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Distributed optimization with Markovian switching targets and stochastic observation noises with applications to DC microgrids

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Abstract A distributed optimization problem with Markovian switching targets and stochastic observation noises is considered in this paper. In order to solve target following and renewable following for microgrid (MG) optimal power balancing, and to attenuate observation noises simultaneously, distributed optimization algorithms are developed. The interaction between observation noises and Markovian switching targets may introduce a fundamental tradeoff in reducing the optimization errors and choosing the step size. Furthermore, under infrequent Markovian switching assumptions, the mean-square optimization error bounds, the switching ordinary differential equation (ODE) limit, and the asymptotic distributions of the optimization errors are established rigorously and comprehensively. A simulation example on a DC MG is presented to show the main results of the paper.

Keywords constraint optimization, noisy observation, output variation, Markovian switching target, distributed algorithm, DC microgrid

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1 Introduction

Distributed estimation, optimization, and control problems have been widely studied in many research areas [1–10]. This paper considers a distributed and contained optimization problem, which is motivated by distributed optimal target tracking problems in DC microgrids (MGs). This optimization problem aims to achieve optimal tradeoff among target tracking (load allocation), power loss reduction, and voltage management. DC MGs are widely used in distribution-level smart grids. Many different optimization algorithms have been developed for DC MG power management problems [11–16], but in these studies stochastic target variations have not been considered.

Distributed optimal target tracking in DC MGs is a highly challenging problem [17, 18]. Constant targets were considered in our previous studies [19–21], by considering a deterministic framework for distributed optimal target tracking. Since the MGs with different target types and distributed renewable generators may contain large target variations, we further studied the impact of stochastic target variations on the optimization error of distributed optimal target tracking algorithms in [22, 23], and optimal step size selection problems in [24].

In [22, 24], we assumed that the target variation is a stochastic process, which includes zero-mean noise, mixing, and martingale difference sequences. Although it is common to use the aforementioned assumptions, to accommodate more practical scenarios, this paper employs discrete-time Markov chain models for the randomly switching targets. Markov chain [25, 26] is a powerful mathematical tool that

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has been extensively used to represent stochastic processes of systems switching among different states. In this case, the time-varying target process takes a finite number of values. The target takes one value from a finite state space at any given time. It stays at this value for a random duration, and jumps into a new value. Next, it stays in this new value for a random duration, and this process continues over time. Many applications such as machine learning, system identification, manufacturing processes, network systems [27–32] used Markov chain to model hybrid systems. These existing results do not deal with the interaction between Markovian switching and observation noises in optimization problems.

This paper considers optimization problems for tracking Markovian switching targets with observation noises in network systems. Distributed optimization algorithms are studied. The main contributions of the paper are given as follows:

(1) Modeling uncertain and typical load variations, demand changes, and optimization targets in DC MGs as Markov chains;

(2) Developing distributed optimization algorithms to track Markovian switching targets and to attenuate observation noises at the same time;

(3) Revealing that the interaction between Markovian switching targets and noisy observations introduces a fundamental tradeoff between reducing optimization errors and choosing the step size of iterative optimization algorithms;

(4) Establishing mean-square optimization error bounds, the switching ordinary differential equation (ODE) limit, and asymptotic distributions of optimization errors rigorously and comprehensively under infrequently Markovian switching target assumptions.

The remainder of the paper is arranged as follows. Section 2 presents the problem formulation, followed by the distributed optimization algorithm in Section 3. The fundamental tradeoff in choosing the step size is given in Section 4. The main results including the mean squares order estimates, asymptotic distributions, and properties of the scaled tracking error sequences are established in Section 5. Section 6 presents simulation studies. Section 7 concludes the paper with a summary of the main findings of this paper and some potential future directions.

2 Problem formulation

This paper investigates a general distributed and constrained optimization problem that can be specified to many application domains. For concreteness and readability, power systems used in this paper are often for interpreting abstract concepts. Throughout the paper, for a matrix $X \in \mathbb{R}^{m\times n}$, $\|X\| = (\lambda_{\max}\{XX^{\mathrm{T}}\})^{\frac{1}{2}}$ denotes the Euclidean norm, and $\lambda_{\max}\{\cdot\}$ denotes the largest eigenvalue. $\mathbb{E}[\cdot]$ denotes the expectation operator, and I_m denotes the $m \times m$-dimensional identity matrix.

Consider a complex system that consists of n agents. Each agent can be a renewable generating site, a community, or a charging station. For $i \in \{1, \dots, n\}$, let u^i denote the local input, and ℓ^i denote the local target. The target can be measured locally but cannot be controlled. It commonly represents a consolidation of all uncontrollable assets. The target is exemplified by time varying loads on a bus of a power grid. The use of time-varying targets implies that the optimal solutions are also time varying. Note that x^{ij} denotes the flow from node i to node j. Also, $u \in \mathbb{R}^n$, $\ell \in \mathbb{R}^n$, $x \in \mathbb{R}^m$ denote the vector forms which containing the control, target, and network state, respectively. An undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ is used to describe the cyber-physical network topology, where $\mathcal{V}$ and $\mathcal{E}$ denote the set of nodes and edges, respectively. This network topology determines the "neighbors" that relate the control input u, the target ℓ, and the network state x. Since we assume that the cyber network is identical to the physical network, within this topology each agent i only communicates with its neighbors.

In many practical applications, the target ℓ_k is a time-varying process. In [22], we assumed that the variation $\delta_{k+1} = \ell_{k+1} - \ell_k$ is a stochastic process, which includes zero-mean noise, mixing, and martingale difference sequences. Although it is common to use such assumptions, to accommodate more practical scenarios, this paper treats ℓ_k as a function of a discrete-time Markov chain.

Assumption 1. Let $\ell_k = \ell(\alpha_k)$ be modulated by a Markovian switching process α_k taking values in a finite set $\mathcal{S} = \{1, 2, \dots, s\}$ with a stationary transition probability matrix (TPM) $P = [p^{ij}] \in \mathbb{R}^{s\times s}$:

$$p^{ij} = P\{\alpha_{k+1} = j | \alpha_k = i\}. \tag{1}$$

The Markov chain α_k is irreducible and aperiodic.

Remark 1. By Assumption 1, there is a $\nu = [\nu_1, \ldots, \nu_s] \in \mathbb{R}^{1\times s}$ with $0 < \nu_i < 1$ for $i = 1, \ldots, s$ such that the n-step transition matrix P^n satisfies

$$P^n \to \mathbb{1}\nu \text{ as } n \to \infty \text{ and } |P^n - \mathbb{1}\nu| \leqslant K\lambda^n, \tag{2}$$

for some $0 < \lambda < 1$, where $\mathbb{1}$ is a column vector with all components being 1. The ergodicity indicates that the n-step transition matrix has a limit with identical rows and the convergence is exponentially fast.

2.1 Performance index

We consider the following performance index that describes power loss between subsystems in power system applications:

$$f(u, x) = \sum_{i\in\mathcal{V}} f^i(u^i) + \sum_{(i,j)\in\mathcal{E}} f^{ij}(x^{ij}). \tag{3}$$

The objective is to minimize the performance index with respect to x and u, i.e., $\min_{u,x} f(u, x)$. The equality constraints $h(u, x, \ell) = 0$ represent the power balancing and other physical network constraints. Since the target ℓ_k is assumed to be a Markovian switching process, the optimal solution for (3) is time-varying and tracks the property of ℓ_k.

Assumption 2. (1) $f^i(u^i)$ and $f^{ij}(x^{ij})$ are smooth and strongly convex.

(2) $\nabla h_u = \frac{\partial h(u,x,\ell)}{\partial u}$ is of rank n. This implies that for any given x and ℓ, u is uniquely determined. This may be symbolically written as $u = g(x, \ell)$. This is a cyber network property and is used in communication and computation. This function is distributed in the sense that u^i is a function of the variables from its own and its neighbors.

(3) $\nabla h_x = \frac{\partial h(u,x,\ell)}{\partial x}$ is of rank m. This implies that for any given u and ℓ, the physical system will generate unique values of x. This is the physical property of the system.

Similar to [22], for concrete technical results, we study the following quadratic performance measure:

$$f(u, x) = \frac{1}{2}(x^{\mathrm{T}}Sx + u^{\mathrm{T}}Ru), \tag{4}$$

where $S = \mathrm{diag}\{S^{ij}\} \in \mathbb{R}^{m\times m}$ and $R = \mathrm{diag}\{R^i\} \in \mathbb{R}^{n\times n}$ are diagonal and positive definite, and $f^i(u^i) = R^i(u^i)^2/2$ and $f^{ij}(x^{ij}) = S^{ij}(x^{ij})^2/2$. The physical system constraints have a linear structure for typical DC MGs given in [20], i.e., $u = Ax + \ell$, where $A \in \mathbb{R}^{n\times m}$. The matrix A is full column rank, because of the physical features of the system, i.e., the network is connected. Thus, Assumption 2 naturally holds, which means that the variable x is generated by the physical system uniquely with a given u.

Note that the computation of u is local when x is given. However, by $x = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}(u - \ell)$, we know that each u^i influences all x; i.e., the calculation of x needs global information from u. But there is no need to calculate x since we can measure x directly. This structure allows us to design distributed algorithms.

2.2 Global optimality conditions

Under Assumption 2, for any given ℓ, the global optimality condition is the following stationarity condition:

$$\begin{cases} \nabla f_u + \widetilde{\lambda}^{\mathrm{T}}\nabla h_u = 0, \\ \nabla f_x + \widetilde{\lambda}^{\mathrm{T}}\nabla h_x = 0, \\ h(u, x, \ell) = 0, \end{cases} \tag{5}$$

where $\widetilde{\lambda}$ is the Lagrange multiplier.

For a linear quadratic problem in (4), the Lagrange function is given as follows:

$$L(u, x, \lambda) = \frac{1}{2}(x^{\mathrm{T}}Sx + u^{\mathrm{T}}Ru) + \widetilde{\lambda}^{\mathrm{T}}(u - Ax - \ell), \tag{6}$$

and the stationarity conditions is

$$\begin{cases}\nabla_x L(u,x,\widetilde{\lambda}) = Sx - A^{\mathrm{T}}\widetilde{\lambda} = 0,\\ \nabla_u L(u,x,\widetilde{\lambda}) = Ru + \widetilde{\lambda} = 0,\\ \nabla_{\widetilde{\lambda}} L(u,x,\widetilde{\lambda}) = u - Ax - \ell = 0.\end{cases}$$

Then the optimal solution is

$$x^* = -M^{-1}A^{\mathrm{T}}R\ell, \quad u^* = Ax^* + \ell, \quad \widetilde{\lambda}^* = -Ru^*,$$

where $M = S + A^{\mathrm{T}}RA \in \mathbb{R}^{m\times m}$.

We need to obtain the inverse of the matrix $M = S + A^{\mathrm{T}}RA$ in order to obtain the optimal solution, which is unfeasible in a distributed manner. In this paper, we will develop a distributed optimization algorithm for updating x and then calculate u accordingly. This implies that each subsystem only needs access to its own and its neighbors' states. We first replace u by x and obtain the following performance index:

$$J(x) = \frac{1}{2}(x^{\mathrm{T}}Sx + (Ax+\ell)^{\mathrm{T}}R(Ax+\ell)), \tag{7}$$

whose gradient is $\nabla_x J(x) = Sx + A^{\mathrm{T}}R(Ax+\ell) = Mx + B\ell$, where $B = A^{\mathrm{T}}R$.

3 Distributed optimization algorithm

This paper treats the load ℓ_k as a time-varying and random variable (see Assumption 1). Following the common practice of pseudo steady-state methods in power systems, we carry out the optimization task with the following objective function conditioned on ℓ_k:

$$\min_x J_k(x) = \mathbb{E}\left[\frac{1}{2}(x^{\mathrm{T}}Sx + (Ax+\ell_k)^{\mathrm{T}}R(Ax+\ell_k))|\mathcal{F}_k\right] = \min_x \frac{1}{2}(x^{\mathrm{T}}Sx + (Ax+\ell_k)^{\mathrm{T}}R(Ax+\ell_k)), \tag{8}$$

where $\mathcal{F}_k$ is the σ-algebra generated by $\{\ell_j : j \leqslant k\}$.

The gradient of this performance index is

$$\nabla_x J_k(x) = Sx + A^{\mathrm{T}}R(Ax+\ell_k) = Mx + B\ell_k,$$

and its optimal solution is

$$x_k^* = -M^{-1}B\ell_k = -M^{-1}B\ell(\alpha_k). \tag{9}$$

As the global optimal solution is not available, the optimal solution must be obtained by iterative searching with local communications among agents. Such communications naturally lead to gradient errors. Consequently, the gradient-based algorithm is

$$x_{k+1} = x_k - \mu(Mx_k + B\ell_k + d_k), \tag{10}$$

where d_k is the observation/computation noise for the gradient, and $\mu > 0$ is the step size. Moreover, it is apparent that the above algorithm is strictly distributed because the update of x_{k+1}^{ij} only involves the information from its own and its neighbors, i.e., $x_k^{ij}, x_k^{iq}, x_k^{pj}$ (q is the neighbor of i, and p is the neighbor of j), and $\ell_k^i, \ell_k^j, S^{ij}, R^i, R^j$.

Noting that ℓ_k is a random process, thus the optimal solution x_k^* conditioned on ℓ_k is also a stochastic process and is sample-path dependent. In addition, the optimization error will be

$$e_k = x_k - x_k^* = x_k + M^{-1}B\ell_k, \tag{11}$$

and have the following relationship:

$$\begin{aligned} e_{k+1} &= x_{k+1} + M^{-1}B\ell_{k+1}\\ &= x_k - \mu(Mx_k + B\ell_k + d_k) + M^{-1}B\ell_{k+1}\\ &= x_k + M^{-1}B\ell_k - \mu[M(x_k + M^{-1}B\ell_k) + d_k] + M^{-1}B\ell_{k+1} - M^{-1}B\ell_k \end{aligned}$$

$$
\begin{aligned}
&= e_k - \mu(Me_k + d_k) + M^{-1}B(\ell_{k+1} - \ell_k) \\
&= (I - \mu M)e_k - \mu d_k + M^{-1}B\delta_{k+1} \\
&= (I - \mu M)^{k+1}e_0 - \mu\sum_{i=0}^{k}(I - \mu M)^{k-i}d_i + \sum_{i=0}^{k}(I - \mu M)^{k-i}M^{-1}B\delta_{i+1},
\end{aligned} \tag{12}
$$

where $\delta_{k+1} = \ell_{k+1} - \ell_k$.

Lemma 1 ([22])**.** There exist $\mu^* > 0$ and $c > 0$, such that for any $\mu \in (0, \mu^*)$, we have $0 \leqslant 1 - \mu c < 1$ and $\|(I - \mu M)^k\| \leqslant (1 - \mu c)^k$.

In Sections 4–6, we assume that $\mu \in (0, \mu^*)$ holds. Note that all proofs are provided in the appendixes for easy reading.

4 Fundamental tradeoff

Since the step size μ can be designed, a fundamental tradeoff in choosing μ exists in practice. First, the step size μ should be reduced to zero gradually in order to reduce the noise effect, and then the proposed optimization algorithm may converge to the optimal solution. But, x_k may be unable to track time-varying targets if the step size μ approaches zero. We will show the fundamental tradeoff in this section under the following assumption on Markovian switching targets and stochastic observation noises.

Assumption 3. The following properties hold on targets and stochastic observation noises:

(1) $e_0 \in \mathbb{R}^m$, $d_k \in \mathbb{R}^m$, and $\ell_k \in \mathbb{R}^n$ are mutually independent.

(2) $\{d_k\}$ is i.i.d with $\mathbb{E}[d_k] = \mathbf{0}_{m\times 1} \in \mathbb{R}^m$ and $\mathbb{E}[d_k d_k^{\mathrm{T}}] = \Sigma^d \in \mathbb{R}^{m\times m}$.

Now, let $\eta_k = \mathbb{E}[e_k]$ denote the error mean, and $\Sigma_k = \mathbb{E}[e_k e_k^{\mathrm{T}}]$ denote the error variance. In Subsections 4.1 and 4.2, we establish the properties of η_∞ and Σ_∞ in order to show the fundamental tradeoff for choosing the step size μ.

4.1 Analysis of η_k

By (12) and Assumption 3, we have

$$
\eta_{k+1} = (I - \mu M)^{k+1}\eta_0 + \sum_{i=0}^{k}(I - \mu M)^{k-i}M^{-1}B\mathbb{E}[\delta_{i+1}]. \tag{13}
$$

Concerning the Markov chain, using repeated applications of conditional expectation and the Markov property, it is readily seen that

$$
\mathbb{E}[\alpha_i] = \sum_{i_0, i_1, \ldots, i_j \in \mathcal{S}} p^{i_j, i} p^{i_{j-1}, i_j} \cdots p^{i_0, i_1} p^{i_0},
$$

where p^{i_0} denotes $P\ (\alpha_0 = i_0)$.

We note that the computation of $\mathbb{E}[\delta_{i+1}]$ depends crucially on that of $\mathbb{E}[\alpha_{i+1} - \alpha_i]$. To simplify the notation, we use the transition matrices. The calculation of the δ_{i+1} essentially involves the difference of the transition matrix $P^{i+1} - P^i$. It is readily seen that $P^{i+1} - P^i \to 0$ exponentially fast as $i \to \infty$. Therefore,

$$
\mathbb{E}[\delta_{i+1}] \to \mathbf{0}_{n\times 1} \text{ as } i \to \infty. \tag{14}
$$

Note that the proof of Theorem 1 is provided in Appendix A.

Theorem 1. Under Assumptions 1 and 3, the mean of error η_k tends to $\mathbf{0}_{m\times 1}$.

4.2 Analysis of Σ_k

Note that

$$
\begin{aligned}
\Sigma_{k+1} &= \mathbb{E}[e_{k+1}e_{k+1}^{\mathrm{T}}] \\
&= (I - \mu M)^{k+1}\mathbb{E}[e_0 e_0^{\mathrm{T}}](I - \mu M)^{k+1}
\end{aligned}
$$

$$+ \mu^2 \mathbb{E}\Bigg[\Bigg(\sum_{i=0}^{k}(I-\mu M)^{k-i}d_i\Bigg)\cdot\Bigg(\sum_{i=0}^{k}(I-\mu M)^{k-i}d_i\Bigg)^{\mathrm{T}}\Bigg]$$
$$+ \mathbb{E}\Bigg[\Bigg(\sum_{i=0}^{k}(I-\mu M)^{i}M^{-1}B\delta_{k-i+1}\Bigg)\cdot\Bigg(\sum_{i=0}^{k}(I-\mu M)^{i}M^{-1}B\delta_{k-i+1}\Bigg)^{\mathrm{T}}\Bigg]. \qquad (15)$$

Let

$$D_k = \mu\sum_{i=0}^{k}(I-\mu M)^{k-i}d_i.$$

By Assumption 3, since d_i is i.i.d. and zero mean, its variance is

$$\Sigma_k^D = \mathbb{E}[D_kD_k^{\mathrm{T}}] = \mu^2\mathbb{E}\Bigg[\Bigg(\sum_{i=0}^{k}(I-\mu M)^{k-i}d_i\Bigg)\cdot\Bigg(\sum_{i=0}^{k}(I-\mu M)^{k-i}d_i\Bigg)^{\mathrm{T}}\Bigg]$$
$$= \mu^2\sum_{i=0}^{k}(I-\mu M)^{i}\mathbb{E}[d_0d_0^{\mathrm{T}}](I-\mu M)^{i}$$
$$= \mu^2\sum_{i=0}^{k}(I-\mu M)^{i}\Sigma^d(I-\mu M)^{i}, \qquad (16)$$

as a result

$$\|\Sigma_k^D\| \leqslant K\mu^2\sum_{i=0}^{\infty}(1-\mu c)^i = O(\mu). \qquad (17)$$

Here let K denote a general positive constant, and K may be different for different cases. Thus, the notation $K+K=K$ and $KK=K$ is understood in an appropriate sense.

It remains to derive the variance of

$$H_k = \sum_{i=0}^{k}(I-\mu M)^{i}M^{-1}B\delta_{k-i+1}. \qquad (18)$$

In fact,

$$\mathbb{E}[H_kH_k^{\mathrm{T}}] = \sum_{j=0}^{k}\sum_{i=0}^{k}(I-\mu M)^{i}M^{-1}B\mathbb{E}[\delta_{k-i+1}\delta_{k-j+1}^{\mathrm{T}}]B^{\mathrm{T}}M^{-1}(I-\mu M)^{j}. \qquad (19)$$

Since α_k is an ergodic Markov chain, it is a bounded ϕ mixing process with mixing rate $O(\lambda^k)$ [33, p. 168]. Consequently,

$$\mathbb{E}[H_kH_k^{\mathrm{T}}] = \widetilde{H}_{k,1} + \widetilde{H}_{k,2},$$

where

$$\widetilde{H}_{k,1} = \sum_{j=0}^{k}\sum_{i=0}^{k}(I-\mu M)^{i}M^{-1}B(\mathbb{E}[\delta_{k-i+1}\delta_{k-j+1}^{\mathrm{T}}] - \mathbb{E}[\delta_{k-i+1}]\mathbb{E}[\delta_{k-j+1}^{\mathrm{T}}])B^{\mathrm{T}}M^{-1}(I-\mu M)^{j},$$
$$\widetilde{H}_{k,2} = \sum_{j=0}^{k}\sum_{i=0}^{k}(I-\mu M)^{i}M^{-1}B\mathbb{E}[\delta_{k-i+1}]\mathbb{E}[\delta_{k-j+1}^{\mathrm{T}}]B^{\mathrm{T}}M^{-1}(I-\mu M)^{j}. \qquad (20)$$

Then, using the well-known mixing inequality [33, p. 170], we have

$$\|\widetilde{H}_{k,1}\| \leqslant K\sum_{j=0}^{\infty}(1-\mu c)^j\sum_{i\geqslant j}\lambda^{i-j} = O(1/\mu). \qquad (21)$$

As for $\widetilde{H}_{k,2}$, first note that

$$\|\mathbb{E}[\delta_{k-i+1}]\| = \|\mathbb{E}[\delta_{k-i+1}] - \mathbb{E}_\nu[\delta_{k-i+1}]\| \leqslant K\lambda^{k-i+1}, \qquad (22)$$

where $\mathbb{E}_\nu[\delta_{k-i+1}]$ denotes the expectation with respect to the stationary measure (and in fact is equal to 0). Thus,

$$\|\widetilde{H}_{k,2}\| \leqslant K\sum_{j=0}^{\infty}\lambda^j\sum_{i=0}^{\infty}\lambda^i = O(1). \tag{23}$$

Thus, Eq. (20) yields

$$\mathbb{E}[H_kH_k^{\mathrm{T}}] = O(1/\mu). \tag{24}$$

In summary, by comparing (17) and (24), we can obtain the fundamental tradeoff in selecting the step size: when $\mu \to 0$, $\mathbb{E}[H_kH_k^{\mathrm{T}}]$ is of the order $O(1/\mu)$, whereas Σ_k^D is of the order $O(\mu)$. Thus, if the step size μ decreases, the tracking error caused by the measurement noise, i.e., Σ_k^D, will decrease. But the tracking error caused by the load variation, i.e., $\mathbb{E}[H_kH_k^{\mathrm{T}}]$, will increase, and vice versa. This indicates the tradeoff between tracking ability and noise sensitivity.

5 Convergence analysis

The tradeoff results on the step size μ only hold on the stationary distribution of the Markov chain, which satisfies Assumptions 1 and 3. However, the convergence rate of the distributed optimization algorithm may depend on the property of matrix P. We now consider infrequently jumping Markovian switching targets, and show the impact of P on the convergence results.

5.1 Mean-square optimization error bounds

Assumption 4. For the Markovian switching process, suppose that there is a small constant $\varepsilon > 0$ such that TPM is

$$P^\varepsilon = I_s + \varepsilon Q, \tag{25}$$

where $Q = [q^{ij}] \in \mathbb{R}^{s\times s}$ is the generator of an irreducible continuous-time Markov chain, which satisfies $\sum_{j=1}^s q^{ij} = 0$ for each $i = 1,\ldots,s$, and $q^{ij} \geqslant 0$ for $i \neq j$. Also, we assume that $P(\alpha_0 = i) = p_0^i$ for each $i = 1,\ldots,s$, where $p_0^i \geqslant 0$ and $\sum_{i=1}^s p_0^i = 1$.

Note that the small constant $\varepsilon > 0$ ensures that the identity matrix I_s dominates. As a result, the process does not switch much frequently. In the following part, we will establish the relationship between ε and the optimization error e_k to show the impact of the matrix P^ε on the convergence rates.

Theorem 2. Under Assumptions 3 and 4, there exists a $K_\mu > 0$ such that for all $k \geqslant K_\mu$,

$$\mathbb{E}[\|e_k\|^2] = O\left(\mu + \varepsilon + \frac{\varepsilon^2}{\mu}\right). \tag{26}$$

Remark 2. The proof of Theorem 2 is given in Appendix B. Note that for the step size μ, there are three possible choices: (1) $\mu = \varepsilon$, (2) $\mu \gg \varepsilon$, and (3) $\mu \ll \varepsilon$. For Case (1), the target and the distributed algorithm update at the similar rate. In this case, $\mathbb{E}[\|e_k\|^2] = O(\mu) = O(\varepsilon)$ holds. For Case (2), the optimization rate of the distributed algorithm is faster than the Markov chain dynamics. Hence, the target process is essentially a constant. We assume that $\varepsilon = \mu^{1+\Delta}$ for some $\Delta > 0$. Then, $\mathbb{E}[\|e_k\|^2] = O(\mu) = O(\varepsilon^{1/(1+\Delta)})$ holds. For Case (3), the Markov chain switches very fast, i.e., the target converges quickly to the stationary distribution. Thus, we assume that $\varepsilon = \mu^\gamma$ for some $1/2 < \gamma < 1$. Then, $\mathbb{E}[\|e_k\|^2] = O(\mu^\gamma \wedge \mu^{2\gamma-1})$, where $a \wedge b = \min\{a, b\}$ for $a, b \in \mathbb{R}$.

5.2 Switching ODE limit

We now develop the asymptotic properties of the distributed optimization algorithm. To proceed, we define piecewise constant interpolation processes for $t \in [\mu k, \mu k + \mu)$ as follows:

$$\begin{cases} \alpha_\mu(t) = \alpha_k, \\ x_\mu(t) = x_k, \\ x_\mu^*(t) = x_k^* = -M^{-1}B\ell(\alpha_k). \end{cases} \tag{27}$$

The asymptotic behavior is different for the above three cases. The proof uses singularly perturbed Markov chain and martingale averaging techniques. Let Assumptions 3 and 4 hold for Theorems 3–8.

Case (1): $\varepsilon = \mu$**.** For this case, due to the presence of the Markovian property of $x^*(t)$, the limit dynamic system (29) randomly changes its regime.

Theorem 3. Assume that $\varepsilon = \mu$. Then, $(\alpha_\mu(\cdot), x_\mu(\cdot), x_\mu^*(\cdot))$ converges weakly to $(\alpha(\cdot), x(\cdot), x^*(\cdot))$ such that $\alpha(\cdot)$ is a continuous-time Markov chain generated by Q and $x^*(\cdot)$ satisfies

$$x^*(t) = -M^{-1}B\ell(\alpha(t)), \tag{28}$$

and

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = M(x^*(t) - x(t)), \quad x(0) = x_0. \tag{29}$$

Since the proofs of Theorems 3–8 are similar to those in [27, 29], we omit them here and summarize the main results. For Theorem 3, by the assumptions, the pair of stochastic processes $(x_\mu(\cdot), x_\mu^*(\cdot))$ is tight and compact. Also, it converges to $(x(\cdot), x^*(\cdot))$ that is the unique solution of a martingale problem.

Case (2): $\mu \gg \varepsilon$**.** In this case, the switching target process is basically a constant since the Markov chain switches slowly. Thus, the limiting continuous-time process satisfies an ODE.

Theorem 4. Assume that $\varepsilon = \mu^{1+\Delta}$ for some $\Delta > 0$. Then, $x_\mu(\cdot)$ converges weakly to $x(\cdot)$ such that $x(\cdot)$ is the solution of the following ODE:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = M(x^* - x(t)), \quad x(0) = x_0, \tag{30}$$

where $x^* = -M^{-1}B\sum_{i=1}^{s}\ell(i)P(\alpha_0 = i)$.

Case (3): $\mu \ll \varepsilon$**.** For this fast-varying Markov chain case, the limit dynamic system depends on the stationary distribution of the Markov chain.

Theorem 5. Assume that $\varepsilon = \mu^\gamma$ for some $1/2 < \gamma < 1$. In addition, assume Q is irreducible. Then, $x_\mu(\cdot)$ converges weakly to $x(\cdot)$ such that $x(\cdot)$ is the solution of the following ODE:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = M(\bar{x} - x(t)), \quad x(0) = x_0, \tag{31}$$

where $\bar{x} = -M^{-1}B\sum_{i=1}^{s}v^i\ell(i)$, and $v = [v^1, \ldots, v^s]$ is the stationary distribution corresponding to the continuous-time Markov chain with the generator matrix Q.

The main ideas for the proofs of Theorems 4 and 5 are summarized here. By the assumptions, the tightness and compactness of $x_\mu(\cdot)$ can be proved as in the previous case. To obtain the limit, we only need to treat the average of the term involving the Markov chain and the stationary distribution of the Markov chain, respectively.

5.3 Asymptotic distributions

Here we consider the asymptotic distribution of the optimization error sequence $\{x_k - x_k^*\}$ in this subsection. Note that the diffusion approximation limit of the tracking error under suitable scales shows the asymptotic convergence rate of the optimization algorithm. For simplicity, we omit the proof, and just state the following theorems.

Case (1): $\varepsilon = \mu$**.** Here we will show that $r_k = (x_k - x_k^*)/\sqrt{\mu}$ is asymptotically normal. From (12), we have

$$r_{k+1} = r_k - \mu M r_k - \sqrt{\mu}d_k + \frac{M^{-1}B\delta_{k+1}}{\sqrt{\mu}}. \tag{32}$$

It can be shown that there exists a positive integer K_μ such that $\{r_k : k \geqslant K_\mu\}$ is tight. To develop the asymptotic properties of the optimization error, we define the piecewise constant interpolation $r_\mu(t) = r_k$, for $t \in [\mu(k - K_\mu), \mu(k - K_\mu + 1))$. For any $t, s > 0$, let

$$r_\mu(t) = r_{K_\mu} - \sum_{k=K_\mu}^{t/\mu-1}(\mu M r_k + \sqrt{\mu}d_k) - \frac{M^{-1}B\ell(\alpha_{t/\mu}) - M^{-1}B\ell(\alpha_{K_\mu})}{\sqrt{\mu}} \tag{33}$$

and $\Sigma = \mathbb{E}[d_0 d_0^{\mathrm{T}}]$.

Theorem 6. Assume that $\varepsilon = \mu$. Then, $r_\mu(\cdot)$ converges weakly to $r(\cdot)$ such that $r(\cdot)$ is the solution of the following stochastic differential equation:

$$\mathrm{d}r = -Mr\mathrm{d}t + \Sigma^{1/2}\mathrm{d}w, \tag{34}$$

where $w(\cdot)$ is a standard Brownian motion.

We know that the tracking error $\{x_k - x_k^*\}$ is asymptotically normal with zero mean and covariance $\mu\widetilde{C}$, where $\widetilde{C}$ is the solution of the following Lyapunov equation:

$$M\widetilde{C} + \widetilde{C}M = -\Sigma. \tag{35}$$

Note that

$$\widetilde{C} = \int_0^\infty \exp(-Mt)\Sigma\exp(-Mt)\mathrm{d}t. \tag{36}$$

Case (2): $\mu \gg \varepsilon$**.** Assume that $\varepsilon = \mu^{1+\Delta}$ for some $\Delta > 0$. Define $v_k = (x_k - x^*)/\sqrt{\mu}, v_\mu(t) = v_k$, for $t \in [\mu(k - K_\mu), \mu(k - K_\mu + 1))$. The following result indicates that $x_k - x^*$ is asymptotically normal with mean 0 and covariance $\mu\widetilde{C}$.

Theorem 7. Assume that $\varepsilon = \mu^{1+\Delta}$ for some $\Delta > 0$. Then, $v_\mu(\cdot)$ converges weakly to $v(\cdot)$ such that $v(\cdot)$ is the solution of the stochastic differential equation:

$$\mathrm{d}v = -Mv\mathrm{d}t + \Sigma^{1/2}\mathrm{d}w, \tag{37}$$

where $w(\cdot)$ is a standard Brownian motion.

Case (3): $\mu \ll \varepsilon$**.** Assume that $\varepsilon = \mu^\gamma$ and $1/2 < \gamma < 1$. Let $z_k = (x_k - \bar{x})/\sqrt{\mu}, z_\mu(t) = z_k$, for $t \in [\mu(k - K_\mu), \mu(k - K_\mu + 1))$.

Theorem 8. Assume that $\varepsilon = \mu^\gamma$ for some $1/2 < \gamma < 1$. Then, $z_\mu(\cdot)$ converges weakly to $z(\cdot)$ such that $z(\cdot)$ is the solution of the stochastic differential equation:

$$\mathrm{d}z = -Mz\mathrm{d}t + \Sigma^{1/2}\mathrm{d}w, \tag{38}$$

where $w(\cdot)$ is a standard Brownian motion.

6 Simulation study

6.1 Testing system and performance measures

We use the Junbaose trolleybus network as an example to demonstrate the theoretical findings; see Figure 1. Certain details were presented in [19, 20, 22, 24]. Here we assume that the communication network topology and the physical network are the same. The line power loss matrix is $S = \mathrm{diag}\{0.4, 0.38, 0.34, 0.31, 0.36\}$, which is calculated from physical system measurements. Figure 1 shows that there are communication lines between 1 and 2, 2 and 3, 2 and 4, 4 and 5, 5 and 6. Let $u = [I^1, I^2, I^3, I^4, I^5, I^6]^\mathrm{T}$, $x = [I^{12}, I^{23}, I^{24}, I^{45}, I^{56}]^\mathrm{T}$, $\ell = [\ell^1, \ell^2, \ell^3, \ell^4, \ell^5, \ell^6]^\mathrm{T}$, and $A = [1\ 0\ 0\ 0\ 0;\ -1\ 1\ 1\ 0\ 0;\ 0\ -1\ 0\ 0\ 0;\ 0\ 0\ -1\ 1\ 0;\ 0\ 0\ 0\ -1\ 1;\ 0\ 0\ 0\ 0\ -1]$. Therefore, we have $u = Ax + \ell$.

The optimization problem has two objectives. First, the currents on the feeders need to be balanced, i.e., $I_0^i \to \bar{I}$ and $\bar{I} = \sum_{i=1}^6 I_0^i/6$; second, the line losses $\sum_{ij}^5 (I^{ij})^2 S^{ij} = x^\mathrm{T}Sx$ need to be reduced. By combining these two objectives, we form the performance index as follows:

$$\min_{x,u} J = \frac{1}{2}\left(x^\mathrm{T}Sx + (u - \bar{I}\mathbb{1})^\mathrm{T}(u - \bar{I}\mathbb{1})\right), \qquad \text{s.t.}\ \ u = Ax + \ell. \tag{39}$$

By $A^\mathrm{T}\mathbb{1} = 0$, we know that

$$J(x) = \frac{1}{2}(x^\mathrm{T}Sx + (Ax + \ell - \bar{I}\mathbb{1})^\mathrm{T}(Ax + \ell - \bar{I}\mathbb{1}))$$

and the gradient is $\nabla J_x = Sx + A^\mathrm{T}(Ax\ell) = (S + A^\mathrm{T}A)x + A^\mathrm{T}\ell$. Thus, we have

$$x^* = -(S + A^\mathrm{T}A)^{-1}A^\mathrm{T}\ell, \quad u^* = Ax^* + \ell.$$

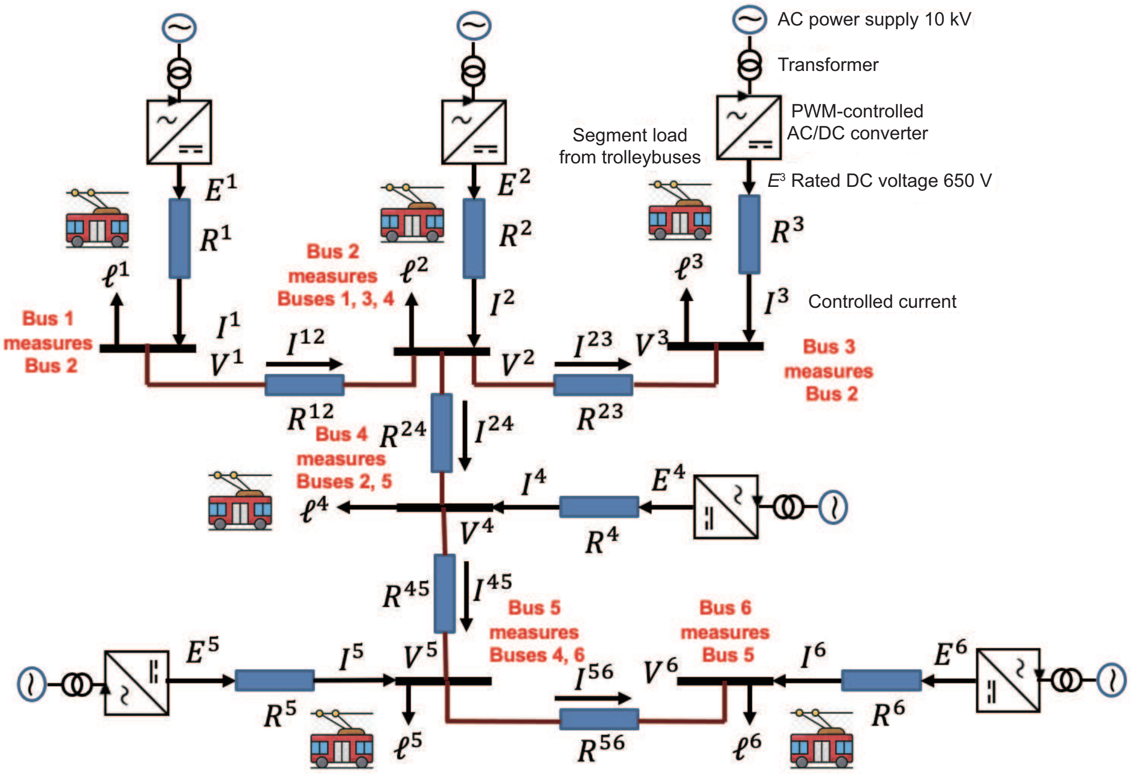

Figure 1 (Color online) DC power network of the Junbaose trolleybus system.

Here we use (10) to track the optimal solution $x^*(k)$:

$$x_{k+1} = x_k - \mu[(S + A^{\mathrm{T}}A)x_k + A^{\mathrm{T}}\ell_k + d_k],$$

where $d_k = [d_k^1, \ldots, d_k^5]^{\mathrm{T}} \in \mathbb{R}^5$. Note that the optimization algorithm is distributed by the structure of the matrix $S + A^{\mathrm{T}}A$, and the adaptation of I^{ij} only needs data from i and j. Also, at time instant k, the control input is $u_k = Ax_k + \ell_k$.

6.2 Evaluation results

In this subsection, we demonstrate convergence properties. By Lemma 1, we know that the step size $\mu \in (0, 0.4370)$ holds. We take $\mu = 0.1$ and $\varepsilon = \mu/2(\varepsilon = O(\mu))$, $\varepsilon = \mu^3$ (slowly time varying Markov chain) and $\varepsilon = \sqrt{\mu}$ (fast time varying Markov chain), respectively. Also, the state space for the target ℓ_k is $\mathcal{L} = \{\ell(1), \ell(2), \ell(3)\} = \{[400; 500; 600; 700; 800; 900], [900; 800; 700; 600; 500; 400], [500; 700; 900; 400; 600; 800]\}A$, and

$$Q = \begin{pmatrix} -0.6 & 0.4 & 0.2 \\ 0.2 & -0.5 & 0.3 \\ 0.4 & 0.1 & -0.5 \end{pmatrix}.$$

Thus, the stationary distribution of Q is $v = [1/3, 1/3, 1/3]$, and

$$\bar{x} = -M^{-1}B\sum_{i=1}^{3} v^i \ell(i) = [35.8257; 60.2734; -40.9584; 28.6050; 40.3694].$$

Let the initial distribution for ℓ_0 be $[3/4, 1/8, 1/8]$. Then we have

$$x^* = -M^{-1}B\sum_{i=1}^{3} \ell(i)P(\ell_0 = \ell(i)) = [99.5392; 25.8298; 125.5644; 132.6112; 93.2674].$$

Also, let the observation noise $d_k^i \sim N(0, 10^2)$ for $i = 1, \ldots, 6$ and $k \geqslant 0$.

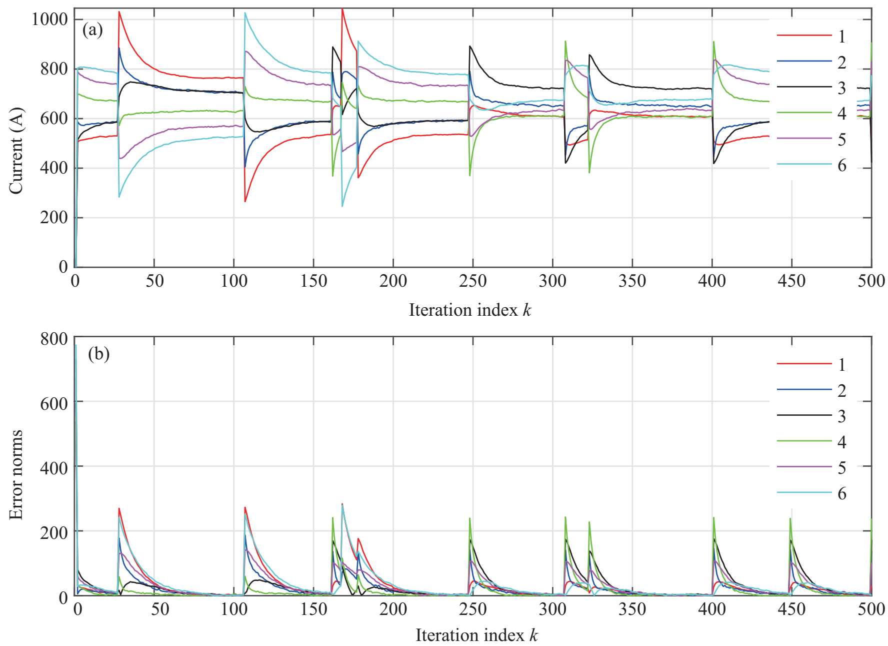

Figure 2 (Color online) Current (a) and error (b) trajectories with $\varepsilon = \mu/2$.

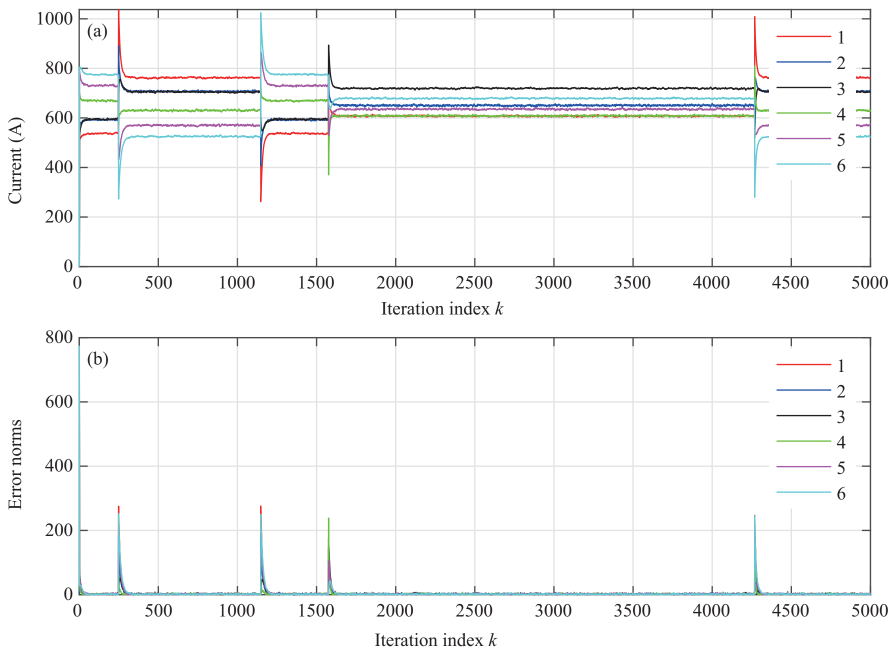

Figure 3 (Color online) Current (a) and error (b) trajectories with $\varepsilon = \mu^3$.

Let $I_0^1 = 713A, I_0^2 = 811A, I_0^3 = 960A, I_0^4 = 844A, I_0^5 = 887A, I_0^6 = 823A$. Then for each I_i, we can get the current input sequence

$$\{I_k^i, \quad k = 1, 2, \ldots, 500\}, \quad i = 1, \ldots, 6,$$

and the error sequence

$$\|I_k^i - I_k^{i,*}\|, \quad k = 1, 2, \ldots, 500, \quad i = 1, \ldots, 6,$$

where $I_k^{i,*}$ is the time-varying optimal solution. The results for $\varepsilon = \mu/2$, $\varepsilon = \mu^3$, and $\varepsilon = \sqrt{\mu}$ are shown in Figures 2–4, respectively. These simulations show that for the slowly varying case, the proposed algorithm performs quite well; i.e., each time the optimal solution jumps, the error trajectory reduces to zero quickly. When $\varepsilon = O(\mu)$, the proposed algorithm still shows a good property. As for the fast

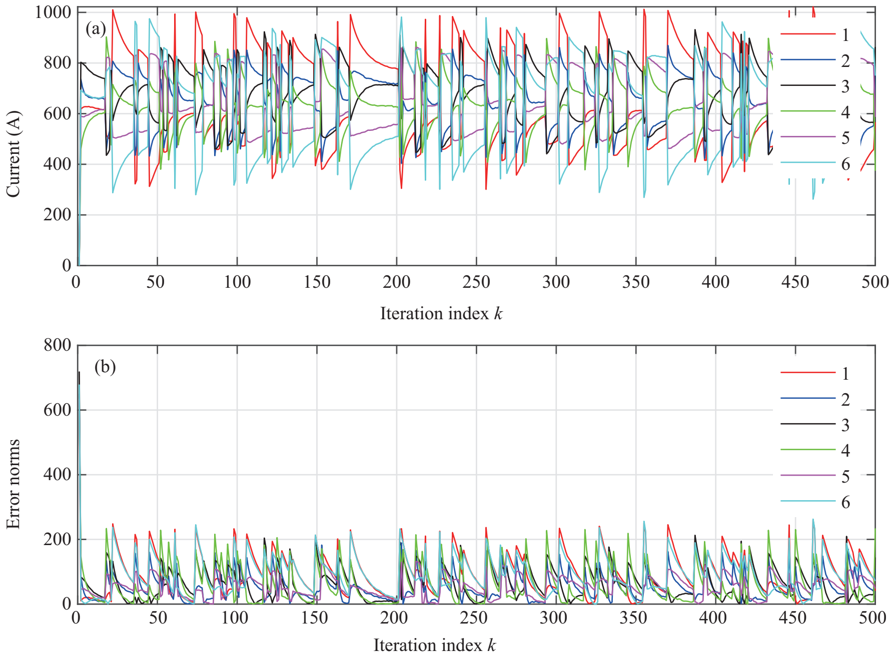

Figure 4 (Color online) Current (a) and error (b) trajectories with $\varepsilon = \sqrt{\mu}$.

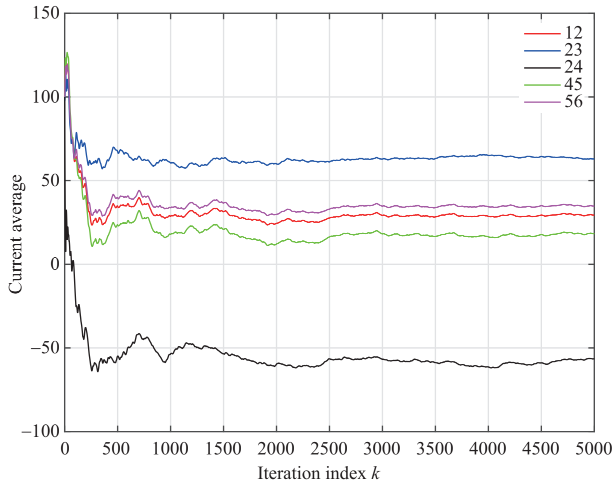

Figure 5 (Color online) Current average curves with $\varepsilon = \sqrt{\mu}$.

changing Markov chain, it is impossible to track the optimal solution. However, for the cumulative average of the sequence x_k, i.e., $(\sum_{j=1}^{k} x_j)/k$, the current average quickly converges to $\bar{x}$; see Figure 5.

7 Concluding remarks

This paper has studied a distributed constrained optimization problem that involves Markovian switching targets and stochastic observation noises. A distributed optimization algorithm has been developed to track Markovian switching targets and to attenuate observation noises simultaneously. We have analyzed the impact of Markovian switching targets on the proposed distributed optimization algorithm and the fundamental tradeoff in choosing the step size. Moreover, mean squares error optimization bounds, switching ODE limit, and asymptotic distributions have been established rigorously under an infrequently Markovian switching target assumption. Some important issues such as considering communication channel interruptions and/or more general AC power systems remain open.

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References

1 Vovos P N, Kiprakis A E, Wallace A R, et al. Centralized and distributed voltage control: impact on distributed generation penetration. IEEE Trans Power Syst, 2007, 22: 476–483
2 Nedic A, Ozdaglar A. Distributed subgradient methods for multi-agent optimization. IEEE Trans Automat Contr, 2009, 54: 48–61
3 Johansson B, Rabi M, Johansson M. A randomized incremental subgradient method for distributed optimization in networked systems. SIAM J Optim, 2010, 20: 1157–1170
4 Sayed A H. Adaptive networks. Proc IEEE, 2014, 102: 460–497
5 Yu W W, Li C J, Yu X H, et al. Economic power dispatch in smart grids: a framework for distributed optimization and consensus dynamics. Sci China Inf Sci, 2018, 61: 012204
6 Li Z H, Ding Z T. Distributed optimization on unbalanced graphs via continuous-time methods. Sci China Inf Sci, 2018, 61: 129204
7 Su S, Lin Z. Distributed consensus control of multi-agent systems with higher order agent dynamics and dynamically changing directed interaction topologies. IEEE Trans Automat Contr, 2016, 61: 515–519
8 Molzahn D K, Dörfler F, Sandberg H, et al. A survey of distributed optimization and control algorithms for electric power systems. IEEE Trans Smart Grid, 2017, 8: 2941–2962
9 Xie S Y, Guo L. A necessary and sufficient condition for stability of LMS-based consensus adaptive filters. Automatica, 2018, 93: 12–19
10 Liang S, Wang L Y, Yin G. Distributed quasi-monotone subgradient algorithm for nonsmooth convex optimization over directed graphs. Automatica, 2019, 101: 175–181
11 Wang Z, Liu F, Chen Y, et al. Unified distributed control of stand-alone DC microgrids. IEEE Trans Smart Grid, 2019, 10: 1013–1024
12 Khorsandi A, Ashourloo M, Mokhtari H. A decentralized control method for a low-voltage DC microgrid. IEEE Trans Energy Convers, 2014, 29: 793–801
13 Baranwal M, Askarian A, Salapaka S, et al. A distributed architecture for robust and optimal control of DC microgrids. IEEE Trans Ind Electron, 2019, 66: 3082–3092
14 Nasirian V, Moayedi S, Davoudi A, et al. Distributed cooperative control of DC microgrids. IEEE Trans Power Electron, 2015, 30: 2288–2303
15 Wang C, Duan J, Fan B, et al. Decentralized high-performance control of DC microgrids. IEEE Trans Smart Grid, 2019, 10: 3355–3363
16 Trip S, Cucuzzella M, Cheng X, et al. Distributed averaging control for voltage regulation and current sharing in DC microgrids. IEEE Control Syst Lett, 2019, 3: 174–179
17 Hosseinzadeh M, Salmasi F R. Power management of an isolated hybrid AC/DC micro-grid with fuzzy control of battery banks. IET Renew Power Generation, 2015, 9: 484–493
18 Dragičević T, Guerrero J M, Vasquez J C, et al. Supervisory control of an adaptive-droop regulated DC microgrid with battery management capability. IEEE Trans Power Electron, 2014, 29: 695–706
19 Zhang D, Jiang J, Wang L Y, et al. Robust and scalable management of power networks in dual-source trolleybus systems: a consensus control framework. IEEE Trans Intell Transp Syst, 2015, 17: 1029–1038
20 Zhang D, Wang L Y, Jiang J, et al. Optimal power management in DC microgrids with applications to dual-source trolleybus systems. IEEE Trans Intell Transp Syst, 2018, 19: 1188–1197
21 Sindi E, Wang L Y, Polis M, et al. Distributed optimal power and voltage management in DC microgrids: applications to dual-source trolleybus systems. IEEE Trans Transp Electrific, 2018, 4: 778–788
22 Xie S Y, Nazari M H, Wang L Y, et al. Impact of stochastic generation/load variations on distributed optimal energy management in DC microgrids for transportation electrification. IEEE Trans Intell Transp Syst, 2022, 23: 7196–7205
23 Nazari M H, Xie S Y, Wang L Y, et al. Impact of communication packet delivery ratio on reliability of optimal load tracking and allocation in DC microgrids. IEEE Trans Smart Grid, 2021, 12: 2812–2821
24 Xie S Y, Nazari M H, Wang L Y, et al. Adaptive step size selection in distributed optimization with observation noise and unknown stochastic target variation. Automatica, 2022, 135: 109940
25 Yin G, Zhang Q. Discrete-time Markov Chains: Two-Time-Scale Methods and Applications. Berlin: Springer, 2005. 55
26 Ethier S N, Kurtz T G. Markov Processes: Characterization and Convergence. Hoboken: John Wiley & Sons, 2009. 282
27 Yin G G, Krishnamurthy V. Least mean square algorithms with Markov regime-switching limit. IEEE Trans Automat Contr, 2005, 50: 577–593
28 Yin G G, Kan S, Wang L Y, et al. Identification of systems with regime switching and unmodeled dynamics. IEEE Trans Automat Contr, 2009, 54: 34–47
29 Yin G G, Hashemi A, Wang L Y. Sign-regressor adaptive filtering algorithms for Markovian parameters. Asian J Control, 2014, 16: 95–106
30 Huang M, Dey S, Nair G N, et al. Stochastic consensus over noisy networks with Markovian and arbitrary switches. Automatica, 2010, 46: 1571–1583
31 Zhang Q, Zhang J F. Distributed parameter estimation over unreliable networks with Markovian switching topologies. IEEE Trans Automat Contr, 2012, 57: 2545–2560
32 Patrinos P, Sopasakis P, Sarimveis H, et al. Stochastic model predictive control for constrained discrete-time Markovian switching systems. Automatica, 2014, 50: 2504–2514
33 Billingsley P. Convergence of Probability Measures. Hoboken: John Wiley & Sons, 1968

Appendix A Proof of Theorem 1

For the proof of Theorem 1, we first state the following lemma.

Lemma A1. Let $\{a_k \in \mathbb{R}^m\}$ be a real vector sequence convergent to $a \in \mathbb{R}^m$. Let $c_{i,k} \in \mathbb{R}^{m\times m}$ (where $0 \leqslant i \leqslant k$) be a matrix sequence that satisfies each elements of $c_{i,k}$ is non-negative and $\lim_{k\to\infty} c_{i,k} = \mathbf{0}_{m\times m}, \forall i$, and $\lim_{k\to\infty}\sum_{i=0}^{k} c_{i,k} = I_m$. Then $\lim_{k\to\infty}\sum_{i=0}^{k} c_{i,k}a_i = a$ holds.

Proof. Since $\lim_{k\to\infty} a_k = a$, we know that there is a constant $N > 0$ such that $|a_k^j - a^j| < N$ holds for any $k \geqslant 0$ and $j \in \{1,\ldots,m\}$, where a_k^j and a^j are the jth element of a_k and a, respectively. Moreover, for any $\varepsilon > 0$, there exists a positive integer K_1 such that for any $k > K_1$ and $j \in \{1,\ldots,m\}$, $|a_k^j - a^j| < \frac{\varepsilon}{3}$ holds.

If we fix K_1, by $\lim_{k\to\infty} c_{i,k} = \mathbf{0}_{m\times m}$, we know that there is a positive integer K_2 such that for any $k > K_2$ and $j,s \in \{1,\ldots,m\}$, $|c_{i,k}^{js}| < \frac{\varepsilon}{3NK_1m}, i = 1,\ldots,K_1$ holds. By $\lim_{k\to\infty}\sum_{i=0}^k c_{i,k} = I_m$, we know that for any $\varepsilon > 0$, there exists a positive integer K_3 and $j,s \in \{1,\ldots,m\}$, such that $\sum_{i=K_3}^{\infty} c_{i,\infty} < \frac{\varepsilon}{3} I_m$.

Note that

$$\sum_{i=0}^k c_{i,k}a_i - a = \sum_{i=0}^k c_{i,k}a_i - \sum_{i=0}^k c_{i,k}a - \sum_{i=k}^{\infty} c_{i,\infty}a = \begin{pmatrix} \sum\limits_{i=0}^k\sum\limits_{j=1}^m c_{i,k}^{1j}(a_i^j - a^j) - \sum\limits_{i=k}^{\infty}\sum\limits_{j=1}^m c_{i,\infty}^{1j}a^j \\ \sum\limits_{i=0}^k\sum\limits_{j=1}^m c_{i,k}^{2j}(a_i^j - a^j) - \sum\limits_{i=k}^{\infty}\sum\limits_{j=1}^m c_{i,\infty}^{2j}a^j \\ \vdots \\ \sum\limits_{i=0}^k\sum\limits_{j=1}^m c_{i,k}^{mj}(a_i^j - a^j) - \sum\limits_{i=k}^{\infty}\sum\limits_{j=1}^m c_{i,\infty}^{mj}a^j \end{pmatrix}.$$

For any $\varepsilon > 0$, there exists a positive integer $K = \max\{K_1,K_2,K_3\}$ such that for any $k \geqslant K$ and each $s \in \{1,\ldots,m\}$,

$$\begin{aligned}\left|\sum_{i=0}^k\sum_{j=1}^m c_{i,k}^{sj}(a_i^j - a^j)\right| &= \sum_{i=0}^k\sum_{j=1}^m c_{i,k}^{sj}|a_i^j - a^j| = \sum_{i=0}^{K_1}\sum_{j=1}^m c_{i,k}^{sj}|a_i^j - a^j| + \sum_{i=K_1+1}^{k}\sum_{j=1}^m c_{i,k}^{sj}|a_i^j - a^j| \\ &\leqslant N\sum_{i=0}^{K_1}\sum_{j=1}^m c_{i,k}^{sj} + \frac{\varepsilon}{2}\sum_{i=K_1+1}^{k}\sum_{j=1}^m c_{i,k}^{sj} \leqslant NK_1m\cdot\frac{\varepsilon}{3NK_1m} + \frac{\varepsilon}{3}\sum_{t=0}^{\infty}\sum_{j=1}^m c_{i,\infty}^{sj} = \frac{2\varepsilon}{3},\end{aligned}$$

and when $\sum_{j=1}^m |a_j| > 0$, we have

$$\left|\sum_{i=k}^{\infty}\sum_{j=1}^m c_{i,\infty}^{sj}a^j\right| < \frac{\varepsilon}{3\sum_{j=1}^m|a^j|}\sum_{j=1}^m |a^j| = \frac{\varepsilon}{3},$$

and when $\sum_{j=1}^m |a_j| = 0$, we have $|\sum_{i=k}^{\infty}\sum_{j=1}^m c_{i,\infty}^{sj}a^j| = 0$. Thus, for any $\varepsilon > 0$, there exists a positive integer $K = \max\{K_1,K_2,K_3\}$, such that for any $k \geqslant K$ and each $s \in \{1,\ldots,m\}$,

$$\left|\sum_{i=0}^k\sum_{j=1}^m c_{i,k}^{sj}(a_i^j - a^j) - \sum_{i=k}^{\infty}\sum_{j=1}^m c_{i,\infty}^{sj}a^j\right| < \varepsilon$$

holds.

Proof of Theorem 1. By (13), we have

$$\eta_{k+1} = (I-\mu M)^{k+1}\eta_0 + \sum_{i=0}^k (I-\mu M)^{k-i}M^{-1}B\mathbb{E}[\delta_{i+1}] = (I-\mu M)^{k+1}\eta_0 + \sum_{i=0}^k (I-\mu M)^{k-i}\mu M\frac{M^{-2}B}{\mu}\mathbb{E}[\delta_{i+1}]. \tag{A1}$$

Note that

$$\lim_{k\to\infty}\sum_{i=0}^k (I-\mu M)^{k-i}\mu M = I_m,$$

and since $(I-\mu M)^{k-i}$ is exponentially stable, we have

$$\lim_{k\to\infty}(I-\mu M)^{k-i}\mu M = \mathbf{0}_{m\times m}$$

holds. Moreover, we know that

$$\lim_{i\to\infty}\frac{M^{-2}B}{\mu}\mathbb{E}[\delta_{i+1}] = \mathbf{0}_{m\times 1}.$$

Then by Lemma A1,

$$\lim_{k\to\infty}\sum_{i=0}^k (I-\mu M)^{k-i}M^{-1}B\mathbb{E}[\delta_{i+1}] = \mathbf{0}_{m\times 1}.$$

Thus, we have $\eta_\infty = \mathbf{0}_{m\times 1}$.

Appendix B Proof of Theorem 2

Define $V(x) = e^{\mathrm{T}}e/2$. Thus,

$$\mathbb{E}_k[V(e_{k+1})] - V(e_k) = \mathbb{E}_k[e_k^{\mathrm{T}}\{-\mu Me_k - \mu d_k + H(\ell_k - \ell_{k+1})\}] + \mathbb{E}_k[\|-\mu Me_k - \mu d_k + H(\ell_k - \ell_{k+1})\|^2], \tag{B1}$$

where $H = -M^{-1}B$. By Assumption 4, we know that

$$\mathbb{E}_k[H(\ell_k - \ell_{k+1})] = H\sum_{i=1}^s \mathbb{E}[\ell(i) - \ell_{k+1}|\ell_k = \ell(i)]I_{\{\ell_k=\ell(i)\}} = H\sum_{i=1}^s\left[\ell(i) - \sum_{j=1}^s \ell(j)(\delta_{ij} + \varepsilon q_{ij})\right]I_{\{\ell_k=\ell(i)\}} = O(\varepsilon). \tag{B2}$$

Also, we can obtain that

$$\mathbb{E}_k[\|H(\ell_k - \ell_{k+1})\|^2] \leqslant \|H\|^2 \sum_{j=1}^{s}\sum_{i=1}^{s} |\ell(i) - \ell(j)| I_{\{\ell_k = \ell(i)\}} P(\ell_{k+1} = \ell(j)|\ell_k = \ell(i))$$
$$= \|H\|^2 \sum_{j=1}^{s}\sum_{i=1}^{s} |\ell(i) - \ell(j)| I_{\{\ell_k = \ell(i)\}} (\delta_{ij} + \varepsilon q_{ij}) = O(\varepsilon). \tag{B3}$$

By noting $\|e_k\| = \|e_k\| \cdot 1 \leqslant (\|e_k\|^2 + 1)/2$, we have

$$O(\varepsilon)\|e_k\| \leqslant O(\varepsilon)(V(e_k) + 1). \tag{B4}$$

By the property of the signal $\{d_k\}$, we have

$$\mathbb{E}_k[\| - \mu M e_k - \mu d_k + H(\ell_k - \ell_{k+1})\|^2] = \mathbb{E}_k[\|H(\ell_k - \ell_{k+1})\|^2] + O(\mu^2 + \mu\varepsilon)(V(e_k) + 1). \tag{B5}$$

Using (B1) and (B5), we know that

$$\mathbb{E}_k[V(e_{k+1})] - V(e_k) = \mathbb{E}_k[e_k^{\mathrm{T}}\{-\mu M e_k - \mu d_k + H(\ell_k - \ell_{k+1})\}] + \|H\|^2\mathbb{E}_k[\|\ell_k - \ell_{k+1}\|^2] + O(\mu^2 + \mu\varepsilon)(V(e_k) + 1)$$
$$= \mathbb{E}_k[e_k^{\mathrm{T}}\{-\mu M e_k + H(\ell_k - \ell_{k+1})\}] + \|H\|^2\mathbb{E}_k[\|\ell_k - \ell_{k+1}\|^2] + O(\mu^2 + \mu\varepsilon)(V(e_k) + 1). \tag{B6}$$

To analyze (B6), we define the following perturbations of the Lyapunov function:

$$V_1^{\varepsilon}(e, k) = \sum_{j=k}^{\infty} e^{\mathrm{T}}\mathbb{E}_k[H(\ell_j - \ell_{j+1})], \tag{B7}$$

$$V_2^{\varepsilon}(k) = \|H\|^2 \sum_{j=k}^{\infty} \mathbb{E}_k[(\ell_k - \ell_{k+1})^{\mathrm{T}}(\ell_j - \ell_{j+1})]. \tag{B8}$$

Since Q is irreducible, there is an integer $N_\varepsilon > 0$ such that for any $N > N_\varepsilon$, $\|(I + \varepsilon Q)^N - 1_s v_\varepsilon\| \leqslant K\varepsilon$, where v_ε is the stationary distribution to $I + \varepsilon Q$. Thus, for all $N_1 \geqslant N \geqslant N_\varepsilon$,

$$\left\|\sum_{j=k}^{N_1} e^{\mathrm{T}}\mathbb{E}_k[H(\ell_j - \ell_{j+1})]\right\| = \|e^{\mathrm{T}} H \mathbb{E}_k[(\ell_k - \ell_{N_1})]\| \leqslant O(\varepsilon)(V(e) + 1).$$

Hence,

$$\|V_1^{\varepsilon}(e, k)\| \leqslant O(\varepsilon)(V(e) + 1). \tag{B9}$$

Also, we have

$$\|V_2^{\varepsilon}(k)\| = O(\varepsilon). \tag{B10}$$

Note also that

$$\mathbb{E}_k[V_1^{\varepsilon}(e_{k+1}, k+1)] - V_1^{\varepsilon}(e_k, k) = \mathbb{E}_k[V_1^{\varepsilon}(e_{k+1}, k+1)] - \mathbb{E}_k[V_1^{\varepsilon}(e_k, k+1)] + \mathbb{E}_k[V_1^{\varepsilon}(e_k, k+1)] - V_1^{\varepsilon}(e_k, k). \tag{B11}$$

It follows that

$$\mathbb{E}_k[V_1^{\varepsilon}(e_k, k+1)] - V_1^{\varepsilon}(e_k, k) = -\mathbb{E}_k[e_k^{\mathrm{T}} H(\ell_k - \ell_{k+1})]. \tag{B12}$$

In addition, we have

$$\mathbb{E}_k[V_1^{\varepsilon}(e_{k+1}, k+1)] - \mathbb{E}_k[V_1^{\varepsilon}(e_k, k+1)] = \sum_{j=k+1}^{\infty} \mathbb{E}_k[(e_{k+1} - e_k)^{\mathrm{T}}]\mathbb{E}_{k+1}[H(\ell_j - \ell_{j+1})]. \tag{B13}$$

By (12), similar to the analysis of (B3), we have

$$\mathbb{E}_k[\|e_{k+1} - e_k\|] \leqslant \mu\mathbb{E}[\|M\|\|e_k\|] + \mu\mathbb{E}_k[\|d_k\|] + O(\varepsilon) = O(\mu)(V(e_k) + 1) + O(\varepsilon) \tag{B14}$$

and

$$\left\|\sum_{j=k+1}^{\infty} \mathbb{E}_k[(e_{k+1} - e_k)^{\mathrm{T}}]\mathbb{E}_{k+1}[H(\ell_j - \ell_{j+1})]\right\| \leqslant O(\mu^2 + \varepsilon^2)(V(e_k) + 1). \tag{B15}$$

Thus, we know that

$$\mathbb{E}_k[V_1^{\varepsilon}(e_{k+1}, k+1)] - V_1^{\varepsilon}(e_k, k) = -\mathbb{E}_k[e_k^{\mathrm{T}} H(\ell_k - \ell_{k+1})] + O(\mu^2 + \varepsilon^2)(V(e_k) + 1), \tag{B16}$$

and

$$\mathbb{E}_k[V_2^{\varepsilon}(k+1) - V_2^{\varepsilon}(k)] = -\|H\|^2\mathbb{E}_k[\|\ell_k - \ell_{k+1}\|^2] + O(\varepsilon^2). \tag{B17}$$

Define

$$W(e,k)=V(e)+V_1^{\varepsilon}(e,k)+V_2^{\varepsilon}(k).$$

Then, by (B6), (B16), (B17), we have

$$\begin{aligned}\mathbb{E}_k[W(e_{k+1},k+1)]-W(e_k,k)&\leqslant-\mu e_k^{\mathrm{T}}Me_k+O(\mu^2+\varepsilon^2)(V(e_k)+1)\\&\leqslant-\lambda\mu V(e_k)+O(\mu^2+\varepsilon^2)(V(e_k)+1)\\&\leqslant-\lambda\mu W(e_k,k)+O(\mu^2+\varepsilon^2)(W(e_k,k)+1),\end{aligned}\tag{B18}$$

for some constant $\lambda>0$.

If μ and ε are chosen to be small enough, there is a constant $\lambda_0>0$ such that $\lambda_0<\lambda$ and $-\lambda\mu+O(\mu^2+\varepsilon^2)\leqslant\lambda_0\mu$. Then we have

$$\mathbb{E}_k[W(e_{k+1},k+1)]\leqslant(1-\lambda_0\mu)W(e_k,k)+O(\mu^2+\varepsilon^2).\tag{B19}$$

By taking expectation on two sides, we have

$$\mathbb{E}[W(e_{k+1},k+1)]\leqslant(1-\lambda_0\mu)^{N-N_\varepsilon}\mathbb{E}[W(e_{N_\varepsilon},N_\varepsilon)]+O\left(\mu+\frac{\varepsilon^2}{\mu}\right).\tag{B20}$$

Letting N be large enough, $(1-\lambda_0\mu)^{N-N_\varepsilon}\leqslant O(\mu)$ holds. Thus, $\mathbb{E}[W(e_{k+1},k+1)]\leqslant O(\mu+\frac{\varepsilon^2}{\mu})$. By (B9) and (B10), we have

$$\mathbb{E}[V(e_{k+1})]\leqslant O\left(\mu+\varepsilon+\frac{\varepsilon^2}{\mu}\right).$$

This completes the proof.