) the following set

$$\mathbb{S} = \mathbb{R}^m \backslash \mathbb{F}$$

is called the set of singular points, and any element $Tx^{(0\sim n-1)}(t-\sigma_j(t))\,|_{j=1\sim\gamma}$ in $\mathbb{S}$ is called a singular point of sub-FAS (63).

Similarly, the concepts of sub-FAS, global FAS, almost FAS and basic FAS associated with non-affine system (63) can be defined in accordance with Definition 2.

To demonstrate the above concepts, let us consider an example.

**Example 3.** Consider the following system in [54]:

$$\begin{cases} \dot{x}_1(t) = x_1(t) + x_2^3(t), \\ \dot{x}_2(t) = u(t), \end{cases}$$

where $x_i(t), i = 1, 2$ are scalar state variables, and $u(t)$ is a scalar input.

Adding a time-varying state delay $\tau(t)$ in the $x_1(t)$ term, we have the following system:

$$\begin{cases} \dot{x}_1(t) = x_1(t-\tau(t)) + x_2^3(t), \\ \dot{x}_2(t) = u(t). \end{cases} \tag{64}$$

From the first equation in (64) we can obtain

$$x_2 = (\dot{x}_1(t) - x_1(t-\tau))^{\frac{1}{3}}. \tag{65}$$

Taking the differential of the first equation in (64) yields

$$\begin{aligned} \ddot{x}_1(t) &= \frac{\mathrm{d}x_1(t-\tau)}{\mathrm{d}t} + 3x_2^2(t)\,\dot{x}_2 \\ &= \frac{\mathrm{d}x_1(t-\tau)}{\mathrm{d}(t-\tau)}(1-\dot{\tau}) + 3x_2^2(t)\,\dot{x}_2 \\ &= \dot{x}_1(t-\tau)(1-\dot{\tau}) + 3x_2^2(t)\,\dot{x}_2. \end{aligned}$$

Substituting (65) and also the second equation in (64) into the above equation, we have the following sub-FAS with time delays:

$$\ddot{x}_1(t) = f(\dot{x}_1(t-\tau), t) + B\left(T\begin{bmatrix} x_1^{(0\sim1)}(t) \\ x_1^{(0\sim1)}(t-\tau) \end{bmatrix}\right)u(t), \tag{66}$$

where

$$f(\dot{x}_1(t-\tau), t) = \dot{x}_1(t-\tau)(1-\dot{\tau}), \tag{67}$$

and

$$B\left(T\begin{bmatrix} x_1^{(0\sim1)}(t) \\ x_1^{(0\sim1)}(t-\tau) \end{bmatrix}\right) = B\left(\dot{x}_1(t), x_1(t-\tau)\right) = 3\left(\dot{x}_1(t) - x_1(t-\tau)\right)^{\frac{2}{3}}, \tag{68}$$

with

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix},\ T_1 = \begin{bmatrix} 0 & 1 \end{bmatrix},\ T_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{69}$$

Thus

$$\eta = 1,\ \tau_1(t) = \tau(t);\ \text{and}\ \gamma = 2,\ \sigma_1(t) = 0, \sigma_2(t) = \tau(t).$$

Clearly, the feasibility condition of the system is

$$\dot{x}_1(t) - x_1(t - \tau(t)) \neq 0.$$

Therefore, the set of feasible points is

$$\mathbb{F} = \left\{ \begin{bmatrix} \dot{x}_1(t) \\ x_1(t-\tau) \end{bmatrix} \middle| \ \dot{x}_1(t) - x_1(t-\tau(t)) \neq 0 \right\}. \tag{70}$$

## 4.2 Multi-order sub-FASs

### 4.2.1 *Affine sub-FAS models*

In this subsection, let us generalize the above single-order time-delay sub-FASs into the case of multi-order ones.

Again, adding a structural matrix $T$ in the system (42), we have the following multi-order affine system with time delays:

$$\begin{aligned} x_k^{(\mu_k)}|_{k=1\sim\eta}(t) \ = \ & f\left(x_k^{(0\sim\mu_k-1)}(t-\tau_j(t))\,|_{k=1\sim\eta,\ j=1\sim\zeta}, t\right) \\ & + B\left(Tx_k^{(0\sim\mu_k-1)}(t-\sigma_j(t))\,|_{k=1\sim\eta,\ j=1\sim\gamma}, t\right)u(t), \end{aligned} \tag{71}$$

where

$$T \in \mathbb{R}^{M\times\gamma\varkappa_0},\ M \leqslant \gamma\varkappa_0,$$

while all the other variables are as stated before.

**Definition 4.** Let $\mathbb{F} \subset \mathbb{R}^M$, with $M \leqslant \gamma\varkappa_0$ be the largest set such that the following full-actuation condition is met:

$$\det B(X,t) \neq 0 \text{ or } \infty, \forall X \in \mathbb{F} \text{ and } t \geqslant 0.$$

Then, we have the following:

(1) the set $\mathbb{F} \subset \mathbb{R}^M$ is called the set of feasible points of the sub-FAS (71), and any element $Tx_k^{(0\sim\mu_k-1)}(t-\sigma_j(t))\,|_{k=1\sim\eta,\ j=1\sim\gamma}$ in $\mathbb{F}$ is called a feasible point of sub-FAS (71); and

(2) the following set

$$\mathbb{S} = \mathbb{R}^M \backslash \mathbb{F}$$

is called the set of singular points, and any element $Tx_k^{(0\sim\mu_k-1)}(t-\sigma_j(t))\,|_{k=1\sim\eta,\ j=1\sim\gamma}$ in $\mathbb{S}$ is called a singular point of sub-FAS (71).

Let $\mathbb{F} \subset \mathbb{R}^M$ be the set of feasible points of system (71). Then the system (71) is called a sub-FAS if $\mathbb{F}$ is not empty. Particularly, the concepts of global FAS, almost FAS and basic FAS associated with system (71) can also be defined in accordance with Definition 4.

Obviously, in the case of $\eta = 1$, the above sub-FAS model (71) reduces to the form of the single-order sub-FAS (62).

#### 4.2.2 *Non-affine sub-FAS models*

Parallel to the above multi-order affine sub-FAS (71), we can also define the following multi-order non-affine one:

$$\begin{aligned} x_k^{(\mu_k)}|_{k=1\sim\eta}(t) = \; & f\left(x_k^{(0\sim\mu_k-1)}(t-\tau_j(t))\,|_{k=1\sim\eta,\; j=1\sim\zeta}, t\right) \\ & +g\left(Tx_k^{(0\sim\mu_k-1)}(t-\sigma_j(t))\,|_{k=1\sim\eta,\; j=1\sim\gamma}, t, u(t)\right), \end{aligned} \tag{72}$$

where $g(\cdot)\in\mathbb{R}^r$ is a nonlinear function, and the other variables are as stated before.

Corresponding to the above Definition 4, we can now give the following one.

**Definition 5.** Let $\mathbb{F}\subset\mathbb{R}^M$, with $M\leqslant\gamma\varkappa_0$ be the largest set such that the mapping

$$\tilde{u}=g(X,t,u)$$

forms a homeomorphism between $u$ and $\tilde{u}$ for all $X\in\mathbb{F}$ and $t\geqslant 0$. Then, we have the following:

(1) the set $\mathbb{F}\subset\mathbb{R}^M$ is called the set of feasible points of the sub-FAS (72), and any element $Tx_k^{(0\sim\mu_k-1)}(t-\sigma_j(t))\,|_{k=1\sim\eta,j=1\sim\gamma}$ in $\mathbb{F}$ is called a feasible point of sub-FAS (72);

(2) the following set

$$\mathbb{S}=\mathbb{R}^M\backslash\mathbb{F}$$

is called the set of singular points, and any element $Tx_k^{(0\sim\mu_k-1)}(t-\sigma_j(t))\,|_{k=1\sim\eta,j=1\sim\gamma}$ in $\mathbb{S}$ is called a singular point of sub-FAS (72).

Again, the concepts of global FAS, almost FAS and basic FAS associated with non-affine system (71) can also be defined similarly to Definition 4.

**Example 4.** In [55], the following simple but important nonlinear system without delays is introduced:

$$\begin{cases} \dot{x}_0 = u, \\ \dot{x}_1 = v, \\ \dot{x}_2 = x_0x_1, \end{cases} \tag{73}$$

where $x_i, i=0,1,2$ are scalar state variables, and $u$ and $v$ are two scalar inputs. Simple alteration of the system produces the following time-delay system:

$$\begin{cases} \dot{x}_0(t) = u(t), \\ \dot{x}_1(t) = x_1(t-\tau(t))+v(t), \\ \dot{x}_2(t) = x_0(t)(x_1(t)-1), \end{cases} \tag{74}$$

where $\tau(t)$ is a bounded nonnegative scalar function.

When $x_1\neq 1$, we have, from the third equation in (74),

$$x_0=\frac{\dot{x}_2}{x_1-1}. \tag{75}$$

Taking the derivative of the third equation in (74) and using the above relation (75) and the first two ones in (74), we have

$$\ddot{x}_2(t)=(x_1(t)-1)u(t)+\frac{\dot{x}_2(t)}{x_1(t)-1}x_1(t-\tau(t))+\frac{\dot{x}_2(t)}{x_1(t)-1}v(t). \tag{76}$$

Finally, by combining (76) with the second equation in (74), we obtain the following sub-FAS model for the system

$$\begin{bmatrix}\dot{x}_1(t)\\ \ddot{x}_2(t)\end{bmatrix}=\begin{bmatrix} f_1(x_1(t-\tau)) \\ f_2(x_1(t),x_1(t-\tau),\dot{x}_2(t))\end{bmatrix}+B\left(T\begin{bmatrix}x_1(t)\\ x_2^{(0\sim1)}(t)\end{bmatrix}\right)\begin{bmatrix}u(t)\\ v(t)\end{bmatrix}, \tag{77}$$

where

$$f_1\left(x_1\left(t-\tau\right)\right) = x_1\left(t-\tau\right), \tag{78}$$

$$f_2\left(x_1\left(t\right), x_1\left(t-\tau\right), \dot{x}_2\left(t\right)\right) = \frac{\dot{x}_2\left(t\right)}{x_1\left(t\right)-1} x_1\left(t-\tau\left(t\right)\right), \tag{79}$$

$$B\left(T\begin{bmatrix} x_1\left(t\right) \\ x_2^{(0\sim 1)}\left(t\right) \end{bmatrix}\right) = B\left(x_1\left(t\right), \dot{x}_2\left(t\right)\right) = \begin{bmatrix} 0 & 1 \\ x_1\left(t\right)-1 & \frac{\dot{x}_2(t)}{x_1(t)-1} \end{bmatrix}, \tag{80}$$

with

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Clearly, the full-actuation condition is

$$x_1\left(t\right) \neq 1,\ \forall t \geqslant 0.$$

Correspondingly, the set of feasible points, determined by the above matrix $B\left(x_1\left(t\right), \dot{x}_2\left(t\right)\right)$, is

$$\mathbb{F} = \left\{ \begin{bmatrix} x_1 \\ \dot{x}_2 \end{bmatrix} \middle| x_1 \neq 1 \right\}.$$

## 5 Controller design

This section considers the design of controllers for the proposed sub-FASs.

### 5.1 Single-order sub-FASs

When the set of feasible points of single-order affine sub-FAS (62) $\mathbb{F} \subset \mathbb{R}^m$ is not empty, it is obvious that $B^{-1}\left(x^{(0\sim n-1)}\left(t-\sigma_j\left(t\right)\right)|_{j=1\sim\gamma}, t\right)$ exists in $\mathbb{F}$. Hence we can immediately obtain the following result.

**Theorem 1.** Let $\mathbb{F} \subset \mathbb{R}^m, m \leqslant \gamma nr$ be the set of feasible points of single-order affine sub-FAS (62), and $A_{0\sim n-1} \in \mathbb{R}^{r\times nr}$ be an arbitrarily given matrix. Design the following controller:

$$\begin{cases} u\left(t\right) = -B^{-1}\left(Tx^{(0\sim n-1)}\left(t-\sigma_j\left(t\right)\right)|_{j=1\sim\gamma}, t\right)\left[f\left(x^{(0\sim n-1)}\left(t-\tau_j\left(t\right)\right)|_{j=1\sim\zeta}, t\right) + u^*\left(t\right)\right], \\ u^*\left(t\right) = A_{0\sim n-1}x^{(0\sim n-1)}\left(t\right) - v\left(t\right),\ t \geqslant 0 \end{cases} \tag{81}$$

for the single-order affine sub-FAS (62), where $v\left(t\right)$ is an external input vector. Then the following constant linear closed-loop system

$$x^{(n)}\left(t\right) + A_{0\sim n-1}x^{(0\sim n-1)}\left(t\right) = v\left(t\right),\ t \geqslant 0 \tag{82}$$

is obtained provided that the following state constraint is met:

$$Tx^{(0\sim n-1)}\left(t-\sigma_j\left(t\right)\right)|_{j=1\sim\gamma} \in \mathbb{F}. \tag{83}$$

For the single-order non-affine sub-FAS (63), a result similar to Theorem 1 still holds. In this case, the controller can be obtained using the differential homeomorphism property of the mapping $g\left(\cdot\right)$ as

$$\begin{cases} u\left(t\right) = g^{-1}\left(Tx^{(0\sim n-1)}\left(t-\sigma_j\left(t\right)\right)|_{j=1\sim\gamma}, t, \tilde{u}\left(t\right)\right), \\ \tilde{u}\left(t\right) = -f\left(x^{(0\sim n-1)}\left(t-\tau_j\left(t\right)\right)|_{j=1\sim\zeta}, t\right) - u^*\left(t\right), \\ u^*\left(t\right) = A_{0\sim n-1}x^{(0\sim n-1)}\left(t\right) - v\left(t\right),\ t \geqslant 0, \end{cases} \tag{84}$$

and eventually, the same closed-loop linear system as (82) is resulted in.

To demonstrate the above proposed controller, let us consider two examples treated in Sections 2 and 4.

**Example 5.** Continuation of Example 1.

Since the FAS (23)-(24) derived in Example 1 is a global FAS, the state constraint (83) in Theorem 1 vanishes. Following Theorem 1, a controller for the system can be designed as

$$\begin{cases} u(t) = -\frac{1}{g(x_1(t-\sigma(t)),t)}\left[\frac{f_2(x_1^{\lceil \tau_{1\sim 2}(t)\rceil}, x_1^{\lceil \sigma(t)\rceil}, \dot{x}_1^{\lceil \tau_3(t)\rceil}, \ddot{x}_1(t), t)}{g(x_1(t-\sigma(t)),t)} + u^*(t)\right], \\ u^* = a_0 x_1 + a_1\dot{x}_1 + a_2\ddot{x}_1 - v, \end{cases} \tag{85}$$

where $v$ is an external input, which can be usually set to zero. The corresponding closed-loop system is given by

$$\dddot{x}_1 + a_2\ddot{x}_1 + a_1\dot{x}_1 + a_0 x_1 = v. \tag{86}$$

The parameters $a_i$, $i = 0, 1, 2$ can be easily determined to make the closed-loop system stable.

**Example 6.** Continuation of Example 3.

Following the above Theorem 1, the controller for the sub-FAS (66)–(69) derived in Example 3 is given as follows:

$$\begin{cases} u(t) = -B^{-1}(\dot{x}_1(t), x_1(t-\tau))\left[f(\dot{x}_1(t-\tau), t) + u^*(t)\right], \\ u^*(t) = a_{0\sim 1}x^{(0\sim 1)}(t) - v(t), \ t \geqslant 0, \end{cases} \tag{87}$$

and the corresponding closed-loop system is given by

$$\ddot{x}_1 + a_1\dot{x}_1 + a_0 x_1 = v, \tag{88}$$

where $a_0$ and $a_1$ are two positive scalars. The initial values of the system need to be carefully chosen such that the following constraint is met:

$$T\begin{bmatrix} x_1^{(0\sim 1)}(t) \\ x_1^{(0\sim 1)}(t-\tau) \end{bmatrix} = \begin{bmatrix} \dot{x}_1(t) \\ x_1(t-\tau) \end{bmatrix} \in \mathbb{F}, \ t \geqslant 0, \tag{89}$$

where $\mathbb{F}$ is the set of feasible points of the system given by (70).

The allowable initial values of the system determined by the above constraint (89) will be further analyzed in Section 6.

### 5.2 Multi-order sub-FASs

As a generalization of Theorem 1, we have the following result.

**Theorem 2.** Let $\mathbb{F} \subset \mathbb{R}^M$ be the set of feasible points of the sub-FAS (71), and $A \in \mathbb{R}^{r\times \varkappa_0}$ be an arbitrarily given matrix. Then the following controller

$$\begin{cases} u(t) = -B^{-1}\left(Tx_k^{(0\sim\mu_k-1)}(t-\sigma_j(t))\,|_{k=1\sim\eta,\ j=1\sim\gamma}, t\right) \\ \qquad\quad \times\left[f\left(x_k^{(0\sim\mu_k-1)}(t-\tau_j(t))\,|_{k=1\sim\eta,\ j=1\sim\zeta}, t\right) - u^*(t)\right], \\ u^*(t) = Ax_k^{(0\sim\mu_k-1)}(t)\,|_{k=1\sim\eta} + v(t), \end{cases} \tag{90}$$

with $v(t)$ being an external input, produces the constant linear system

$$x_k^{(\mu_k)}|_{k=1\sim\eta}(t) = Ax_k^{(0\sim\mu_k-1)}(t)\,|_{k=1\sim\eta} + v(t), \tag{91}$$

provided that the following state constraint is met:

$$Tx_k^{(0\sim\mu_k-1)}(t-\sigma_j(t))\,|_{k=1\sim\eta,\ j=1\sim\gamma} \in \mathbb{F}. \tag{92}$$

For the multi-order non-affine FAS (72), a result similar to Theorem 2 still holds. In this case, the controller is given as

$$\begin{cases} u(t) = -g^{-1}\left(Tx_k^{(0\sim\mu_k-1)}(t-\sigma_j(t))\,|_{k=1\sim\eta,\ j=1\sim\gamma}, t, \tilde{u}(t)\right), \\ \tilde{u}(t) = f\left(x_k^{(0\sim\mu_k-1)}(t-\tau_j(t))\,|_{k=1\sim\eta,\ j=1\sim\zeta}, t\right) - u^*(t), \\ u^*(t) = Ax_k^{(0\sim\mu_k-1)}(t)\,|_{k=1\sim\eta} + v(t), \end{cases} \tag{93}$$

and the same closed-loop system as (91) is achieved.

**Remark 2.** Particularly, if we choose in the above controllers (90) and (93)

$$A = \mathrm{Blockdiag}\left([A_i]_{0\sim\mu_i-1}, i = 1, 2, \dots, \eta\right),$$

where $[A_i]_{0\sim\mu_i-1} \in \mathbb{R}^{r_i\times\mu_i r_i}$, $i = 1, 2, \dots, \eta$, then the following set of decoupled constant linear closed-loop systems is obtained:

$$x_i^{(\mu_i)} + [A_i]_{0\sim\mu_i-1} x_i^{(0\sim\mu_i-1)} = v_i, \quad i = 1, 2, \dots, \eta, \tag{94}$$

with $v$ being an external input vector given by

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_\eta \end{bmatrix}, \quad v_i \in \mathbb{R}^{r_i}. \tag{95}$$

For a systematic parametric method to solve the controller parameter matrix $A$ such that the closed-loop linear system possesses a desired eigenstructure, please refer to [49].

**Remark 3.** Let

$$\Phi(A_{0\sim n-1}) = \begin{bmatrix} 0 & I & & \\ & & \ddots & \\ & & & I \\ -A_0 & -A_1 & \cdots & -A_{n-1} \end{bmatrix}, \ B_{\mathrm{c}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_r \end{bmatrix}, \tag{96}$$

and then it can be easily checked that the closed-loop system (82) can be expressed by the following state-space form:

$$\dot{x}^{(0\sim n-1)} = \Phi(A_{0\sim n-1}) x^{(0\sim n-1)} + B_{\mathrm{c}} v. \tag{97}$$

Further note that

$$\Phi(A_{0\sim n-1}) = \Phi(0_{0\sim n-1}) - B_{\mathrm{c}} A_{0\sim n-1}, \tag{98}$$

the solution to the gain matrix $A_{0\sim n-1} \in \mathbb{R}^{r\times nr}$ in the controllers (81) and (84) can then be easily carried out by conducting the eigenstructure assignment in the matrix pair $(\Phi(0_{0\sim n-1}), B_{\mathrm{c}})$. Similarly, the gain matrix $A \in \mathbb{R}^{r\times\gamma\varkappa_0}$ in (93), can also be solved. We point out that specific parametric eigenstructure assignment designs have been proposed in [47], which give the general expressions of these gain matrices, and provide all the degrees of freedom in the designs. The full degrees of freedom can be further utilized to achieve additional system performance.

In order to demonstrate the above result, let us consider two examples treated in Sections 3 and 4.

**Example 7.** Continuation of Example 2.

Since the system (60) derived in Example 2 is a global FAS, its set of feasible points is the whole state space. Hence the state constraint (92) vanishes. According to Theorem 2, a controller for the FAS (60) can be designed as

$$\begin{cases} u(t) = -B^{-1}\left(\begin{bmatrix} f_1\left(x_1^{\lceil\tau_1(t)\rceil}, x_3^{(0\sim1)}(t-\tau_2(t))\right) \\ f_2\left(x_1^{\lceil\tau_3(t)\rceil}, x_3^{(0\sim1)}(t), x_3^{\lceil\tau_4(t)\rceil}\right) \end{bmatrix} + \begin{bmatrix} u_1^*(t) \\ u_2^*(t) \end{bmatrix}\right), \\ u_1^* = a x_1, \\ u_2^* = a_0 x_3 + a_1 \dot{x}_3, \end{cases} \tag{99}$$

where $a_0, a_1$ and $a$ are real positive scalars. The closed-loop system is clearly given as follows:

$$\begin{cases} \dot{x}_1 + a x_1 = 0, \\ \ddot{x}_3 + a_1 \dot{x}_3 + a_0 x_3 = 0. \end{cases} \tag{100}$$

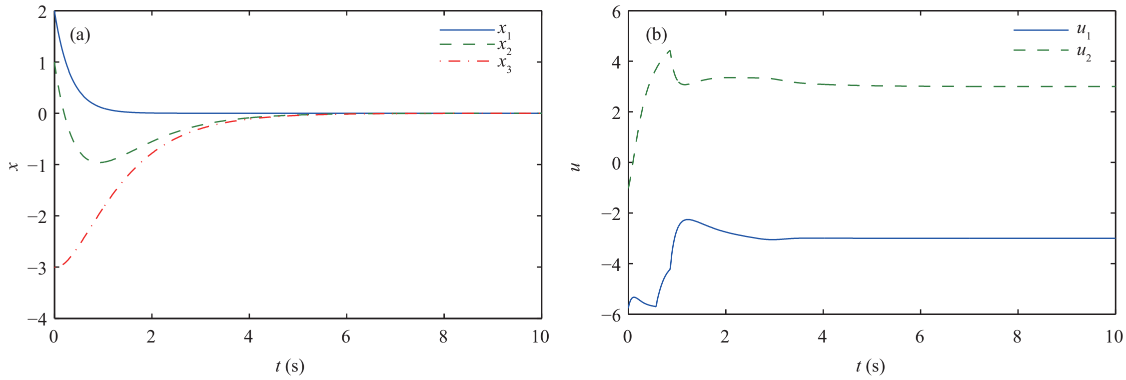


**Figure 1** (Color online) Simulation result of Example 7. (a) State variables; (b) control inputs.

Please note that the initial values of the closed-loop system, namely, $x_1(0)$ and $x_3^{(0\sim1)}(0)$ are not arbitrarily chosen, and they are generated by the system initial values. Furthermore, the original state $x_2$ can be immediately obtained via (54) when $x_3$ and $\dot{x}_3$ are obtained.

In our simulation, the closed-loop parameters are chosen as $a_0 = 2$, $a_1 = a = 3$, and the time-delays are chosen as

$$\begin{cases} \tau_1(t) = 0.5\,|\sin t|\,, \\ \tau_2(t) = \mathrm{e}^{-t}, \\ \tau_3(t) = 1 + \cos 2t, \\ \tau_4(t) = 10. \end{cases} \tag{101}$$

With the following initial value functions:

$$x_1(s) = 2,\ \ s \leqslant 0,$$
$$x_2(s) = 1,\ \ x_3(s) = -3,\ \ s \leqslant 0,$$

which correspond to

$$\dot{x}_3(s) = -0.1\,(1 + \tanh 3)\,,\ \ s \leqslant 0,$$

the simulation of the system is carried out and the result is shown in Figure 1.

**Example 8.** Continuation of Example 4.

For control of the sub-FAS (77)–(80) derived in Example 4, let us adopt the decoupled design approach mentioned in Remark 2. Hence a controller for the system can be designed as

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = -B^{-1}(x_1(t), \dot{x}_2(t)) \left( \begin{bmatrix} f_1(x_1(t-\tau)) \\ f_2(x_1(t), x_1(t-\tau), \dot{x}_2(t)) \end{bmatrix} + \begin{bmatrix} a x_1(t) \\ a_0 x_2(t) + a_1 \dot{x}_2(t) \end{bmatrix} \right), \tag{102}$$

where $a_0, a_1$ and $a$ are three positive scalars. As a result, the closed-loop system is given by the two decoupled subsystems

$$\dot{x}_1 + a x_1 = 0, \tag{103}$$

and

$$\ddot{x}_2 + a_1 \dot{x}_2 + a_0 x_2 = 0, \tag{104}$$

where the subsystem (103) is subject to the following constraint:

$$x_1(t) \neq 1,\ \ t \geqslant 0.$$

In order to meet this constraint and simultaneously guarantee the stability of the system at the origin, it is easily seen that it is necessary and sufficient to restrict the initial value $x(0)$ of the subsystem (103) to be less than 1.

When

$$a_0 = 3,\ \ a_1 = 2,\ \ a = 1,\ \text{ and }\ \tau(t) = 1 + \cos 2t,$$

with the following initial value functions:

$$x_1(s) = 0.95,\ \ s \in (-c_1, 0],\ \text{ and }\ x_2(0) = 4,\ \ \dot{x}_2(0) = -1,$$

the simulation of the system is carried out and the result is shown in Figure 2.

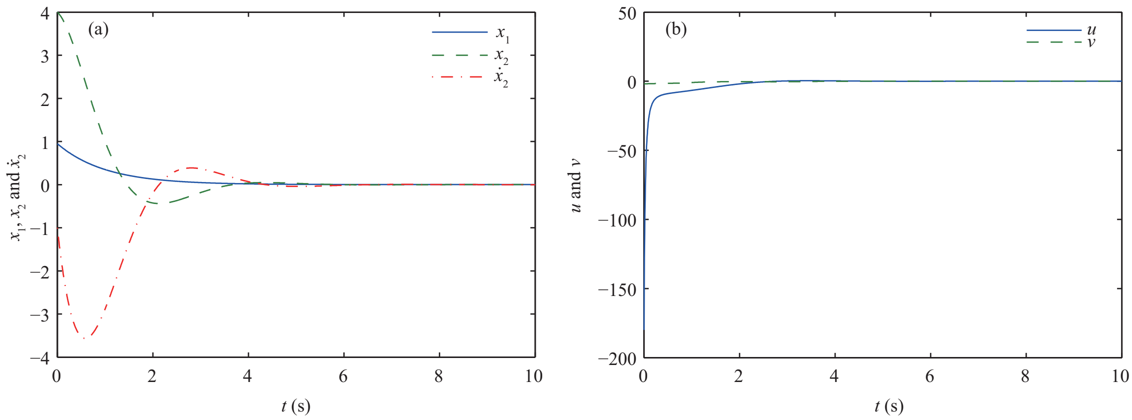


**Figure 2** (Color online) Simulation result of Example 8. (a) State variables; (b) control inputs.

### 5.3 Controllability and stabilizability

Let us start with the controllability of the following standard linear system:

$$\dot{x} = Ax + Bu, \tag{105}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^r$ are the state vector and the input vector, respectively, $A$ and $B$ are the coefficient matrices. It is well-known that the system (105) is controllable if and only if one of the following conditions is satisfied:

(1) there exists a feedback controller to drive an arbitrary initial value $x_0 = x(0)$ to the origin within finite time;

(2) the controllability matrix of the system is of full-row rank;

(3) the PBH criterion is met; and

(4) the closed-loop eigen-polynomial, or equivalently, the closed-loop eigenvalues, can be arbitrarily assigned by state feedback.

Which of the above can be generalized to the nonlinear system case, or even the nonlinear time-delay system case?

The first one can indeed be generalized to the nonlinear system case, but in general the results up to date are not at all satisfactory. For nonlinear time-delay systems, the generalization is unimaginably difficult.

The second and the third ones may help in the so-called local controllability of nonlinear systems, but in nature they cannot be generalized to the nonlinear system case.

Now how about the last one? As a matter of fact, it can be generalized to nonlinear system cases, or even the nonlinear time-delay system cases. Along with this direction, the controllability of a general nonlinear system with time-delays may be defined as follows:

A general dynamical system, no matter linear or nonlinear, no matter time-varying or not, no matter involving time delays or not, is controllable if there exists a full state feedback controller for the system such that the closed-loop system is turned into a constant linear one with an arbitrarily assignable eigenstructure (or eigen-polynomial).

It clearly follows from Theorem 2 that, a full state feedback controller for a sub-FAS can be designed such that the closed-loop system (94) is a constant linear one locally on the set of feasible points. When expanded, each subsystem in the above linear closed-loop system (94) can be written as

$$x_k^{(\mu_k)} + [A_k]_{\mu_k-1}x^{(\mu_k-1)} + \cdots + [A_k]_1\dot{x} + [A_k]_0x = v_k,$$

whose eigen-polynomial matrix obviously is

$$P_k(s) = I_{r_k}s^{\mu_k} + [A_k]_{\mu_k-1}s^{\mu_k-1} + \cdots + [A_k]_1s + [A_k]_0.$$

Due to the arbitrariness of the series of matrices $[A_k]_i,\ i = 0, 1, \ldots, \mu_k - 1,\ k = 1, 2, \ldots, \eta$, the eigen-polynomial matrices of each subsystem in the linear closed-loop system (94) can be arbitrarily assigned. Therefore, a linear closed-loop system with desired performance can be obtained by properly selecting these matrices. Particularly, the closed-loop eigenvalues can be arbitrarily assigned by properly selecting these matrices. Such a property of FASs totally reveals the meaning of controllability of control systems in the sense mentioned above.

Introduce the following general nonlinear system:

$$F\left(z_k^{(0\sim\mu_k-1)}\left(t-\tau_i\left(t\right)\right)\Big|_{i=1\sim\phi,k=1\sim\omega}, u\left(t-\sigma_i\left(t\right)\right)|_{i=1\sim\varphi}, t\right)=0, \tag{106}$$

where $\phi, \varphi$ and $\omega$ are some nonnegative integers, $F\left(\cdot\right)$ is a vector function of appropriate dimension, and $z$ and $u$ are respectively the state and input vectors of appropriate dimensions.

Based on the above discussion, we can propose the following definition for the controllability of the system (106).

**Definition 6.** The dynamical system (106) is said to be

- completely controllable if it can be equivalently converted into a global FAS in the form of (71) or (72);
- almost completely controllable if it can be equivalently converted into an almost FAS in the form of (71) or (72);
- basically completely controllable if it can be equivalently converted into a basic FAS in the form of (71) or (72); and
- sub-controllable if it can be equivalently converted into a sub-FAS in the form of (71) or (72).

The above definition obviously contains Definition 3 in [39] and Definition 3.2 in [47] as special cases.

**Remark 4.** With this definition, the following is obvious:

(1) all dynamical control systems, not only those in state-space representations, possess the concepts of controllability;

(2) the problem of controllability analysis and the problem of control are now highly consistent in the sense that, once the controllability of a system is verified by obtaining the FAS model of the system, the design problem is then easily solved in view of Theorems 1 and 2.

Like controllability, stabilizability is also one of the most important concepts in control systems theory. This subsection further looks into the problem of stabilizability.

It is well-known that a stabilizable linear system can always be decomposed into a controllable subsystem and an uncontrollable subsystem, but with the uncontrollable one stable. Inspired by such a fact, we can now introduce the following definition.

**Definition 7.** Suppose that the dynamical control system (106) can be equivalently converted into a system in the form of

$$\begin{cases} x_k^{(\mu_k)}|_{k=1\sim\eta}\left(t\right)=f\left(x_k^{(0\sim\mu_k-1)}\left(t-\tau_j\left(t\right)\right)|_{k=1\sim\eta,\ j=1\sim\zeta}, z_k^{(0\sim\nu_k-1)}\left(t-\tau_j\left(t\right)\right)|_{k=1\sim\iota,\ j=1\sim\xi}, t\right) \\ \qquad\qquad + B\left(Tx_k^{(0\sim\mu_k-1)}\left(t-\sigma_j\left(t\right)\right)|_{k=1\sim\eta,\ j=1\sim\gamma}, z_k^{(0\sim\nu_k-1)}\left(t-\tau_j\left(t\right)\right)|_{k=1\sim\iota,\ j=1\sim\xi}, t\right)u\left(t\right), \\ z_k^{(\nu_k)}|_{k=1\sim\iota}\left(t\right)=g\left(z_k^{(0\sim\nu_k-1)}\left(t-\tau_j\left(t\right)\right)|_{k=1\sim\iota,\ j=1\sim\xi}, t\right), \end{cases} \tag{107}$$

where $\xi, \iota$, and $\nu_k, k=1,2,\ldots,\iota$, are nonnegative integers, $z_k, k=1,2,\ldots,\iota$, are vectors of appropriate dimensions. If the first subsystem is an FAS, which is called the completely controllable subsystem, the second subsystem, which is called the uncontrollable subsystem, is stable in a certain sense, then the dynamical control system (106), and also (107), is said to be stabilizable in the same sense.

The stability of nonlinear dynamical systems is complicated. It could be globally stable or locally stable, and could be asymptotically stable or critically stable. Furthermore, the equilibrium status may also be complicated. It is the type of stability of the uncontrollable subsystem that determines the type of stabilizability of the whole system.

When the first subsystem in (107) is replaced with a sub-FAS, the concept of sub-stabilizability can be similarly defined.

For more comments on controllability and stabilizability of nonlinear systems without time delays, please refer to [47].

## 6 State constrained control

Recall Theorem 1 that the control of a single-order sub-FAS is reduced to the solution of the matrix $A_{0\sim n-1}$ and proper system initial values $x^{(0\sim n-1)}(0)$ such that the following constrained system is globally asymptotically stable:

$$\begin{cases} x^{(n)}(t) + A_{0\sim n-1}x^{(0\sim n-1)}(t) = v, \ t \geqslant 0, \\ Tx^{(0\sim n-1)}(t-\sigma_j(t))\,|_{j=1\sim\gamma} \in \mathbb{F}, \end{cases} \tag{108}$$

where $v$ is often taken to be a constant signal.

Let $X(t) = x^{(0\sim n-1)}(t)$. Then the problem is turned into the following one:

$$\begin{cases} \dot{X}(t) = \Phi(A_{0\sim n-1})X(t) + B_{\mathrm{c}}v, \\ TX(t-\sigma_j(t))\,|_{j=1\sim\gamma} \in \mathbb{F}. \end{cases}$$

Similarly, recall Theorem 2 that the control of a multi-order sub-FAS is reduced to the solution of the matrix $A$ and proper initial values $x_k^{(0\sim\mu_k-1)}(0)\,|_{k=1\sim\eta}$ such that the following constrained system is globally asymptotically stable:

$$\begin{cases} x_k^{(\mu_k)}|_{k=1\sim\eta}(t) = Ax_k^{(0\sim\mu_k-1)}(t)\,|_{k=1\sim\eta} + v, \\ Tx_k^{(0\sim\mu_k-1)}(t-\sigma_j(t))\,|_{k=1\sim\eta,\ j=1\sim\gamma} \in \mathbb{F}. \end{cases} \tag{109}$$

Let $X(t) = x_k^{(0\sim\mu_k-1)}(t)\,|_{k=1\sim\eta}$. Then the problem can be turned into the following one:

$$\begin{cases} \dot{X}(t) = \tilde{\Phi}(A)X(t) + \tilde{B}_{\mathrm{c}}v, \\ TX(t-\sigma_j(t))\,|_{j=1\sim\gamma} \in \mathbb{F}, \end{cases}$$

where $\tilde{\Phi}(A)$ and $\tilde{B}_{\mathrm{c}}$ are two properly determined constant matrices.

With the above background, in this section let us consider the state constrained control problem for the linear system

$$\dot{y} = Ay + Bu, \tag{110}$$

where $y \in \mathbb{R}^n$ and $u \in \mathbb{R}^r$ are the state vector and the control input vector, respectively, $A$ and $B$ are real matrices of appropriate dimensions.

**Problem 1.** Let $\mathbb{F} \subset \mathbb{R}^m$. Find a controller for the linear system (110), in the form of

$$u = Ky + v,\ K \in \mathbb{R}^{r\times n}, \tag{111}$$

such that the closed-loop system

$$\dot{y} = A_{\mathrm{c}}y + Bv, \quad A_{\mathrm{c}} = A + BK \tag{112}$$

is stable, and meanwhile, satisfies the following state constraint:

$$Ty(t-\sigma_j(t))\,|_{j=1\sim\gamma} \in \mathbb{F}, \tag{113}$$

where $T \in \mathbb{R}^{m\times\gamma n}$ is a distribution matrix.

### 6.1 Preliminary

Consider the stable linear system

$$\dot{x} = A_{\mathrm{c}}x. \tag{114}$$

In this subsection, let us present three formulas for estimation of the number $\kappa$ satisfying

$$\|Tx(t)\| \leqslant \kappa\,\|x(0)\|,\ t \geqslant 0. \tag{115}$$

Before doing this, let us first make some preparation.

**Definition 8.** Let

$$Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} \in \mathbb{R}^{m\times n},$$

and

$$1 \leqslant k_1 < k_2 < \cdots < k_l \leqslant n \tag{116}$$

be a series of integers. Then the following matrix

$$\mathrm{csel}_{\langle k_{1\sim l}\rangle}(Q) = \begin{bmatrix} q_{k_1} & q_{k_2} & \cdots & q_{k_l} \end{bmatrix} \in \mathbb{R}^{m\times l}$$

is called a column selection of $Q$ by ordering (116). Particularly, if $n = \theta m$, where $\theta$ is some integer, and

$$\begin{cases} k_1 = 1, \\ k_2 = m+2, \\ k_3 = 2m+3, \\ \quad\vdots \\ k_\theta = (\theta-1)\,m+\theta = n, \end{cases}$$

then $\mathrm{csel}_{\langle k_{1\sim\theta}\rangle}(Q)$ is called the basic column selection of the matrix $Q$, and is simply denoted by $\mathrm{bcs}(Q)$.

With the help of the above definition, we can now state the following result whose proof is given in Appendix A.

**Theorem 3.** Let $A_\mathrm{c}$ be stable, $T \in \mathbb{R}^{m\times n}$ be a given matrix. Then, for the system (114) there exists a constant number $\kappa > 0$ satisfying (115). Particularly,

(1) when $T = I_n$, an estimate of $\kappa$ can be given as

$$\kappa = \sqrt{\|P\|\,\|P^{-1}\|}, \tag{117}$$

where $P$ is the solution to the following Lyapunov equation:

$$A_\mathrm{c}^\mathrm{T} P + P A_\mathrm{c} = -I_n; \tag{118}$$

(2) when $A_\mathrm{c}$ is non-defective, that is, it has a diagonal Jordan matrix, an estimate of $\kappa$ can be given as

$$\kappa = \|TV\|\,\left\|V^{-1}\right\|, \tag{119}$$

where $V$ is an eigenvector matrix of the matrix $A_\mathrm{c}$;

(3) when $A_\mathrm{c}$ is non-defective, an estimate of $\kappa$ can also be given as

$$\kappa = \sqrt{n}\left\|\mathrm{bcs}\left(V^{-\mathrm{T}} \otimes (TV)\right)\right\|_*, \ * = 2 \text{ or F}. \tag{120}$$

Recall that $A_\mathrm{c} = A + BK$, all the estimates (117)–(120) are dependent on the feedback gain $K$. Particularly, when parametric eigenstructure assignment is applied to the solution of $K$, the design degrees of freedom can be used to minimize the number $\kappa$ through formulas (119) and (120).

**Example 9.** Continuation of Example 6.

For the sub-FAS (66)–(69) derived in Example 3, the controller (87) is designed in Example 6, and the corresponding closed-loop system is given by

$$\ddot{x}_1 + a_1\dot{x}_1 + a_0 x_1 = v \tag{121}$$

with the states being subject to the constraint $x_1^{(0\sim1)} \in \mathbb{F}$.

This system (121) is clearly converted into the following state-space form:

$$\dot{x}_1^{(0\sim1)} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} x_1^{(0\sim1)} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v.$$

When $a_0 = 2$ and $a_1 = 1$, corresponding to the choices of

$$T = I_2, \ T = T_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \text{ and } T = T_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \tag{122}$$

we have computed the values of $\kappa$ using the different formulas in Theorem 3. For comparison, the same set of values for the case of $a_0 = 1$ and $a_1 = 2.5$ is also computed. All these values are provided in Table 1.

**Table 1** Values of $\kappa$

| $T$ | Case of $a_0 = 2, a_1 = 1$ | | | Case of $a_0 = 1, a_1 = 2.5$ | | |
|---|---|---|---|---|---|---|
| | by (117) | by (119) | by (120) | by (117) | by (119) | by (120) |
| $I_2$ | 1.6180 | 1.6684 | 1.6036 | 2.8508 | 3.0000 | 2.3570 |
| $T_1$ | – | 1.5882 | 1.5882 | – | 2.2361 | 2.2836 |
| $T_2$ | – | 1.1230 | 1.1230 | – | 2.2361 | 2.2836 |

### 6.2 Case of a single delay in $B(\cdot)$

Let us now start with the simpler case of $\gamma = 1$. In this case, the constraint (113) turns into

$$Ty(t-\sigma(t)) \in \mathbb{F},\ \forall t \geqslant \sigma_{\max} \text{ and some } \sigma(t) \geqslant 0, \tag{123}$$

where

$$\sigma_{\max} = \sup\{\sigma(t), t \geqslant 0\}.$$

It is well-known that, when the linear system (112) is stable and $v = 0$, there holds

$$\|Ty(t)\| \leqslant \kappa \|y(0)\|,\ t \geqslant 0, \tag{124}$$

for some scalar $\kappa > 0$.

**Theorem 4.** Let $v = 0$, $d$ be a positive scalar, $T \in \mathbb{R}^{m\times n}$ be a given matrix, $\mathbb{F} \subset \mathbb{R}^m$ be a region containing the origin, and

$$\mathbb{B} = \{z \mid \|z\| \leqslant d,\ z \in \mathbb{R}^m\} \subset \mathbb{F}.$$

If Eq. (124) holds for some $\kappa > 0$, then

$$y(0) \in \mathbb{B}_0 \implies Ty(t-\sigma(t)) \in \mathbb{B},\quad \forall t \geqslant 0, \tag{125}$$

where

$$\mathbb{B}_0 = \left\{y \mid \|y\| \leqslant \frac{d}{\kappa},\ y \in \mathbb{R}^n\right\}.$$

*Proof.* Note that

$$y(0) \in \mathbb{B}_0 \Longleftrightarrow \kappa\|y(0)\| \leqslant d \implies \|Ty(t)\| \leqslant d \implies \|Ty(t-\sigma(t))\| \leqslant d \Longleftrightarrow Ty(t-\sigma(t)) \in \mathbb{B}.$$

The conclusion immediately follows.

The meaning of the above result is that, in order that the constraint (123) is satisfied it is sufficient to select the system initial values within the allowable region $\mathbb{B}_0$. Furthermore, it can be easily observed that, the smaller the number $\kappa$ is, the larger the allowable region $\mathbb{B}_0$ is. Therefore, the nature in the above state constrained problem, i.e., Problem 1, is to find a gain matrix $K$ such that the closed-loop system is stable and meanwhile a smaller $\kappa$ is provided. This can be done by the well-known parametric eigenstructure assignment technique with which a parameterization of the feedback gain $K$ that stabilizes the open-loop system is given, and the design degrees of freedom can be further used to seek a smaller $\kappa$.

In the case that $v$ is a nonzero constant, the equilibrium point of the system (112) is

$$y_{\mathrm{e}} = -A_{\mathrm{c}}^{-1}Bv. \tag{126}$$

Let

$$z = y - y_{\mathrm{e}},$$

and then the system (112) is transformed into

$$\dot{z} = A_{\mathrm{c}}z. \tag{127}$$

Now the relation (124) becomes

$$\|T(y(t) - y_{\mathrm{e}})\| \leqslant \kappa\|y(0) - y_{\mathrm{e}}\|. \tag{128}$$

Now applying Theorem 4 to the system (127), we have the following more general result.

**Theorem 5.** Let $v \in \mathbb{R}^r$ be a constant vector, $y_{\rm e}$ be given by (126), $d$ be a positive scalar, $T \in \mathbb{R}^{m\times n}$, $\mathbb{F} \subset \mathbb{R}^m$, and

$$\mathbb{B} = \{z \mid \ \|z - Ty_{\rm e}\| \leqslant d,\ z \in \mathbb{R}^m\} \subset \mathbb{F}.$$

If Eq. (128) holds for some $\kappa > 0$, then

$$y(0) \in \mathbb{B}_0 \implies y(t - \sigma(t)) \in \mathbb{B},\quad \forall t \geqslant \sigma_{\max}, \tag{129}$$

where

$$\mathbb{B}_0 = \left\{ y \mid \ \|y - y_{\rm e}\| \leqslant \frac{d}{\kappa},\ y \in \mathbb{R}^n \right\}.$$

The above result may suit the case that the set $\mathbb{F} \subset \mathbb{R}^n$ may not contain the origin.

### 6.3 Case of multiple delays in $B(\cdot)$

Let

$$\|T_j y(t - \sigma_j(t))\| \leqslant \kappa_j \|y(0)\|,\ j = 1, 2, \ldots, \gamma \tag{130}$$

hold for some $\kappa_j > 0$, $j = 1, 2, \ldots, \gamma$, and

$$T_{\rm E} = \text{Blockdiag}(T_1, T_2, \ldots, T_\gamma) \in \mathbb{R}^{m\times n\gamma}, \tag{131}$$

where

$$T_i \in \mathbb{R}^{m_i\times n}, i = 1, 2, \ldots, \gamma, \text{ and } m = m_1 + m_2 + \cdots + m_\gamma. \tag{132}$$

Then, we have

$$\|T_{\rm E} y(t - \sigma_j(t))|_{j=1\sim\gamma}\|^2 \leqslant \sum_{j=1}^{\gamma} \|T_j y(t - \sigma_j(t))\|^2 \leqslant \sum_{j=1}^{\gamma} \kappa_j^2 \|y(0)\|^2,$$

which gives

$$\|T_{\rm E} y(t - \sigma_j(t))|_{j=1\sim\gamma}\| \leqslant \kappa_{\rm s} \|y(0)\|,\quad \kappa_{\rm s} = \sqrt{\sum_{j=1}^{\gamma} \kappa_j^2}. \tag{133}$$

Therefore,

$$\|y(0)\| \leqslant \frac{d}{\kappa_{\rm s}} \Longleftrightarrow \kappa_{\rm s} \|y(0)\| \leqslant d \Longrightarrow \|T_{\rm E} y(t - \sigma_j(t))|_{j=1\sim\gamma}\| \leqslant d.$$

This gives the following result.

**Theorem 6.** Let $v = 0$, $d$ be a positive scalar, $T_{\rm E}$ be defined by (131) and (132), $\mathbb{F} \subset \mathbb{R}^m$ be a region containing the origin, and

$$\mathbb{B} = \{z \mid \ \|z\| \leqslant d,\ z \in \mathbb{R}^m\} \subset \mathbb{F}.$$

Furthermore, assume that Eq. (124) holds for some $\kappa > 0$. Then

$$y(0) \in \mathbb{B}_0 \implies T_{\rm E} y(t - \sigma_j(t))|_{j=1\sim\gamma} \in \mathbb{B},\quad \forall t \geqslant 0, \tag{134}$$

where

$$\mathbb{B}_0 = \left\{ y \mid \ \|y\| \leqslant \frac{d}{\kappa_{\rm s}},\ y \in \mathbb{R}^n \right\}.$$

Corresponding to Theorem 5, we also have the following more general result.

**Theorem 7.** Let $v \in \mathbb{R}^r$ be a constant vector, $y_{\rm e}$ be given by (126), $z_{\rm e}$ be defined as

$$z_{\rm e} = \begin{bmatrix} T_1 y_{\rm e} \\ \vdots \\ T_\gamma y_{\rm e} \end{bmatrix},$$

$d$ be a positive scalar, $T_{\rm E}$ be defined by (131) and (132), $\mathbb{F} \subset \mathbb{R}^m$, and

$$\mathbb{B} = \{z \mid \ \|z - z_{\rm e}\| \leqslant d,\ z \in \mathbb{R}^m\} \subset \mathbb{F}.$$

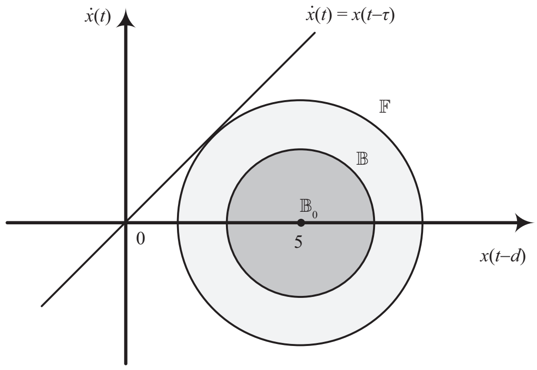


**Figure 3** The allowable state regions.

Furthermore, assume that Eq. (130) holds. Then

$$y\,(0) \in \mathbb{B}_0 \implies T_{\mathrm{E}} y\,(t-\sigma_j\,(t))\,|_{j=1\sim\gamma} \in \mathbb{B}, \quad \forall t \geqslant 0, \tag{135}$$

where

$$\mathbb{B}_0 = \left\{ y \mid \|y - y_{\mathrm{e}}\| \leqslant \frac{d}{\kappa_{\mathrm{s}}},\ y \in \mathbb{R}^n \right\}. \tag{136}$$

The above result may suit the case that the set $\mathbb{F} \subset \mathbb{R}^n$ may not contain the origin.

**Example 10.** Continuation of Example 9.

Consider again the system (64) treated in Examples 3, 6 and 9, whose sub-FAS is given by (66)–(69). Define

$$T_{\mathrm{E}} = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix},$$

with the matrices $T_1$ and $T_2$ given by (122). Then we have

$$T_{\mathrm{E}} \begin{bmatrix} x_1^{(0\sim 1)}\,(t) \\ x_1^{(0\sim 1)}\,(t-\tau) \end{bmatrix} = \begin{bmatrix} \dot{x}_1\,(t) \\ x_1\,(t-\tau) \end{bmatrix}.$$

Eventually, the state constraint can be expressed as

$$T_{\mathrm{E}} \begin{bmatrix} x_1^{(0\sim 1)}\,(t) \\ x_1^{(0\sim 1)}\,(t-\tau) \end{bmatrix} \in \mathbb{F}.$$

When $a_0 = 2$, $a_1 = 1$, it follows from the results in Table 1 that, corresponding to the matrices $T_1$ and $T_2$, the values of $\kappa$ are $\kappa_1 = 1.5882$ and $\kappa_2 = 1.1230$, respectively. Therefore,

$$\kappa_{\mathrm{s}} = \sqrt{1.5882^2 + 1.1230^2} = 1.9451.$$

Further, choosing $v = 10$ yields

$$x_{\mathrm{e}} = -\begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} v = -\frac{1}{a_0} \begin{bmatrix} 1 \\ 0 \end{bmatrix} v = \begin{bmatrix} 5 \\ 0 \end{bmatrix}.$$

An illustration is given in Figure 3. It then is clearly seen from Figure 3 that the largest radius $d = \frac{5\sqrt{2}}{2}$. Hence, it follows from (136) that the allowable initial value region is given by

$$\mathbb{B}_0 = \left\{ x \mid \|x - x_{\mathrm{e}}\| \leqslant 1.8177,\ x \in \mathbb{R}^2 \right\}. \tag{137}$$

When the time delay in the system is chosen to be

$$\tau\,(t) = 1 + \mathrm{e}^{-t},$$

and the system initial values are chosen as

$$x_1\,(s) = 1.2724,\ \dot{x}_1\,(s) = -1.2981, \quad s \in [-2, 0],$$

the simulation of the system is carried out and the result is shown in Figure 4.

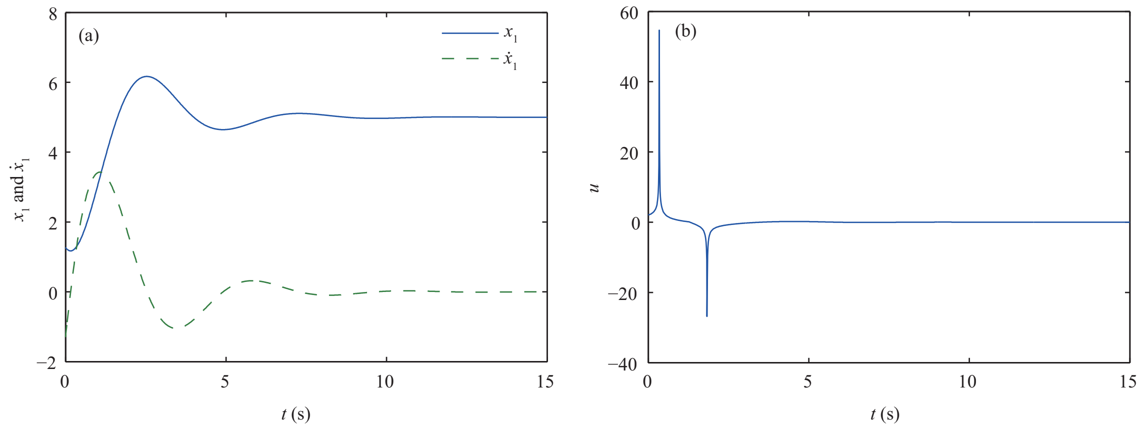


**Figure 4** (Color online) Simulation result of Example 10. (a) State variables; (b) control input.

# 7 Further generalizations

## 7.1 Generalized FAS models

In this subsection, we generalize the introduced multi-order FAS models with time delays into the case that both the nonlinear functions $f$ and $B$ containing delayed inputs. The generalized FAS model appears as follows:

$$\begin{aligned} x_k^{(\mu_k)}|_{k=1\sim\eta}(t) = &\; f\left(x_k^{(0\sim\mu_k-1)}(t-\tau_j(t))\,|_{k=1\sim\eta,\ j=1\sim\zeta}, u(t-\omega_j(t))\,|_{j=1\sim\xi}, t\right) \\ &+ B\left(x_k^{(0\sim\mu_k-1)}(t-\sigma_j(t))\,|_{k=1\sim\eta,\ j=1\sim\gamma}, u(t-\omega_j(t))\,|_{j=1\sim\xi}, t\right) u(t), \end{aligned} \tag{138}$$

where $\xi$ is an integer, and $\omega_j(t), j = 1, 2, \ldots, \xi$ are a set of positive bounded scalar functions which represents the time delays in the control input $u(t)$, while the other variables are as stated before.

Recall that $\varkappa_0 = \sum_{k=1}^{\eta} \mu_k r_k$. We still call the above system (138) a (global) FAS with time delays if the following assumption is met.

**Assumption A5.** $\det B(X, Y, t) \neq 0$ or $\infty$, $\forall X \in \mathbb{R}^{\gamma\varkappa_0}, Y \in \mathbb{R}^{r\xi}$ and $t \geqslant 0$.

Under Assumption A5, we can introduce a new control vector

$$\tilde{u}(t) = B\left(x_k^{(0\sim\mu_k-1)}(t-\sigma_j(t))\,|_{k=1\sim\eta,\ j=1\sim\gamma}, u(t-\omega_j(t))\,|_{j=1\sim\xi}, t\right) u(t), \tag{139}$$

and now the (global) FAS (138) can be written in the following standard form:

$$x_k^{(\mu_k)}|_{k=1\sim\eta}(t) = f\left(x_k^{(0\sim\mu_k-1)}(t-\tau_j(t))\,|_{k=1\sim\eta,\ j=1\sim\zeta}, u(t-\omega_j(t))\,|_{j=1\sim\xi}, t\right) + \tilde{u}(t). \tag{140}$$

Parallel to the above multi-order affine FAS (138), we can also introduce the following multi-order non-affine generalized one:

$$\begin{aligned} x_k^{(\mu_k)}|_{k=1\sim\eta}(t) = &\; f\left(x_k^{(0\sim\mu_k-1)}(t-\tau_j(t))\,|_{k=1\sim\eta,\ j=1\sim\zeta}, u(t-\omega_j(t))\,|_{j=1\sim\xi}, t\right) \\ &+ g\left(x_k^{(0\sim\mu_k-1)}(t-\sigma_j(t))\,|_{k=1\sim\eta,\ j=1\sim\gamma}, u(t-\omega_j(t))\,|_{j=1\sim\xi}, t, u(t)\right). \end{aligned} \tag{141}$$

The above system (141) is called a non-affine (global) FAS if the function $g(\cdot)$ satisfies the following assumption.

**Assumption A6.** The following mapping

$$\tilde{u}(t) = g(X, Y, t, u(t)) \tag{142}$$

forms a differential homeomorphism from $u$ to $\tilde{u}$ for all $X \in \mathbb{R}^{\gamma\varkappa_0}$, $Y \in \mathbb{R}^{r\xi}$ and $t \geqslant 0$.

Due to Assumption A6, we can similarly introduce the following control vector transformation:

$$\tilde{u}(t) = g\left(x_k^{(0\sim\mu_k-1)}(t-\sigma_j(t))\,|_{k=1\sim\eta,\ j=1\sim\gamma}, u(t-\omega_j(t))\,|_{j=1\sim\xi}, t, u(t)\right), \tag{143}$$

under which the multi-order non-affine (global) FAS (141) is also turned into the standard form of (140).

### 7.2 Controller design

As a direct extension to Theorem 2, the following important fact about the control of the multi-order FAS (138) can be given.

**Theorem 8.** Let Assumption A5 be satisfied. Further, let $[A_i]_{0\sim\mu_i-1} \in \mathbb{R}^{r_i\times\mu_i r_i},\ i=1,2,\ldots,\eta$ be a set of arbitrarily given matrices. Then the following controller for system (138):

$$\begin{cases} u(t) = -B^{-1}\left(x_k^{(0\sim\mu_k-1)}(t-\sigma_j(t))\,|_{k=1\sim\eta,\ j=1\sim\gamma}, u(t-\omega_j(t))\,|_{j=1\sim\xi}, t\right)\tilde{u}(t), \\ \tilde{u}(t) = f\left(x_k^{(0\sim\mu_k-1)}(t-\tau_j(t))\,|_{k=1\sim\eta,\ j=1\sim\zeta}, u(t-\omega_j(t))\,|_{j=1\sim\xi}, t\right) + u^*(t), \\ u^*(t) = \begin{bmatrix} [A_1]_{0\sim\mu_1-1}\, x_1^{(0\sim\mu_1-1)}(t) \\ [A_2]_{0\sim\mu_2-1}\, x_2^{(0\sim\mu_2-1)}(t) \\ \vdots \\ [A_\eta]_{0\sim\mu_\eta-1}\, x_\eta^{(0\sim\mu_\eta-1)}(t) \end{bmatrix} - v(t), \end{cases} \tag{144}$$

where $v$ is an external input vector, produces the constant linear closed-loop system (94).

Parallelly, the controller for the non-affine FAS (141) is given as

$$\begin{cases} u(t) = -g^{-1}\left(x_k^{(0\sim\mu_k-1)}(t-\sigma_j(t))\,|_{k=1\sim\eta,\ j=1\sim\gamma}, u(t-\omega_j(t))\,|_{j=1\sim\xi}, t, \tilde{u}(t)\right), \\ \tilde{u}(t) = f\left(x_k^{(0\sim\mu_k-1)}(t-\tau_j(t))\,|_{k=1\sim\eta,\ j=1\sim\zeta}, u(t-\omega_j(t))\,|_{j=1\sim\xi}, t\right) + u^*(t), \\ u^*(t) = \begin{bmatrix} [A_1]_{0\sim\mu_1-1}\, x_1^{(0\sim\mu_1-1)}(t) \\ [A_2]_{0\sim\mu_2-1}\, x_2^{(0\sim\mu_2-1)}(t) \\ \vdots \\ [A_\eta]_{0\sim\mu_\eta-1}\, x_\eta^{(0\sim\mu_\eta-1)}(t) \end{bmatrix} - v(t), \end{cases} \tag{145}$$

and the same closed-loop system as in (94) is achieved.

For the standard multi-order FAS (140), the controller (144) reduces to

$$\begin{cases} u(t) = -f\left(x_k^{(0\sim\mu_k-1)}(t-\tau_j(t))\,|_{k=1\sim\eta,\ j=1\sim\zeta}, u(t-\omega_j(t))\,|_{j=1\sim\xi}, t\right) - u^*(t), \\ u^*(t) = \begin{bmatrix} [A_1]_{0\sim\mu_1-1}\, x_1^{(0\sim\mu_1-1)}(t) \\ [A_2]_{0\sim\mu_2-1}\, x_2^{(0\sim\mu_2-1)}(t) \\ \vdots \\ [A_\eta]_{0\sim\mu_\eta-1}\, x_\eta^{(0\sim\mu_\eta-1)}(t) \end{bmatrix} - v(t), \end{cases} \tag{146}$$

which can also be written as

$$\begin{aligned} u_i(t) = &-f_i\left(x_k^{(0\sim\mu_k-1)}(t-\tau_j(t))\,|_{k=1\sim\eta,\ j=1\sim\zeta}, u(t-\omega_j(t))\,|_{j=1\sim\xi}, t\right) \\ &-[A_i]_{0\sim\mu_i-1}\, x_i^{(0\sim\mu_i-1)}(t) + v_i(t), \quad i=1,2,\ldots,\eta. \end{aligned} \tag{147}$$

To conclude this section, we remark that the concepts and results proposed in this section can also be generalized into the case of sub-FASs. Also, the concepts of controllability and stabilizability of the generalized FASs (138) and (141) can also be defined.

## 8 Concluding remarks

Nonlinear system control has already been a very difficult problem in the state-space theories. When time delays, particularly time-varying delays, are added, the problem becomes more difficult to solve within the state-space scheme.

Very recently, a different simple and effective approach has been proposed for solving nonlinear control problems, which is termed as the FAS approach [38–46, 48–50]. This paper demonstrates that the FAS approach can be effectively extended to solve the control of nonlinear time-delay systems.

State-space models are not the best ones for control problems. While the FAS models, from which the control vectors can be solved analytically and explicitly, serve as better control models.

It is shown that the general single-order and multi-order FAS and sub-FAS models for continuous-time nonlinear systems can be naturally generalized into corresponding models for continuous-time nonlinear time-delay systems. It is also demonstrated that once a general dynamical time-delay system is represented as an FAS model, rather than a state-space one, a controller for the system can be easily designed so that the closed-loop system is a constant linear one with arbitrarily assignable eigenstructure.

For a sub-FAS with time delays, the set of feasible points is defined. Like the case of global FASs with time delays, a controller for sub-FASs with time delays can be designed to transform the closed-loop systems into constant and linear ones with arbitrarily assignable eigenstructures. In contrast to global FASs with time delays, the linear closed-loop system needs to work only within the feasible set of the system. Such a requirement can be turned, as shown, into constraints on the initial values.

This paper focuses on the considered systems with only state delays, whereas the second part will address control of continuous-time systems with both state delays and input delays.

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## Appendix A Proof of Theorem 3

**Proof of conclusion (1).**

Let

$$V(x) = x^{\mathrm{T}} P x,$$

and then by the Lyapunov equation it is easily shown $\dot{V}(x) < 0$. Therefore, we have

$$V(x(t)) = x^{\mathrm{T}}(t) P x(t) \leqslant x^{\mathrm{T}}(0) P x(0) = V(x(0)). \tag{A1}$$

Since

$$\lambda_{\min}(P) \|x(t)\|^2 \leqslant x^{\mathrm{T}}(t) P x(t),$$
$$x^{\mathrm{T}}(0) P x(0) \leqslant \lambda_{\max}(P) \|x(0)\|^2,$$

we have from (A1)

$$\lambda_{\min}(P) \|x(t)\|^2 \leqslant \lambda_{\max}(P) \|x(0)\|^2,$$

which gives

$$\|x(t)\| \leqslant \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \|x(0)\| = \sqrt{\|P\| \|P^{-1}\|} \|x(0)\|.$$

With this the first conclusion is proven.

**Proof of conclusion (2).**

Let

$$A_{\rm c} = V\Lambda V^{-1},$$

where $\Lambda$ is a diagonal matrix with the diagonal elements being the eigenvalues of the matrix $A_{\rm c}$, and $V$ the eigenvector matrix. Then we have

$$x(t) = {\rm e}^{A_{\rm c}t}x(0) = V{\rm e}^{\Lambda t}V^{-1}x(0), \tag{A2}$$

which gives

$$\begin{aligned} \|Tx(t)\| &\leqslant \|TV\| \left\|V^{-1}\right\| \left\|{\rm e}^{\Lambda t}\right\| \|x(0)\| \\ &\leqslant \|TV\| \left\|V^{-1}\right\| \|x(0)\|. \end{aligned}$$

With this the second conclusion is also proven.

**Proof of conclusion (3).**

It follows from (A2) that

$$\begin{aligned} \|Tx(t)\| &\leqslant \left\|TV{\rm e}^{\Lambda t}V^{-1}\right\|_F \|x(0)\| \\ &= \left\|{\rm vec}\left(TV{\rm e}^{\Lambda t}V^{-1}\right)\right\| \|x(0)\|. \end{aligned} \tag{A3}$$

Noting that

$$\begin{aligned} {\rm vec}\left(TV{\rm e}^{\Lambda t}V^{-1}\right) &= \left[V^{-{\rm T}} \otimes (TV)\right] {\rm vec}\left({\rm e}^{\Lambda t}\right) \\ &= {\rm bcs}\left(V^{-{\rm T}} \otimes (TV)\right) \begin{bmatrix} {\rm e}^{\lambda_1 t} \\ {\rm e}^{\lambda_2 t} \\ \vdots \\ {\rm e}^{\lambda_n t} \end{bmatrix}, \end{aligned} \tag{A4}$$

where $\lambda_i, i = 1, 2, \ldots, n$ are the eigenvalues of matrix $A_{\rm c}$, we have

$$\begin{aligned} {\rm vec}\left(TV{\rm e}^{\Lambda t}V^{-1}\right) &\leqslant \left\|{\rm bcs}\left(V^{-{\rm T}} \otimes (TV)\right)\right\|_* \left\| \begin{bmatrix} {\rm e}^{\lambda_1 t} \\ {\rm e}^{\lambda_2 t} \\ \vdots \\ {\rm e}^{\lambda_n t} \end{bmatrix} \right\| \\ &\leqslant \left\|{\rm bcs}\left(V^{-{\rm T}} \otimes (TV)\right)\right\|_* \sqrt{\sum_{i=1}^{n} {\rm e}^{2\lambda_i t}} \\ &\leqslant \left\|{\rm bcs}\left(V^{-{\rm T}} \otimes (TV)\right)\right\|_* \sqrt{n}, \end{aligned}$$

where * represents either 2 or F.

Combining the above relation with (A3) immediately proves this conclusion.