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Positivity and stability of timescale-type linear singular systems with time delays

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Abstract This paper investigates positivity and stability problems of timescale-type delayed linear singular systems (LSSs). The existing results put an extremely strict constraint on the time-delay function. By introducing a novel function, this constraint is successfully removed, which generalizes the scope of the considered systems. Then, some necessary and sufficient criteria are proposed for the positivity of LSSs with bounded and infinite time-varying delays. Finally, the exponential (asymptotical) stability of LSSs with bounded (infinite) time-varying delays is analyzed. The derived results are also applicable to timescale-type differential-difference systems (DDSs). Compared with the existing stability criteria of DDSs with bounded time-varying delays, the strict limit on the parameter related to the convergence rate is eliminated. Hence, the conservatism of the existing results can be reduced. Moreover, when investigating stability of DDSs with infinite time-varying delays, this paper proposes a less conservative stability theorem. To illustrate the validity of the derived results, an example is presented regarding LSSs with bounded and infinite time-varying delays.

Keywords positivity, stability, time delay, linear singular systems, timescale-type systems

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1 Introduction

In recent decades, singular or descriptor systems have drawn continuous attention from many researchers and engineers due to their significant roles in real-world applications, such as aerospace engineering and chemical processing [1–4]. When singular systems are investigated, the first issue is impulse-controllable or impulse-free judgment. If this judgment can be made, singular systems can be transformed into a class of standard differential-algebraic systems. With this transformation, some problems of singular systems will be conveniently addressed [5–7]. When the singular matrices are special, singular systems can be transformed into differential-difference systems (DDSs). Recently, with the development of research on positive systems, positive singular systems have attracted researchers' interest [8–10]. A representative example of positive descriptor systems is the Leontief model [11]. Thus far, few studies on positive singular systems with time delays have been published. In [9, 10], based on the properties of the Drazin inverse, some interesting results regarding the positivity and stability of continuous delayed singular systems were proposed. However, the results in [9, 10] only focused on asymptotical stability of systems with bounded delays. Comparatively speaking, the research on systems with infinite or unbounded delays is more challenging since limited tools are available for their analysis and synthesis. Additionally, solutions to such systems are sensitive to the initial conditions [12]. For positive singular systems with infinite delays, the research progress is relatively slow. When investigating such systems, the time-delay functions must satisfy a restrictive condition [13, 14]. Additionally, since the time-delay functions are not bounded, the exponential stability of such systems is hardly achieved, given the algebraic parts.

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Many complex models in engineering cannot be directly described by continuous or discrete systems, such as multi-agent systems with intermittent information transmissions among agents [15]. The reason is that the time domain may be neither continuous nor uniformly discrete. Therefore, establishing mathematical models under a more general framework is desirable and necessary. Fortunately, this task can be completed via the timescale theory proposed by Hilger [16]. Recently, fruitful studies [17–22] on timescale-type systems have been published. Although timescale-type systems are generalizations of continuous and discrete cases, there are essential differences between the former and the latter two. For example, if the time domain is the set of real numbers $\mathbb{R}$ or integer numbers $\mathbb{Z}$, the term $t-\tau(t)\in\mathbb{R}$ [23] or $t-\tau(t)\in\mathbb{Z}$ is always satisfied for arbitrary time-delay function $\tau(t)\in\mathbb{R}$ or $\tau(t)\in\mathbb{Z}$. In other words, the system states are definitely meaningful at $t-\tau(t)$. However, this statement may be untrue for an arbitrary time scale (please see Remark 1 for more details). Thus, timescale-type systems are not simple extensions of continuous or discrete cases, and the studies on these kinds of systems with time delays are more challenging. In recent years, increasingly more researchers have applied their attention to timescale-type positive systems [24–27] or singular systems [28, 29]. Unfortunately, no studies concerning timescale-type positive singular systems have been published. In [21], timescale-type DDSs with bounded time delays were investigated, and some interesting results were developed regarding the positivity and exponential stability of the considered systems. As mentioned above, DDSs can be considered special cases of singular systems. Thus, it is natural to study timescale-type positive singular systems, which partly motivates our work.

Inspired by [9, 10, 21, 30], this paper first investigates timescale-type linear singular systems (LSSs) with time delays, which take DDSs as their special cases. Compared with [9, 10, 21, 30], some improved results are proposed. The main novelties of this paper are listed as follows.

• Ref. [9, 10] studied the asymptotical stability of continuous positive singular systems with bounded time delays. Different from [9, 10], the considered systems here are more general and complicated since they can include not only continuous or discrete cases but also systems on hybrid time domains. In addition, exponential stability problems are addressed in this paper. Thus, the presented results here are generalizations of those in [9, 10].

• Refs. [21, 30] first investigated timescale-type DDSs with bounded and infinite time-varying delays, respectively. In contrast to these studies, the improved results developed in our paper can be summarized from three aspects. (i) To make system states meaningful, Refs. [21, 30] strictly constrained the time-delay function. By introducing an auxiliary function, our work successfully removes this constraint, which widens the scope of the considered systems. (ii) The stability criterion in [21] requires the parameter related to the convergence rate to be extremely small, which is fairly restrictive. Given the properties of the timescale-type exponential function, the requirement on the parameter related to the convergence rate is excluded here. Thus, the stability criteria in our paper are less conservative than those in [21]. (iii) The stability condition in [30] depends on the graininess function (please see Definition 1), which is difficult to use. In this paper, a less conservative stability theorem is proposed (please see Remark 4 for more details).

The remainder of this paper is organized as follows. In Section 2, some definitions and lemmas of the time scale are introduced. In Section 3, the positivity and exponential stability of timescale-type LSSs with bounded time delays are investigated. In Section 4, positivity and asymptotical stability problems of timescale-type LSSs with infinite time delays are addressed. In Section 5, the derived results are applied to timescale-type DDSs. In Section 6, some simulation examples are given to demonstrate the effectiveness of the derived results.

Notations. In this paper, the following notations will be used. $\|\cdot\|$ denotes the usual Euclidean norm. $Q\succ 0$ ($\succeq 0, \prec 0, \preceq 0$) denotes a matrix in which all elements are positive (nonnegative, negative, nonpositive). A Metzler matrix has only nonnegative off-diagonal elements. Q is a Schur matrix if its spectral radius is less than one. $\det(Q)$ represents the determinant of matrix Q. $\varrho(Q)=\max\{\mathrm{Re}(\lambda):\lambda\in\varpi(Q)\}$, and $\varpi(Q)$ is the set of all eigenvalues of matrix Q.

2 Basic concepts of time scale

Definition 1 ([17]). A time scale is a nonempty closed subset of the real number set $\mathbb{R}$, which takes the sets $\mathbb{Z}$ and $\mathbb{R}$ as its special cases. Assuming $\mathbb{T}$ is a time scale and $t\in\mathbb{T}$, when $t<\sup\mathbb{T}$, the forward jump operator $\sigma:\mathbb{T}\to\mathbb{T}$ is defined as $\sigma(t)=\inf\{s\in\mathbb{T}: s>t\}$; when $t>\inf\mathbb{T}$, the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined as $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$. The graininess function $\mu : \mathbb{T} \to \mathbb{R}^+$ is defined as $\mu(t) := \sigma(t) - t$. If $\mu(t) > 0$, t is said to be right scattered; if $\mu(t) = 0$, t is said to be right dense. If $\rho(t) = t$, then t is left dense, while if $\rho(t) < t$, then t is left scattered. Along with the set $\mathbb{T}$, the set $\mathbb{T}^\kappa$ is defined as follows: if $\mathbb{T}$ contains the left-scattered maximum M, then $\mathbb{T}^\kappa = \mathbb{T} \setminus \{M\}$; otherwise, $\mathbb{T}^\kappa = \mathbb{T}$. The definition of timescale-type derivative $f^\Delta(t)$ of the function $f : \mathbb{T} \to \mathbb{R}$ and its properties can be found in [17].

Definition 2 ([17])**.** A function (matrix-valued function) is rd-continuous if it is continuous at right-dense points and its left-sided limits exist (finite) at left-dense points. The set of all rd-continuous functions (matrix-valued functions) is denoted as $C_{rd}(\mathbb{T}, \mathbb{R})$ ($C_{rd}(\mathbb{T}, \mathbb{R}^{n\times n})$).

Definition 3 ([17])**.** A function (matrix-valued function) $p : \mathbb{T} \to \mathbb{R}$ ($P : \mathbb{T} \to \mathbb{R}^{n\times n}$) is regressive, if for all $t \in \mathbb{T}^\kappa, 1 + \mu(t)p(t) \neq 0$ ($I + \mu(t)P(t)$ is invertible). $\mathcal{R}$ ($\mathcal{R}^{n\times n}$) denotes the set of all regressive and rd-continuous functions (matrix-valued functions). $p : \mathbb{T} \to \mathbb{R}$ is positive (nonnegative) regressive, if for all $t \in \mathbb{T}^\kappa, 1 + \mu(t)p(t) > 0$ ($\geqslant 0$). $\mathcal{R}^+$ ($\mathcal{R}_0^+$) denotes the set of all positive (nonnegative) regressive and rd-continuous functions.

Definition 4 ([17])**.** If $p \in \mathcal{R}$, then the exponential function is defined as $e_p(t, s) = \exp(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau)$, where $s, t \in \mathbb{T}$, $\xi_\mu(p) = \frac{1}{\mu}\log(1 + \mu p)$. If $P \in \mathcal{R}^{n\times n}$, then the matrix exponential function is defined as $e_P(t, s)$, where $s, t \in \mathbb{T}$. If $\mathbb{T} = \mathbb{R}$, $e_p(t, s) = e^{\int_s^t p(r)\mathrm{d}r}$; moreover, if p and P are constant, $e_p(t, s) = e^{p(t-s)}$ and $e_P(t, s) = e^{P(t-s)}$. If $\mathbb{T} = \mathbb{Z}$, $e_p(t, s) = \prod_{r=s}^{t-1}(1 + p(r))$; moreover, if p and P are constant, $e_p(t, s) = (1 + p)^{t-s}$ and $e_P(t, s) = (I + P)^{t-s}$.

Lemma 1 ([17])**.** If $p \in \mathcal{R}$, $s, t, r \in \mathbb{T}$, then (i) $e_0(t, s) = e_p(t, t) = 1$; (ii) $e_p(t, s)e_p(s, r) = e_p(t, r)$; (iii) $e_p(\sigma(t), s) = (1+\mu(t)p(t))e_p(t, s)$; (iv) $e_{\ominus p}(s, t) = e_p(t, s) = \frac{1}{e_p(s,t)}$, where the function $\ominus p = \frac{-p}{1+\mu(t)p}$. If $P \in \mathcal{R}^{n\times n}$, $s, t, r \in \mathbb{T}$, then (i) $e_0(t, s) = e_P(t, t) = I$; (ii) $e_P(t, s)e_P(s, r) = e_P(t, r)$; (iii) $e_P(\sigma(t), s) = (I + \mu(t)P(t))e_P(t, s)$.

3 Systems with bounded delays

Consider the following timescale-type LSSs:

$$Ex^\Delta(t) = Ax(t) + Bx_t,\ t \in \mathbb{T}, \tag{1}$$

where $x \in \mathbb{R}^n$, $x_t = x(\zeta(t - \tau(t)))$, $0 \in \mathbb{T}$, $\mathbb{T}$ is any time scale, the function $\zeta : \mathbb{R} \to \mathbb{T}$ is defined as $\zeta(s) = \inf\{t \in \mathbb{T} : t \geqslant s\}$ [18], $\tau(t)$ is the time delay satisfying $0 < \tau_1 \leqslant \tau(t) \leqslant \tau_2$ with τ_1, τ_2 being constants, E is a singular matrix with rank $r < n$, the initial condition is $\phi(t)$, $t \in [-\tau_2, 0]_\mathbb{T} := [-\tau_2, 0] \bigcap \mathbb{T}$, $A \in \mathcal{R}^{n\times n}$, and the matrix pair (E, A) is regular and impulse free. The matrix pair (E, A) is regular if $\det(sE - A)$ is not identically zero. And the pair (E, A) is impulse free if the degree of the polynomial $\det(sE - A)$ satisfies $\deg(\det(sE - A)) = \mathrm{rank}(E)$ [31].

Remark 1. System (1) is of generality since it can be transformed into many different types of systems in light of various time scale $\mathbb{T}$. And system (1) is not just a simple extension or unification of the classical continuous and discrete cases. For example, when $\mathbb{T}$ is the real set $\mathbb{R}$, system (1) turns into continuous system $E\dot{x}(t) = Ax(t) + Bx_t$. When $\mathbb{T}$ is the integer set $\mathbb{Z}$, system (1) gives rise to discrete system $E(x(t + 1) - x(t)) = Ax(t) + Bx_t$. And when $\mathbb{T} = \{q^k : k \in \mathbb{Z}\} \bigcup\{0\}$, the q-difference systems [17, 32] can be formulated from (1). For continuous ($\mathbb{T} = \mathbb{R}$) or discrete ($\mathbb{T} = \mathbb{Z}$) systems, the condition $\tau(t) \in \mathbb{R}$ or $\tau(t) \in \mathbb{Z}$ is acquiescent, and hence the phenomenon $t - \tau(t) \notin \mathbb{T}$ will never occur. So it is unnecessary to introduce the function $\zeta(t)$. However, in some cases, this phenomenon may be true. For example, if $\mathbb{T} = \bigcup_{k\in\mathbb{Z}}[2k, 2k + 1]$ and $\tau(t) \in \mathbb{T}$, it can be true that $t - \tau(t) \notin \mathbb{T}$. Let us take $\tau(t) = 0.5$, and then $t - 0.5 \notin \mathbb{T}$ for any $t \in [2k, 2k + 0.5)$. Hence, the state $x(t - \tau(t))$ is not always meaningful for any function $\tau(t) \in \mathbb{T}$. Moreover, if we impose an extra condition on the function $\tau(t)$ that $t - \tau(t) \in \mathbb{T}$ [21, 30], then $\tau(t) \equiv 2 \in \mathbb{T}$ must be true, which is conservative. To avoid this phenomenon, Ref. [27] introduced the function $\chi(t) := \sup\{s : s \leqslant t, s \in \mathbb{T}\}$ to make $x(\chi(t - \tau(t)))$ meaningful. Supposing $0 \in \mathbb{T}$ and $-\tau(0) \notin \mathbb{T}$, then $x(\chi(-\tau(0)))$ is meaningless. Therefore, the function $\chi(t)$ in [27] is not so effective. Fortunately, the function $\zeta(t)$ in system (1) can always make $x(\zeta(t - \tau(t)))$ meaningful for any function $\tau(t) \in \mathbb{T}$.

Since (E, A) is regular and impulse-free, there exist two nonsingular matrices P and $Q \succeq 0$, such that $PEQ = \left(\begin{smallmatrix} I_r & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and $PAQ = \left(\begin{smallmatrix} A_r & 0 \\ 0 & I_{n-r} \end{smallmatrix}\right)$ [1, 33]. Let $Q^{-1}x(t) = \hat{x}(t) = \left(\begin{smallmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{smallmatrix}\right)$ and $PBQ = \left(\begin{smallmatrix} \hat{B}_{11} & \hat{B}_{12} \\ \hat{B}_{21} & \hat{B}_{22} \end{smallmatrix}\right)$.

Then, system (1) yields

$$\begin{cases} \hat{x}_1^{\Delta}(t) = A_r\hat{x}_1(t) + \hat{B}_{11}\hat{x}_1(\zeta(t-\tau(t))) + \hat{B}_{12}\hat{x}_2(\zeta(t-\tau(t))), \\ \hat{x}_2(t) = -\hat{B}_{21}\hat{x}_1(\zeta(t-\tau(t))) - \hat{B}_{22}\hat{x}_2(\zeta(t-\tau(t))), \end{cases} \tag{2}$$

where $A_r \in \mathcal{R}^{r\times r}$, and the initial condition is $\hat{\phi}_1(t)$, $\hat{\phi}_2(t)$, $t \in [-\tau_2, 0]_{\mathbb{T}}$.

Clearly, positivity and stability of system (1) are equivalent to that of system (2). In the following sections, system (2) is mainly investigated.

By Theorem 5.24 in [17], the solution to the first equation of (2) is

$$\begin{aligned} \hat{x}_1(t) \;&=\; e_{A_r}(t,s)\hat{x}_1(s) + \int_s^t e_{A_r}(t,\sigma(r))\big(\hat{B}_{11}\hat{x}_1(\zeta(r-\tau(r))) \\ &\quad +\hat{B}_{12}\hat{x}_2(\zeta(r-\tau(r)))\big)\Delta r \\ &:= e_{A_r}(t,s)\hat{x}_1(s) + F(t,s),\ t,s\in\mathbb{T},\ t\geqslant s. \end{aligned} \tag{3}$$

3.1 Positivity analysis

In this subsection, we consider the positivity of system (2). As stated in [21,34], it is necessary to assume that $-\hat{B}_{22}$ is a Schur matrix. Therefore, we also make this assumption throughout this paper.

The following definition is similar to that in [21].

Definition 5. System (2) is said to be positive if for any initial value $\hat{\phi}_1 \succeq 0$ and $\hat{\phi}_2 \succeq 0$, it holds that $x(t) \succeq 0$ and $y(t) \succeq 0$, $t \in [0,\infty)_{\mathbb{T}}$.

Lemma 2 ([21]). Let $A = (a_{ij})_{n\times n} \in \mathcal{R}^{n\times n}$. For any $t,s\in\mathbb{T},\ t\geqslant s,\ e_A(t,s)\succeq 0$ if and only if $A \in \mathcal{M}(\mathbb{T})$, where $\mathcal{M}(\mathbb{T})$ denotes the set of all Metzler matrices satisfying $a_{ii} \in \mathcal{R}_0^+(\mathbb{T})$.

Lemma 3. Let $A \in \mathcal{M}(\mathbb{T})$ and $g(t) \succeq 0$. Then, for any $t,s\in\mathbb{T},\ t\geqslant s,\ \int_s^t e_A(t,\sigma(r))g(r)\Delta r \succeq 0$.

Proof. If t is right dense, then $\sigma(t) = t$ and $\sigma(r) \leqslant t$ for any $r < t$. Then, by Lemma 2, we have $e_A(t,\sigma(r)) \succeq 0$ for any $r \leqslant t$. Thus, $\int_s^t e_A(t,\sigma(r))g(r)\Delta r \succeq 0$, $t \geqslant s$.

If t is right scattered and left scattered, then $\rho(t) < t$ and $\sigma(\rho(t)) = t$. Moreover, by Theorem 1.75 in [17], we have $\int_{\rho(t)}^t e_A(t,\sigma(r))g(r)\Delta r = \mu(\rho(t))e_A(t,t)g(\rho(t))$. Then, it yields

$$\begin{aligned} \int_s^t e_A(t,\sigma(r))g(r)\Delta r \;&=\; \int_s^{\rho(t)} e_A(t,\sigma(r))g(r)\Delta r + \int_{\rho(t)}^t e_A(t,\sigma(r))g(r)\Delta r \\ &= \int_s^{\rho(t)} e_A(t,\sigma(r))g(r)\Delta r + \mu(\rho(t))e_A(t,t)g(\rho(t)) \\ &= \int_s^{\rho(t)} e_A(t,\sigma(r))g(r)\Delta r + \mu(\rho(t))g(\rho(t)). \end{aligned}$$

By observing $\sigma(r) \leqslant t$ for any $r \in [s,\rho(t)]_{\mathbb{T}}$, and from Lemma 2, we can get $\int_s^t e_A(t,\sigma(r))g(r)\Delta r \succeq 0$. If t is right scattered and left dense, then two cases should be considered. Case I: there exists a constant $\varepsilon > 0$ such that $[t-\varepsilon,t]_{\mathbb{T}} = [t-\varepsilon,t]$. Case II: for any $\epsilon > 0$, $[t-\epsilon,t]_{\mathbb{T}} \neq [t-\epsilon,t]$ but there exists an increasing sequence $\{t_k, k\in\mathbb{Z}^+\} \subset \mathbb{T}$ (where $\mathbb{Z}^+$ is the set of all positive integers), such that $t_k < t$ for any $k \in \mathbb{Z}^+$ and $\sup_{k\in\mathbb{Z}^+}\{t_k\} = t$.

Consider case I. It holds that

$$\begin{aligned} \int_s^t e_A(t,\sigma(r))g(r)\Delta r \;&=\; \int_s^{t-\varepsilon} e_A(t,\sigma(r))g(r)\Delta r + \int_{t-\varepsilon}^t e_A(t,\sigma(r))g(r)\Delta r \\ &= \int_s^{t-\varepsilon} e_A(t,\sigma(r))g(r)\Delta r + \int_{t-\varepsilon}^t e^{A(t-r)}g(r)\mathrm{d}r. \end{aligned}$$

Since $\sigma(r) < t$ for any $r \in [s,t-\varepsilon]_{\mathbb{T}}$, and from Lemma 2, it holds that $\int_s^t e_A(t,\sigma(r))g(r)\Delta r \succeq 0$.

Consider case II. In this case,

$$\int_s^t e_A(t,\sigma(r))g(r)\Delta r = \int_s^{t_1} e_A(t,\sigma(r))g(r)\Delta r + \int_{t_1}^t e_A(t,\sigma(r))g(r)\Delta r.$$

For any $r \in [s, t_1]_{\mathbb{T}}$, $\sigma(r) < t$. By Lemma 2, $e_A(t, \sigma(r)) \succeq 0$. And following Theorem 1.79 in [17], we can derive that $\int_{t_1}^{t} e_A(t, \sigma(r))g(r)\Delta r = \sum_{s\in[t_1,t)_{\mathbb{T}}} \mu(s)e_A(t, \sigma(s))g(s)$. For any $s \in [t_1, t)_{\mathbb{T}}$, $\sigma(s) \leqslant t$. Following Lemma 2, we have $e_A(t, \sigma(s)) \succeq 0$. Therefore, we can get $\int_{s}^{t} e_A(t, \sigma(r))g(r)\Delta r \succeq 0$. Based on the above analysis, the proof is completed.

Now, we are in a position to propose the following result.

Theorem 1. System (2) is positive if and only if $A_r \in \mathcal{M}(\mathbb{T})$, $\hat{B}_{1j} \succeq 0$, and $\hat{B}_{2j} \preceq 0$, $j = 1, 2$.

Proof. (Sufficiency): First, we consider the claim $t \in [0, \zeta(\tau_1)]_{\mathbb{T}} = [0, \tau_1]_{\mathbb{T}} \cup (\tau_1, \zeta(\tau_1)]_{\mathbb{T}}$.

Supposing $t \in [0, \tau_1]_{\mathbb{T}}$, then $\zeta(r - \tau(r)) \leqslant \zeta(t - \tau_1) \leqslant 0$ for any $r \leqslant t$. Since $\hat{\phi}_j(t) \succeq 0$ $(j = 1, 2)$ and $\hat{B}_{1j} \succeq 0$, $j = 1, 2$, it follows from Lemmas 2 and 3 that $e_{A_r}(t, 0) \succeq 0$ and $F(t, 0) \succeq 0$. By (3), we have $\hat{x}_1(t) \succeq 0$, $t \in [0, \tau_1]_{\mathbb{T}}$. In light of $\hat{B}_{2j} \preceq 0$, $j = 1, 2$, it is apparent that $\hat{x}_2(t) \succeq 0$, $t \in [0, \tau_1]_{\mathbb{T}}$. Thus, $\hat{x}(t) \succeq 0$ for any $t \in [0, \tau_1]_{\mathbb{T}}$.

Supposing $t \in (\tau_1, \zeta(\tau_1)]_{\mathbb{T}}$, two cases should be considered. (i) $\tau_1 \in \mathbb{T}$, and (ii) $\tau_1 \notin \mathbb{T}$. If case (i) is true, then $\zeta(\tau_1) = \tau_1$, and $t \in [0, \zeta(\tau_1)]_{\mathbb{T}} = [0, \tau_1]_{\mathbb{T}}$, which directly yields $\hat{x}(t) \succeq 0$. If case (ii) is true, then $\rho(\zeta(\tau_1)) < \tau_1 < \zeta(\tau_1)$ and $(\tau_1, \zeta(\tau_1)]_{\mathbb{T}} = \{\zeta(\tau_1)\}$. Following the fact

$$\begin{aligned}
\hat{x}_1(\zeta(\tau_1)) =& \hat{x}_1(\rho(\zeta(\tau_1))) + \mu(\rho(\zeta(\tau_1)))\hat{x}_1^{\Delta}(\rho(\zeta(\tau_1))) \\
=& (I + \mu(\rho(\zeta(\tau_1)))A_r)\hat{x}_1(\rho(\zeta(\tau_1))) \\
&+ \mu(\rho(\zeta(\tau_1)))(\hat{B}_{11}\hat{x}_1(\zeta(\rho(\zeta(\tau_1)) - \tau(\rho(\zeta(\tau_1))))) \\
&+ \hat{B}_{12}\hat{x}_2(\zeta(\rho(\zeta(\tau_1)) - \tau(\rho(\zeta(\tau_1)))))) \\
=& e_{A_r}(\zeta(\tau_1), \rho(\zeta(\tau_1)))\hat{x}_1(\rho(\zeta(\tau_1))) \\
&+ \mu(\rho(\zeta(\tau_1)))(\hat{B}_{11}\hat{x}_1(\zeta(\rho(\zeta(\tau_1)) - \tau(\rho(\zeta(\tau_1))))) \\
&+ \hat{B}_{12}\hat{x}_2(\zeta(\rho(\zeta(\tau_1)) - \tau(\rho(\zeta(\tau_1)))))),
\end{aligned} \tag{4}$$

and by noting $\rho(\zeta(\tau_1)) \in [0, \tau_1]_{\mathbb{T}}$ and $\zeta(\rho(\zeta(\tau_1)) - \tau(\rho(\zeta(\tau_1)))) \leqslant 0$, we can get from Lemmas 2 and 3 that $\hat{x}_1(\zeta(\tau_1)) \succeq 0$.

Since $\zeta(\tau_1) - \tau(\zeta(\tau_1)) \leqslant \zeta(\tau_1) - \tau_1 < \zeta(\tau_1)$, it holds that $\zeta(\zeta(\tau_1) - \tau(\zeta(\tau_1))) \leqslant \zeta(\tau_1)$. If $\zeta(\zeta(\tau_1) - \tau(\zeta(\tau_1))) < \zeta(\tau_1)$, then $\zeta(\zeta(\tau_1) - \tau(\zeta(\tau_1))) \in [-\tau_2, \tau_1]_{\mathbb{T}}$. By the foregoing analysis and $\hat{B}_{2j} \preceq 0$, $j = 1, 2$, we have

$$\begin{aligned}
\hat{x}_2(\zeta(\tau_1)) =& -\hat{B}_{21}\hat{x}_1\big(\zeta(\zeta(\tau_1) - \tau(\zeta(\tau_1)))\big) \\
&- \hat{B}_{22}\hat{x}_2\big(\zeta(\zeta(\tau_1) - \tau(\zeta(\tau_1)))\big) \\
\succeq& 0.
\end{aligned} \tag{5}$$

If $\zeta(\zeta(\tau_1) - \tau(\zeta(\tau_1))) = \zeta(\tau_1)$, then

$$\hat{x}_2(\zeta(\tau_1)) = -(I + \hat{B}_{22})^{-1}\hat{B}_{21}\hat{x}_1(\zeta(\tau_1)).$$

Since $-\hat{B}_{22}$ is a Schur matrix, it yields from the Laplace transformation that $(I + \hat{B}_{22})^{-1} \succeq 0$. Following $\hat{x}_1(\zeta(\tau_1)) \succeq 0$ and $\hat{B}_{21} \preceq 0$, we have $\hat{x}_2(\zeta(\tau_1)) \succeq 0$. Thus, $\hat{x}(t) \succeq 0$ for any $t \in (\tau_1, \zeta(\tau_1)]_{\mathbb{T}}$. To sum up, $\hat{x}(t) \succeq 0$ for any $t \in [0, \zeta(\tau_1)]_{\mathbb{T}}$.

Next, we consider the claim $t \in [\zeta(\tau_1), \zeta(2\tau_1)]_{\mathbb{T}} = [\zeta(\tau_1), 2\tau_1]_{\mathbb{T}} \cup (2\tau_1, \zeta(2\tau_1)]_{\mathbb{T}}$.

Supposing $t \in [\zeta(\tau_1), 2\tau_1]_{\mathbb{T}}$, by (3) we have

$$\begin{aligned}
\hat{x}_1(t) =\ & e_{A_r}(t, \zeta(\tau_1))\hat{x}_1(\zeta(\tau_1)) + \int_{\zeta(\tau_1)}^{t} e_{A_r}(t, \sigma(r)) \\
&\times \big(\hat{B}_{11}\hat{x}_1(\zeta(r - \tau(r))) + \hat{B}_{12}\hat{x}_2(\zeta(r - \tau(r)))\big)\Delta r.
\end{aligned}$$

For any $r \leqslant t$, it holds that $r - \tau(r) \leqslant t - \tau_1 \leqslant \tau_1$, which implies $\zeta(r - \tau(r)) \leqslant \zeta(\tau_1)$. Then, $\hat{x}_j(\zeta(r - \tau(r))) \succeq 0$, $j = 1, 2$. By Lemmas 2 and 3, we have $\hat{x}_1(t) \succeq 0$ for any $t \in [\zeta(\tau_1), 2\tau_1]_{\mathbb{T}}$. Similarly, we can obtain $\hat{x}_2(t) \succeq 0$ in light of $\hat{B}_{2j} \preceq 0$, $j = 1, 2$. Thus, for any $t \in [\zeta(\tau_1), 2\tau_1]_{\mathbb{T}}$, $\hat{x}(t) \succeq 0$.

Supposing $t \in (2\tau_1, \zeta(2\tau_1)]_{\mathbb{T}}$ and $2\tau_1 \in \mathbb{T}$, then $(2\tau_1, \zeta(2\tau_1)]_{\mathbb{T}} = \{2\tau_1\}$. Apparently, $\hat{x}(t) \succeq 0$. If $2\tau_1 \notin \mathbb{T}$, then $(2\tau_1, \zeta(2\tau_1)]_{\mathbb{T}} = \{\zeta(2\tau_1)\}$. Just like (4), we have

$$\begin{aligned}\hat{x}_1(\zeta(2\tau_1)) =& e_{A_r}(\zeta(2\tau_1), \rho(\zeta(2\tau_1)))\hat{x}_1(\rho(\zeta(2\tau_1))) \\ &+\mu(\rho(\zeta(2\tau_1)))(\hat{B}_{11}\hat{x}_1(\zeta(\rho(\zeta(2\tau_1)) - \tau(\rho(\zeta(2\tau_1))))) \\ &+\hat{B}_{12}\hat{x}_2(\zeta(\rho(\zeta(2\tau_1)) - \tau(\rho(\zeta(2\tau_1)))))).\end{aligned}$$

Observing $\rho(\zeta(2\tau_1)) \in [\zeta(\tau_1), 2\tau_1]_{\mathbb{T}}$ and $\zeta(\rho(\zeta(2\tau_1)) - \tau(\rho(\zeta(2\tau_1)))) \leqslant \zeta(\tau_1)$, it holds that $\hat{x}_1(\zeta(2\tau_1)) \succeq 0$. Moreover, following the same step of (5), it yields $\hat{x}_2(\zeta(2\tau_1)) \succeq 0$. Based on the above discussion, we have $\hat{x}(t) \succeq 0$ for any $t \in [\zeta(\tau_1), \zeta(2\tau_1)]_{\mathbb{T}}$.

Repeating the same process, we can derive that $\hat{x}(t) \succeq 0$ for any $t \in [0, \infty)_{\mathbb{T}}$. The sufficiency part is finished.

(Necessity): Firstly, we prove $\hat{B}_{12} \succeq 0$ and $\hat{B}_{22} \preceq 0$. Choose $\hat{\phi}_1(t) = 0$, and then by (2) we have

$$\hat{x}_1^{\Delta}(0) = \hat{B}_{12}\hat{x}_2(\zeta(-\tau(0))).$$

Assuming $\hat{B}_{12} \not\succeq 0$ is true, there maybe exist $k \in \{1, \ldots, r\}$ such that $\hat{x}_{1,k}^{\Delta}(0) < 0$. If 0 is right scattered, then

$$\hat{x}_{1,k}(\sigma(0)) = \hat{x}_{1,k}(0) + \mu(0)\hat{x}_{1,k}^{\Delta}(0) < 0,$$

which is a contradiction. If 0 is right dense, there must exist $t^* \in U^+(0)$ such that $\hat{x}_{1,k}(t^*) < 0$, which is also a contradiction. Thus, $\hat{B}_{12} \succeq 0$.

From (2) and $\hat{\phi}_1(t) = 0$, we have

$$\hat{x}_2(0) = -\hat{B}_{22}\hat{x}_2(\zeta(-\tau(0))).$$

By the proof by contradiction, we can obtain $\hat{B}_{22} \preceq 0$.

Similarly, we can prove $\hat{B}_{11} \succeq 0$ and $\hat{B}_{21} \preceq 0$ by choosing $\hat{\phi}_2(t) = 0$. So it is omitted here.

Finally, we prove $A_r \in \mathcal{M}(\mathbb{T})$. Choose $\hat{x}_1(t) = 0$, $t \in [-\tau_2, 0)_{\mathbb{T}}$ and $\hat{\phi}_2(t) = 0$, and then by (3), it holds that

$$\begin{aligned}\hat{x}_1(t) =& e_{A_r}(t, 0)\hat{x}_1(0) + \int_0^t e_{A_r}(t, \sigma(r))(\hat{B}_{11}\hat{x}_1(\zeta(r - \tau(r))) \\ &+\hat{B}_{12}\hat{x}_2(\zeta(r - \tau(r))))\Delta r.\end{aligned}$$

If 0 is right dense, there always exists $\epsilon \in U^+(0) \bigcap [0, \tau_1]_{\mathbb{T}}$ such that

$$\hat{x}_1(\epsilon) = e_{A_r}(\epsilon, 0)\hat{x}_1(0) + \int_0^{\epsilon} \mathrm{e}^{A_r(\epsilon - r)}\hat{B}_{11}\hat{x}_1(\zeta(r - \tau(r)))\mathrm{d}r.$$

Supposing $A_r \notin \mathcal{M}(\mathbb{T})$, then by Lemma 2, $e_{A_r}(\epsilon, 0) \succeq 0$ may be unsatisfied. Thus, $\hat{x}_1(\epsilon) \not\succeq 0$ may be true by choosing proper initial condition $\hat{x}_1(0)$, which contradicts with positivity of system (2). And hence $A_r \in \mathcal{M}(\mathbb{T})$. If 0 is right scattered, then

$$\begin{aligned}\hat{x}_1(\sigma(0)) =& e_{A_r}(\sigma(0), 0)\hat{x}_1(0) + \int_0^{\sigma(0)} e_{A_r(\sigma(0), \sigma(r))} \\ &\times(\hat{B}_{11}\hat{x}_1(\zeta(r - \tau(r))) + \hat{B}_{12}\hat{x}_2(\zeta(r - \tau(r))))\Delta r \\ =& (I + \mu(0)A_r)\hat{x}_1(0) + \mu(0)\hat{B}_{11}\hat{x}_1(\zeta(-\tau(0))).\end{aligned}$$

Following the above process, we can also get $A_r \in \mathcal{M}(\mathbb{T})$, which finishes the necessity part. Therefore, the proof of Theorem 1 is completed.

Remark 2. In the proof of Theorem 1, we use the step method [35]. Following this way, the existence and uniqueness of solutions of system (1) can be guaranteed under the condition that the matrix pair (E, A) is regular and impulse free.

3.2 Stability analysis

In this subsection, exponential stability of system (2) is investigated. Throughout this subsection, it is assumed that $A_r \in \mathcal{M}(\mathbb{T})$, $\hat{B}_{1j} \succeq 0$, and $\hat{B}_{2j} \preceq 0$, $j = 1, 2$.

To proceed with the main result, the following hypothesis is essential.

Hypothesis 2. $\sup_{t\in\mathbb{T}}\{\mu(\rho(t))\} \leqslant \tau_1$.

Remark 3. Many kinds of time scales satisfy Hypothesis 2. To illustrate this fact, three examples are given. (a) If $\mathbb{T} = \mathbb{R}$, then $\mu(t) \equiv 0 < \tau_1$ for any t. (b) If $\mathbb{T} = \mathbb{Z}$, then $\mu(t) \equiv 1$ for any $t \in \mathbb{Z}$. And for discrete time-delay systems, $\tau_1 \geqslant 1$ is acquiescent. (c) Construct such a time scale satisfying $\rho(t) \equiv t$ for any t, and then $\mu(\rho(t)) \equiv 0$ and Hypothesis 2 is always true. Moreover, Hypothesis 2 is independent of any right-scattered or right-dense points. Thus, Hypothesis 2 is mild.

Theorem 3. Let Hypothesis 2 hold. For a given positive constant γ, system (2) is exponentially stable if there exist two positive vectors $\alpha = (\alpha_1, \ldots, \alpha_r)^{\mathrm{T}}$, $\beta = (\beta_1, \ldots, \beta_{n-r})^{\mathrm{T}}$, such that

$$(A_r + \gamma I)\alpha + \mathrm{e}^{\gamma\tau_2}(\hat{B}_{11}\alpha + \hat{B}_{12}\beta) \prec 0, \tag{6}$$

and

$$\hat{B}_{21}\alpha + (\mathrm{e}^{-\gamma\tau_2} I + \hat{B}_{22})\beta \succ 0. \tag{7}$$

Proof. In the sequel, we consider the claim:

$$\begin{aligned} S(t) : \hat{x}_1(t) &\preceq M e_{\ominus\gamma}(t, 0)\alpha, \\ \hat{x}_2(t) &\preceq M e_{\ominus\gamma}(t, 0)\beta,\ t \in [0, \infty)_{\mathbb{T}}, \end{aligned}$$

where $M = \frac{\|\hat{\phi}_1\|\|\hat{\phi}_2\|}{\min\{\alpha_i,\beta_j\}}$, $i = 1, \ldots, r$, $j = 1, \ldots, n - r$.

Clearly, $S(0)$ is true.

Suppose the claim $S(t)$ is not satisfied and let $t^* = \inf\{t \in (0, \infty)_{\mathbb{T}} : S(t)$ is not true$\}$. Then, $S(t)$ holds for any $t \in [0, t^*)_{\mathbb{T}}$ and there exist $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, n - r\}$ such that at least one of the following statements is true:

$$\hat{x}_{1,i}(t^*) \geqslant M e_{\ominus\gamma}(t^*, 0)\alpha_i, \tag{8}$$

$$\hat{x}_{2,j}(t^*) \geqslant M e_{\ominus\gamma}(t^*, 0)\beta_j. \tag{9}$$

In what follows, three cases are considered.

Case I. Eq. (8) is true. Then, $\hat{x}_2(t^*) \preceq M e_{\ominus\gamma}(t^*, 0)\beta$.

If t^* is left dense, then $\hat{x}_{1,i}(t^*) = M e_{\ominus\gamma}(t^*, 0)\alpha_i$ and t^* is right dense. Moreover, for any $m \neq i$, $\hat{x}_{1,m}(t^*) \leqslant M e_{\ominus\gamma}(t^*, 0)\alpha_m$. By (2), we have

$$\begin{aligned} \hat{x}^{\Delta}_{1,i}(t^*) =& \sum_{k=1}^{r} A_r(i, k)\hat{x}_{1,k}(t^*) + \sum_{k=1}^{r} \hat{B}_{11}(i, k)\hat{x}_{1,k}(\zeta(t^* - \tau(t^*))) \\ &+ \sum_{k=1}^{n-r} \hat{B}_{12}(i, k)\hat{x}_{2,k}(\zeta(t^* - \tau(t^*))) \\ =& M e_{\ominus\gamma}(t^*, 0)A_r(i, i)\alpha_i + \sum_{k\neq i}^{r} A_r(i, k)\hat{x}_{1,k}(t^*) + \sum_{k=1}^{r} \hat{B}_{11}(i, k)\hat{x}_{1,k}(\zeta(t^* - \tau(t^*))) \\ &+ \sum_{k=1}^{n-r} \hat{B}_{12}(i, k)\hat{x}_{2,k}(\zeta(t^* - \tau(t^*))) \\ \leqslant& M e_{\ominus\gamma}(t^*, 0)\left[\sum_{k=1}^{r} A_r(i, k)\alpha_k \right. \\ &\left. + e_{\ominus\gamma}(\zeta(t^* - \tau(t^*)), t^*)\left(\sum_{k=1}^{r} \hat{B}_{11}(i, k)\alpha_k + \sum_{k=1}^{n-r} \hat{B}_{12}(i, k)\beta_k\right)\right]. \end{aligned} \tag{10}$$

By Lemma 1, we have

$$e_{\ominus\gamma}(\zeta(t^*-\tau(t^*)),t^*)=e_{\gamma}(t^*,\zeta(t^*-\tau(t^*)))\leqslant e^{\gamma\tau_2}.$$

Then, it follows from (6) that

$$\sum_{k=1}^{r}A_r(i,k)\alpha_k+e_{\ominus\gamma}(\zeta(t^*-\tau(t^*)),t^*)\times\left(\sum_{k=1}^{r}\hat{B}_{11}(i,k)\alpha_k+\sum_{k=1}^{n-r}\hat{B}_{12}(i,k)\beta_k\right)<0.$$

In view of (10), we can obtain $\hat{x}_{1,i}^{\Delta}(t^*)<0$. Since t^* is right dense, there always exists some $\varepsilon>0$ such that $\hat{x}_{1,i}(t^*+\varepsilon)<\hat{x}_{1,i}(t^*)$, which contradicts with the definition of t^*.

If t^* is left scattered, then $\hat{x}_{1,i}(t^*)\geqslant Me_{\ominus\gamma}(t^*,0)\alpha_i$ and $\hat{x}_1(t)\preceq Me_{\ominus\gamma}(t,0)\alpha$ for any $t\in[0,\rho(t^*)]_{\mathbb{T}}$. Similar to (10), we have

$$\begin{aligned}\hat{x}_{1,i}^{\Delta}(\rho(t^*))\leqslant&A_r(i,i)\hat{x}_{1,i}(\rho(t^*))+Me_{\ominus\gamma}(\rho(t^*),0)\left[\sum_{k\neq i}^{r}A_r(i,k)\alpha_k\right.\\&\left.+e^{\gamma\tau_2}\left(\sum_{k=1}^{r}\hat{B}_{11}(i,k)\alpha_k+\sum_{k=1}^{n-r}\hat{B}_{12}(i,k)\beta_k\right)\right].\end{aligned}\tag{11}$$

Thus, it yields from (11) that

$$\begin{aligned}\hat{x}_{1,i}(t^*)=&\hat{x}_{1,i}(\rho(t^*))+\mu(\rho(t^*))\hat{x}_{1,i}^{\Delta}(\rho(t^*))\\\leqslant&Me_{\ominus\gamma}(\rho(t^*),0)\left[\alpha_i+\mu(\rho(t^*))\left(\sum_{k=1}^{r}A_r(i,k)\alpha_k\right.\right.\\&\left.\left.+e^{\gamma\tau_2}\left(\sum_{k=1}^{r}\hat{B}_{11}(i,k)\alpha_k+\sum_{k=1}^{n-r}\hat{B}_{12}(i,k)\beta_k\right)\right)\right].\end{aligned}\tag{12}$$

By noting $\ominus\gamma=\frac{-\gamma}{1+\mu(t)\gamma}\geqslant-\gamma$, and from (6) and (12), we have

$$\begin{aligned}\hat{x}_{1,i}(t^*)&<Me_{\ominus\gamma}(\rho(t^*),0)(1+\mu(\rho(t^*))\ominus\gamma)\alpha_i\\&=Me_{\ominus\gamma}(\rho(t^*),0)e_{\ominus\gamma}(t^*,\rho(t^*))\alpha_i\\&=Me_{\ominus\gamma}(t^*,0)\alpha_i,\end{aligned}$$

which is a contradiction. Thus, case I is not satisfied.

Case II. Eq. (9) is true. Then, $\hat{x}_1(t^*)\preceq Me_{\ominus\gamma}(t^*,0)\alpha$.

If t^* is left dense, then $\hat{x}_{2,j}(t^*)=Me_{\ominus\gamma}(t^*,0)\beta_j$ and $\hat{x}_{2,p}(t^*)\leqslant Me_{\ominus\gamma}(t^*,0)\beta_p$ for any $p\neq j$. By observing $\zeta(t^*-\tau(t^*))\leqslant t^*$, it follows from (2) that

$$\begin{aligned}\hat{x}_{2,j}(t^*)=&-\sum_{k=1}^{r}\hat{B}_{21}(j,k)\hat{x}_{1,k}(\zeta(t^*-\tau(t^*)))\\&-\sum_{k=1}^{n-r}\hat{B}_{22}(j,k)\hat{x}_{2,k}(\zeta(t^*-\tau(t^*)))\\\leqslant&Me_{\ominus\gamma}(t^*,0)e^{\gamma\tau_2}\left(-\sum_{k=1}^{r}\hat{B}_{21}(j,k)\alpha_k-\sum_{k=1}^{n-r}\hat{B}_{22}(j,k)\beta_k\right).\end{aligned}\tag{13}$$

By (7), Eq. (13) yields $\hat{x}_{2,j}(t^*)<Me_{\ominus\gamma}(t^*,0)\beta_j$, which is a contradiction.

If t^* is left scattered, then $t^*>\rho(t^*)$ and $\hat{x}_{2,j}(t^*)\geqslant Me_{\ominus\gamma}(t^*,0)\beta_j$. By Hypothesis 2, we have $t^*-\rho(t^*)\leqslant\tau_1\leqslant\tau(t^*)$, which yields $\zeta(t^*-\tau(t^*))\leqslant\rho(t^*)<t^*$. Similar to (13), we can get $\hat{x}_{2,j}(t^*)<Me_{\ominus\gamma}(t^*,0)\beta_j$, which is also a contradiction. Therefore, case II does not hold.

Case III. Both Eqs. (8) and (9) are true.

If t^* is left dense, then $\hat{x}_{1,i}(t^*) = Me_{\ominus\gamma}(t^*,0)\alpha_i$ and $\hat{x}_{2,j}(t^*) = Me_{\ominus\gamma}(t^*,0)\beta_j$. In addition, $\hat{x}_{1,p}(t^*) \leqslant Me_{\ominus\gamma}(t^*,0)\alpha_p$ for any $p \neq i$, and $\hat{x}_{2,q}(t^*) \leqslant Me_{\ominus\gamma}(t^*,0)\beta_q$ for any $q \neq j$. Similar to case II, a contradiction occurs.

If t^* is left scattered, then for any $t \in [0,t^*)_{\mathbb{T}}$, we have $\hat{x}_1(t) \preceq Me_{\ominus\gamma}(t,0)\alpha$ and $\hat{x}_2(t) \preceq Me_{\ominus\gamma}(t,0)\beta$. Following the same steps in case II, we can also obtain a contradiction. Hence, case III is not satisfied. Based on the foregoing discussion, $S(t)$ is true. Therefore, for any $t \in [0,\infty)_{\mathbb{T}}$, $\|\hat{x}(t)\| \leqslant Me_{\ominus\gamma}(t,0)(\|\alpha\| + \|\beta\|)$, which means that system (2) is exponentially stable. The proof of Theorem 3 is completed.

4 Systems with infinite delays

Consider the following system:

$$Ex^{\Delta}(t) = Ax(t) + Bx_t + \omega,\ t \in \mathbb{T}, \tag{14}$$

where the time-delay function $\tau(t)$ satisfies $\tau(t) \geqslant \tau_1 > 0$ and $t - \tau(t) \to \infty$ as $t \to \infty$ [30], the vector $\omega = (\omega_1^{\mathrm{T}}, \omega_2^{\mathrm{T}})^{\mathrm{T}} \succeq 0$, and other parameters are the same as those in (1).

Similarly, system (14) can be transformed into the following form:

$$\begin{cases} \hat{x}_1^{\Delta}(t) = A_r\hat{x}_1(t) + \hat{B}_{11}\hat{x}_1(\zeta(t-\tau(t))) + \hat{B}_{12}\hat{x}_2(\zeta(t-\tau(t))) + \omega_1, \\ \hat{x}_2(t) = -\hat{B}_{21}\hat{x}_1(\zeta(t-\tau(t))) - \hat{B}_{22}\hat{x}_2(\zeta(t-\tau(t))) + \omega_2, \end{cases} \tag{15}$$

where $A_r \in \mathcal{R}^{r\times r}$, and the initial condition is $\hat{\phi}_1(t)$, $\hat{\phi}_2(t)$, $t \in (-\infty, 0]_{\mathbb{T}}$.

First, a necessary and sufficient criterion about the positivity of system (15) is developed in the following theorem.

Theorem 4. For any $\omega_i \succeq 0$ $(i = 1, 2)$, system (15) is positive if and only if $A_r \in \mathcal{M}(\mathbb{T})$, $\hat{B}_{1j} \succeq 0$, and $\hat{B}_{2j} \preceq 0$, $j = 1, 2$.

Proof. (Sufficiency): Consider the claim:

$$S(t) : \hat{x}_1(t) \succeq 0,\ \hat{x}_2(t) \succeq 0,\ t \in [0,\infty)_{\mathbb{T}}.$$

Suppose the claim $S(t)$ is not satisfied and let $t^* = \inf\{t \in (0,\infty)_{\mathbb{T}} : S(t) \text{ is not true}\}$. Then, $S(t)$ holds for any $t \in [0,t^*)_{\mathbb{T}}$ and there exist $i \in \{1,\ldots,r\}$ and $j \in \{1,\ldots,n-r\}$ such that at least one of the following statements is satisfied:

$$\hat{x}_{1,i}(t^*) \leqslant 0, \tag{16}$$

$$\hat{x}_{2,j}(t^*) \leqslant 0. \tag{17}$$

In what follows, three cases are considered.

Case I. Eq. (16) is true. Then, $\hat{x}_2(t^*) \succeq 0$.

If t^* is left dense, then $\hat{x}_{1,i}(t^*) = 0$ and t^* is right dense. Moreover, for any $m \neq i$, $\hat{x}_{1,m}(t^*) \geqslant 0$. Following (15), we have

$$\begin{aligned} \hat{x}_{1,i}(t^*) = {} & e_{A_r(i,i)}(t^*,0)\hat{x}_{1,i}(0) + \int_0^{t^*} e_{A_r(i,i)}(t^*,\sigma(r)) \\ & \times \Bigg[\sum_{k\neq i}^{r} A_r(i,k)\hat{x}_{1,k}(\zeta(r-\tau(r))) \\ & + \sum_{k=1}^{r} \hat{B}_{11}(i,k)\hat{x}_{1,k}(\zeta(r-\tau(r))) \\ & + \sum_{k=1}^{n-r} \hat{B}_{12}(i,k)\hat{x}_{2,k}(\zeta(r-\tau(r))) + \omega_{1,i}\Bigg]\Delta r. \end{aligned}$$

By Lemmas 2 and 3, we have $\hat{x}_{1,i}(t^*) > 0$, which is a contradiction.

If t^* is left scattered, then $\hat{x}_{1,i}(t^*) \leqslant 0$, and for any $m \neq i$, $\hat{x}_{1,m}(t^*) \geqslant 0$. From (15), we have

$$\begin{aligned}\hat{x}_{1,i}(t^*) =\ & e_{A_r(i,i)}(t^*,0)\hat{x}_{1,i}(0) + \int_0^{\rho(t^*)} e_{A_r(i,i)}(t^*,\sigma(r)) \\ & \times\left[\sum_{k\neq i}^{r} A_r(i,k)\hat{x}_{1,k}(\zeta(r-\tau(r)))\right. \\ & + \sum_{k=1}^{r}\hat{B}_{11}(i,k)\hat{x}_{1,k}(\zeta(r-\tau(r))) \\ & \left. + \sum_{k=1}^{n-r}\hat{B}_{12}(i,k)\hat{x}_{2,k}(\zeta(r-\tau(r))) + \omega_{1,i}\right]\Delta r \\ & + \sum_{k\neq i}^{r} A_r(i,k)\hat{x}_{1,k}(\zeta(\rho(t^*)-\tau(\rho(t^*)))) \\ & + \sum_{k=1}^{r}\hat{B}_{11}(i,k)\hat{x}_{1,k}(\zeta(\rho(t^*)-\tau(\rho(t^*)))) \\ & + \sum_{k=1}^{n-r}\hat{B}_{12}(i,k)\hat{x}_{2,k}(\zeta(\rho(t^*)-\tau(\rho(t^*)))) + \omega_{1,i}.\end{aligned}$$

By Lemmas 2 and 3, we have $\hat{x}_{1,i}(t^*) > 0$, which is also a contradiction. Thus, case I is not true.

Case II. Eq. (17) is true. Then, $\hat{x}_1(t^*) \succeq 0$.

If t^* is left dense, then $\hat{x}_{2,j}(t^*) = 0$, and for any $q \neq j$, $\hat{x}_{2,q}(t^*) \geqslant 0$. By (15), we can obtain

$$\begin{aligned}\hat{x}_{2,j}(t^*) =\ & -\sum_{k=1}^{r}\hat{B}_{21}(j,k)\hat{x}_{1,k}(\zeta(t^*-\tau(t^*))) \\ & -\sum_{k=1}^{n-r}\hat{B}_{22}(j,k)\hat{x}_{2,k}(\zeta(t^*-\tau(t^*))) + \omega_{2,j}.\end{aligned}$$

When $\zeta(t^*-\tau(t^*)) = t^*$, we have

$$\begin{aligned}(1+\hat{B}_{22}(j,j))\hat{x}_{2,j}(t^*) =\ & -\sum_{k=1}^{r}\hat{B}_{21}(j,k)\hat{x}_{1,k}(\zeta(t^*-\tau(t^*))) \\ & -\sum_{k\neq j}^{n-r}\hat{B}_{22}(j,k)\hat{x}_{2,k}(\zeta(t^*-\tau(t^*))) + \omega_{2,j}.\end{aligned}$$

Since $-\hat{B}_{22}$ is a Schur matrix, $\hat{x}_{2,j}(t^*) > 0$. When $\zeta(t^*-\tau(t^*)) < t^*$, we can also get $\hat{x}_{2,j}(t^*) > 0$, which contradicts with the fact $\hat{x}_{2,j}(t^*) = 0$.

If t^* is left scattered, then $\hat{x}_{2,j}(t^*) \leqslant 0$ and for any $q \neq j$, $\hat{x}_{2,q}(t^*) \geqslant 0$. Similarly, a contradiction also occurs. Thus, case II is not satisfied.

Case III. Both Eqs. (16) and (17) are true. Then, $\hat{x}_{1,i}(t^*) \leqslant 0$ and $\hat{x}_{2,j}(t^*) \leqslant 0$. Moreover, for any $m \neq i$, $q \neq j$, $\hat{x}_{1,m}(t^*) \geqslant 0$ and $\hat{x}_{2,q}(t^*) \geqslant 0$. Taking the similar analysis to case I, case III is false.

Based on the three cases, the claim $S(t)$ is true. Therefore, the sufficiency part is finished.

(Necessity): Firstly, we prove $\hat{B}_{12} \succeq 0$ and $\hat{B}_{22} \preceq 0$. Choose $\hat{\phi}_1(t) = 0$, and then by (15) we have

$$\hat{x}_1^{\Delta}(0) = \hat{B}_{12}\hat{x}_2(\zeta(-\tau(0))) + \omega_1.$$

Assuming $\hat{B}_{12} \not\succeq 0$ is true, there maybe exist $k \in \{1,\ldots,r\}$ such that $\hat{x}_{1,k}^{\Delta}(0) < 0$. If 0 is right scattered, then

$$\hat{x}_{1,k}(\sigma(0)) = \hat{x}_{1,k}(0) + \mu(0)\hat{x}_{1,k}^{\Delta}(0) < 0,$$

which is a contradiction. If 0 is right dense, there must exist $t^* \in U^+(0)$ such that $\hat{x}_{1,k}(t^*) < 0$, which is also a contradiction. Thus, $\hat{B}_{12} \succeq 0$.

From (15) and $\hat{\phi}_1(t) = 0$, we have

$$\hat{x}_2(0) = -\hat{B}_{22}\hat{x}_2(\zeta(-\tau(0))) + \omega_2.$$

By the contradiction, we can obtain $\hat{B}_{22} \preceq 0$.

Similarly, we can prove $\hat{B}_{11} \succeq 0$ and $\hat{B}_{21} \preceq 0$ by choosing $\hat{\phi}_2(t) = 0$. So it is omitted here.

Finally, we prove $A_r \in \mathcal{M}(\mathbb{T})$. Choose $\hat{x}_1(t) = 0$, $t \in (-\infty, 0)_{\mathbb{T}}$ and $\hat{\phi}_2(t) = 0$, and then by (15), we have

$$\begin{aligned}\hat{x}_1(t) &= e_{A_r}(t,0)\hat{x}_1(0) + \textstyle\int_0^t e_{A_r}(t,\sigma(r))(\hat{B}_{11}\hat{x}_1(\zeta(r-\tau(r))) \\ &\quad + \hat{B}_{12}\hat{x}_2(\zeta(r-\tau(r))) + \omega_1)\Delta r.\end{aligned}$$

If 0 is right dense, there always exists $\epsilon \in U^+(0) \bigcap [0, \tau_1]_{\mathbb{T}}$ such that

$$\hat{x}_1(\epsilon) = e_{A_r}(\epsilon, 0)\hat{x}_1(0) + \int_0^{\epsilon} \mathrm{e}^{A_r(\epsilon - r)}(\hat{B}_{11}\hat{x}_1(\zeta(r-\tau(r))) + \omega_1)\mathrm{d}r.$$

Supposing $A_r \notin \mathcal{M}(\mathbb{T})$, then by Lemma 2, $e_{A_r}(\epsilon, 0) \succeq 0$ may be unsatisfied. Thus, $\hat{x}_1(\epsilon) \not\succeq 0$ may be true by choosing proper initial condition $\hat{x}_1(0)$, which contradicts with the positivity of system (15). And hence $A_r \in \mathcal{M}(\mathbb{T})$.

If 0 is right scattered, then

$$\begin{aligned}\hat{x}_1(\sigma(0)) &= e_{A_r}(\sigma(0), 0)\hat{x}_1(0) + \int_0^{\sigma(0)} e_{A_r(\sigma(0),\sigma(r))} \\ &\quad \times(\hat{B}_{11}\hat{x}_1(\zeta(r-\tau(r))) + \hat{B}_{12}\hat{x}_2(\zeta(r-\tau(r))) + \omega_1)\Delta r \\ &= (I + \mu(0)A_r)\hat{x}_1(0) + \mu(0)(\hat{B}_{11}\hat{x}_1(\zeta(-\tau(0))) + \omega_1).\end{aligned}$$

Following the above process, we can also get $A_r \in \mathcal{M}(\mathbb{T})$, which finishes the necessity part. Therefore, the proof of Theorem 4 is completed.

Next, the asymptotical stability problem of system (15) is addressed. In the sequel, it is assumed that $\omega_1 = 0$ and $\omega_2 = 0$.

Lemma 4. Let A be a Metzler matrix and $B_i \succeq 0$, $i = 1, \ldots, 4$. Then, the following statements are equivalent:

(i) B_4 is a Schur matrix and $\varrho(A + B_1 + B_2(I - B_4)^{-1}B_3) < 0$;

(ii) $(A + B_1)p + B_2 q \prec 0$ and $B_3 p + B_4 q \prec q$ for some $p \succ 0$, $q \succ 0$.

Proof. Since A is a Metzler matrix and $B_i \succeq 0$, $i = 1, \ldots, 4$, it holds that $A + B_1 + B_2(I - B_4)^{-1}B_3$ is a Metzler matrix. The remainder of the proof is similar to Lemma II.2 in [36], so it is omitted here.

Theorem 5. Let $A_r \in \mathcal{M}(\mathbb{T})$ and $\hat{B}_{1,j} \succeq 0$, $\hat{B}_{2,j} \preceq 0$, $j = 1, 2$. System (15) is asymptotically stable if $\hat{B}_{2,2}$ is a Schur matrix and $\varrho(A_r + \hat{B}_{11} - \hat{B}_{12}(I + \hat{B}_{22})^{-1}\hat{B}_{21}) < 0$.

Proof. Since $\hat{B}_{2,2}$ is a Schur matrix and $\varrho(A_r + \hat{B}_{11} - \hat{B}_{12}(I + \hat{B}_{22})^{-1}\hat{B}_{21}) < 0$, it follows from Lemma 4 that

$$(A_r + \hat{B}_{11})\varphi_1 + \hat{B}_{12}\varphi_2 \prec 0, \tag{18}$$

$$-\hat{B}_{21}\varphi_1 - \hat{B}_{22}\varphi_2 \prec \varphi_2, \tag{19}$$

where φ_1 and φ_2 are positive constant vectors. Let $y_i(t) = \varphi_i - \hat{x}_i(t, \varphi_1, \varphi_2)$, $i = 1, 2$. Then, by (15), we have

$$\begin{cases} y_1^{\Delta}(t) = A_r y_1(t) + \hat{B}_{11} y_1(\zeta(t-\tau(t))) \\ \qquad\qquad + \hat{B}_{12} y_2(\zeta(t-\tau(t))) - (A_r + \hat{B}_{11})\varphi_1 - \hat{B}_{12}\varphi_2, \\ y_2(t) = -\hat{B}_{21} y_1(\zeta(t-\tau(t))) - \hat{B}_{22} y_2(\zeta(t-\tau(t))) + \hat{B}_{21}\varphi_1 + \hat{B}_{22}\varphi_2 + \varphi_2. \end{cases}$$

By (18) and (19), it holds from Theorem 4 that $\varphi_i \succeq \hat{x}_i(t, \varphi_1, \varphi_2)$ for $t \in [0, \infty)_{\mathbb{T}}$.

Then, from (15), we can get $\hat{x}_1^{\Delta}(t) \leqslant A_r \hat{x}_1(t) + \varphi$, where $\varphi = \hat{B}_{11}\varphi_1 + \hat{B}_{12}\varphi_2$.

Since $A_r \in \mathcal{M}(\mathbb{T})$, by Lemma 2, it holds that the argument $\mu(t)(A_r u + \varphi) + u = e_A(\sigma(t), t)u + \mu(t)\varphi$ is non-decreasing with vector u. Thus, following the timescale-type comparison principle Theorem 4.1.2 in [37], we have $\hat{x}_1(t, \varphi_1, \varphi_2) \preceq z(t, \varphi_1)$, where $z(t)$ is the solution of the comparison system $z^{\Delta}(t) = A_r z(t) + \varphi$ with initial condition $z(0) = \varphi_1$.

In view of the term (ii) of Lemma 4 and Theorem II.1 in [36], we have $\varrho(A_r) < 0$ and $A_r^{-1} \preceq 0$. Introducing $v(t) = z(t, \varphi_1) + A_r^{-1}\varphi$, then $v^{\Delta}(t) = A_r v(t)$. Since $\varrho(A_r) < 0$, the timescale-type Lyapunov inequality $A_r^{\mathrm{T}} Q + QA_r + \mu(t) A_r^{\mathrm{T}} Q A_r < 0$ can be true for some $Q > 0$. Then, by Theorem 4.2 in [38], it yields $\lim_{t\to\infty} v(t) = 0$, which implies $\lim_{t\to\infty} \hat{x}_1(t, \varphi_1, \varphi_2) \preceq \lim_{t\to\infty} z(t, \varphi_1) = -A_r^{-1}\varphi \prec \varphi_1$.

By (15) and (19), we have $\varphi_2 \succ \hat{x}_2(t, \varphi_1, \varphi_2)$. Thus, there exist $t^* \in [0, \infty)_{\mathbb{T}}$ and a constant $\eta \in (0, 1)$, such that $\hat{x}_i(t, \varphi_1, \varphi_2) \preceq \eta\varphi_i$ for any $t \in [t^*, \infty)_{\mathbb{T}}$. The rest of the proof is similar to Theorem 2 in [30], so we omit here.

Remark 4. Due to the unboundedness of the time-delay function $\tau(t)$, we think it is impossible to make the algebraic equation of (15) achieve exponential stability. Thus, we consider asymptotical stability of system (15) in Theorem 5.

5 Application to timescale-type DDSs

As an application, we apply the foregoing results about system (1) and system (14) to DDSs. For simplicity, we only consider the former.

For system (1), let $x = (y^{\mathrm{T}}, z^{\mathrm{T}})^{\mathrm{T}}$, $E = \left(\begin{smallmatrix} I & \\ & 0 \end{smallmatrix}\right)$, $A = \left(\begin{smallmatrix} C & 0 \\ D & -I \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} 0 & G \\ 0 & H \end{smallmatrix}\right)$, where matrices C, D, G and H are with proper dimensions, then system (1) can be rewritten as the following timescale-type DDSs:

$$\begin{cases} y^{\Delta}(t) = Cy(t) + Gz_t, \\ z(t) = Dy(t) + Hz_t. \end{cases} \tag{20}$$

In addition, if $B = \left(\begin{smallmatrix} G_1 & G_2 \\ H_1 & H_2 \end{smallmatrix}\right)$, where matrices G_i and H_i $(i = 1, 2)$ are with proper dimensions, then system (14) can be rewritten as the following DDSs:

$$\begin{cases} y^{\Delta}(t) = Cy(t) + G_1 y_t + G_2 z_t, \\ z(t) = Dy(t) + H_1 y_t + H_2 z_t. \end{cases} \tag{21}$$

It can be observed that $\det(sE - A) = \det(sI - C)$. Then, it is assumed that (I, C) is regular and impulse-free throughout this section. Similar to system (1), systems (20) and (21) can be also transformed into a system like (2). Therefore, the positivity and stability problems of systems (20) and (21) can be addressed by Theorems 1, 3–5.

In [21], the positivity and stability of system (20) with bounded time-delays are also investigated. The related results are introduced in the following theorems.

Theorem 6 ([21])**.** System (20) is positive if and only if $C \in \mathcal{M}(\mathbb{T})$, $G \succeq 0$, $D \succeq 0$, and $H \succeq 0$.

Theorem 7 ([21])**.** System (20) is exponentially stable if there are two positive vectors α and β such that the following conditions hold:

$$C\alpha + G\beta \prec 0,$$

$$D\alpha + (H - I)\beta \prec 0,$$

and

$$(I - H)^{-1} D\alpha \prec \beta.$$

Moreover, system states $y(t)$ and $z(t)$ respectively satisfy $y(t) \preceq Ke_{\ominus\gamma}(t, 0)\alpha$ and $z(t) \preceq Ke_{\ominus\gamma}(t, 0)\beta$, where $t \in [0, \infty)_{\mathbb{T}}$, constant $K > 0$, and constant $\gamma > 0$ is related to convergence rate.

However, in the proof of Theorem 7, the actually-used stability criteria are as follows:

$$C\alpha + e_{\ominus\gamma}(\lambda - \tau(\lambda), \lambda) G\beta \prec (\ominus\gamma)\alpha, \tag{22}$$

$$D\alpha + e_{\ominus\gamma}(\lambda - \tau(\lambda), \lambda) H\beta \prec \beta, \tag{23}$$

where $\lambda \in [0, \infty)_{\mathbb{T}}$.

The following theorem states the positivity and stability criteria of system (21) with infinite time-delays in [30].

Theorem 8 ([20, 30]). Let $\mathbb{RD}$ be the set of all right-scattered points in $\mathbb{T}$. Consider system $w^{\Delta}(t) = Mw(t)$. If M is Hurwitz, the set $\mathbb{S} = \{t \in \mathbb{RD} : \varrho(I + \mu(t)M) \geqslant 1\}$ is finite, and $\varrho(I + \mu(t)M) \leqslant \epsilon < 1$ for any $t \in \mathbb{RD} \cap \mathcal{C}_{\mathbb{T}}\mathbb{S}$. Then, $\lim_{t\to\infty} w(t) = 0$.

The comparison between the studies [21, 30] and our work is made in the following remark.

Remark 5. Compared with the results in [21, 30], our work shows its advantages in the following aspects.

(i) As is stated in Remark 1, the restriction on time-delay function $\tau(t)$ is relaxed. In addition, systems (20) and (21) are special cases of system (1), which means the scope of the studied system in this paper is wider than those in [21, 30]. Therefore, our results are generalizations of those in [21, 30].

(ii) In light of the condition $D\alpha + (H - I)\beta \prec 0$ in Theorem 7 and $e_{\ominus\gamma}(\lambda - \tau(\lambda), \lambda) = e_{\gamma}(\lambda, \lambda - \tau(\lambda))$ (obtained by the term (iv) in Lemma 1), the condition for (23) is that the term $e_{\gamma}(\lambda, \lambda - \tau(\lambda))$ should sufficiently approximate to 1. Since the time-delay function $\tau(t)$ is given in advance, the constant γ must be sufficiently small to make (23) hold. As a result, the stability conditions in Theorem 7 are quite strict. And by noting Definition 4 and $\ominus\gamma = \frac{-\gamma}{1+\mu(t)\gamma}$, both conditions (22) and (23) rely on the time-delay function $\tau(t)$. Moreover, the condition (22) further depends on the function $\mu(t)$. Thus, Eqs. (22) and (23) are conservative. Comparatively, the inequalities (6) and (7) in Theorem 3 are independent of $\mu(t)$ and $\tau(t)$. And Eqs. (6) and (7) are easier to be satisfied since all the parameters are constant. In addition, it can be observed from (6) and (7) that the constant γ does not necessarily have to be sufficiently small. Thus, the stability conditions in Theorem 3 are less conservative than those in Theorem 7.

(iii) In [30], Theorem 8 is fairly important to analyze the positivity and asymptotical stability of system (21). However, the conditions that $\varrho(I + \mu(t)M) \leqslant \epsilon < 1$ and the set $\mathbb{S}$ is finite are not easy to be verified. The reason is that both of them rely on the function $\mu(t)$. Let us consider some types of time scales. If time scale $\mathbb{T}$ is uniform (e.g., $\mathbb{T} = h\mathbb{Z}$ with constant $h > 0$), then $\mu(t) \equiv h$. If $\mathbb{T}$ is some special kind of non-uniform time domain, such as $\mathbb{T} = \bigcup_{k\in\mathbb{Z}}[2k, 2k+1]$, then $\mu(t) \equiv 1$ for all right-scattered points $2k+1$. For these cases, the above two conditions can be easily verified because $\mu(t)$ is constant. However, in some general cases, such as $\mathbb{T} = 2^{\mathbb{Z}} := \{2^k : k \in \mathbb{Z}\}$, or time scale $\mathbb{T}$ with $\mu(t) \leqslant \mu^*$ for some constant μ^*, the function $\mu(t)$ is time varying. As a result, the conditions that $\varrho(I + \mu(t)M) \leqslant \epsilon < 1$ and the set $\mathbb{S}$ is finite are difficult to be satisfied. Thus, we consider Theorem 8 is not so effective for the positivity and asymptotical stability analysis of system (21). Moreover, this issue has also been pointed out in Remark 5 of [30]. Actually, the proof of Theorem 8 in [20, 30] is based on the timescale-type eigenvalue criterion. In our recent work [38], asymptotical stability is analyzed by the timescale-type Lyapunov's second method. The only requirement on the function $\mu(t)$ is bounded above, i.e., $\mu(t) \leqslant \mu^*$ for some constant μ^*. With this treatment, the conservative issue of Theorem 8 can be relaxed.

6 Illustrated examples

In this section, we consider system (1) with the following parameters [21]:

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} -0.9 & 0.2 & 0 \\ 0.3 & -0.8 & 0 \\ 0.5 & 0.3 & -1 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & 0.4 \\ 0 & 0 & 0.6 \\ 0 & 0 & 0.3 \end{pmatrix}, \quad \mathbb{T} = \bigcup_{k\in\mathbb{Z}}[2k, 2k+1].$$

Obviously, the pair (E, A) is regular and impulse free. In the sequel, the considered system with bounded and infinite time delays is respectively investigated.

6.1 Bounded time delay

First, the time-delay function is the same as the example 2 in [21], i.e., $\tau(t) = 1$ if $t = 2k+1$, and $\tau(t) = \{t\}$ if $t \in [2k, 2k+1)$, where $\{t\}$ is the decimal part of t. When $t \in [2k, 2k+1)$, $\sup_{t\in\mathbb{T}}\{\mu(\rho(t))\} = 0 \leqslant \tau(t)$. And when $t = 2k+1$, $\sup_{t\in\mathbb{T}}\{\mu(\rho(t))\} = 1 \leqslant \tau(t)$. Thus, Hypothesis 2 is satisfied.

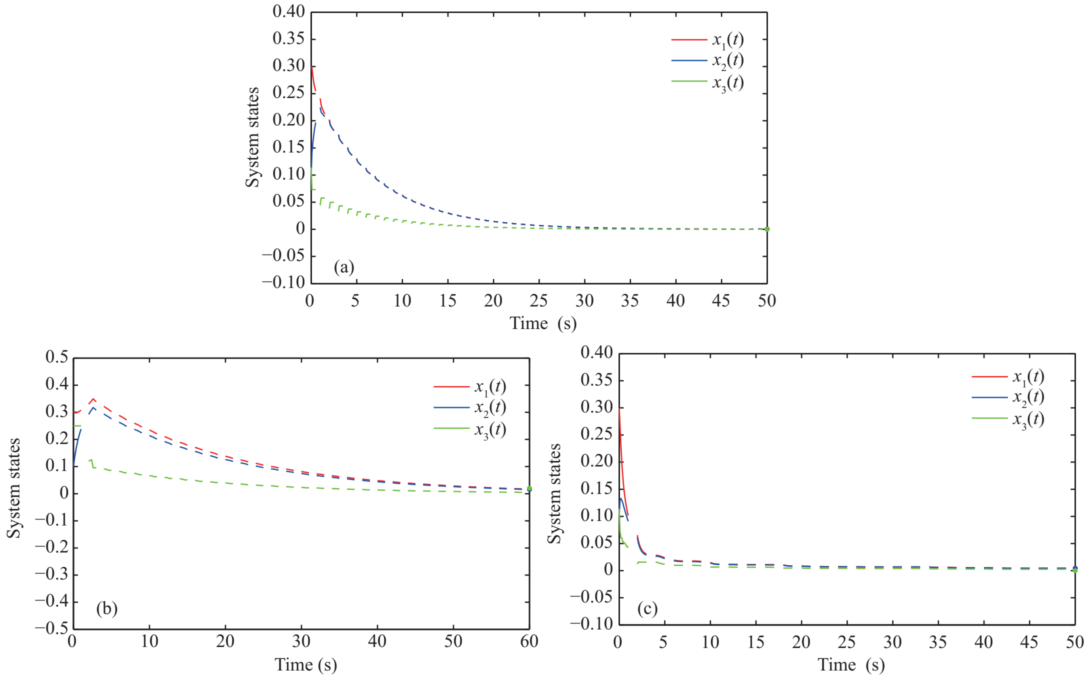

Figure 1 (Color online) Trajectories of system states. (a) and (b) Systems with bounded time delay; (c) system with infinite time delay.

Let nonsingular matrices P, Q be

$$P = \begin{pmatrix} 1.5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{3} & 0.3 & 1 \end{pmatrix}.$$

Then, we have

$$PEQ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad PAQ = \begin{pmatrix} -0.9 & 0.3 & 0 \\ 0.2 & -0.8 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad PBQ = \begin{pmatrix} 0.2 & 0.18 & 0.6 \\ 0.2 & 0.18 & 0.6 \\ -0.1 & -0.09 & -0.3 \end{pmatrix}.$$

Clearly, $A_r \in \mathcal{M}(\mathbb{T})$, $\hat{B}_{1j} \succeq 0$, $\hat{B}_{2j} \preceq 0$, $j = 1, 2$, and $-\hat{B}_{22}$ is Schur. Thus, the considered system is positive.

Choose the same initial condition as that in the example 2 of [21] and let $\gamma = 0.025$. Then, solving the LMIs (6) and (7) by means of the Matlab toolbox, we can get $\alpha = (23.1007, 23.0988)^{\mathrm{T}}$ and $\beta = 6.6454$. Therefore, all conditions of Theorem 3 are satisfied, which yields that the considered system is exponentially stable. Figure 1(a) shows the simulation result.

Next, the time-delay function is chosen as $\tau(t) = 1.3 + 0.2\sin t$. In this case, $t - \tau(t) \notin \mathbb{T}$ can be surely true for some t. Thus, the stability problem of considered system cannot be addressed by the approaches in [21]. Fortunately, this problem can be solved by Theorem 3. By noting $\sup_{t\in\mathbb{T}}\{\mu(\rho(t))\} = 1 \leqslant \tau_1$, Hypothesis 2 is satisfied.

By choosing the forgoing parameters P, Q, γ, α, and β, all conditions of Theorem 3 are also satisfied. Hence, the considered system is exponentially stable. Figure 1(b) shows the simulation result.

6.2 Infinite time delay

Let the time-delay function be $\tau(t) = \frac{t}{2}$. By some calculations, we can get $\varrho(A_r + \hat{B}_{11} - \hat{B}_{12}(I + \hat{B}_{22})^{-1}\hat{B}_{21}) = -0.0571 < 0$. Thus, all conditions of Theorem 5 are satisfied, which yields that the considered system is asymptotically stable. Figure 1(c) shows the simulation result.

7 Conclusion

In this paper, the positivity and stability problems of timescale-type LSSs have been investigated. To analyze the positivity of LSSs with bounded and infinite time delays, some necessary and sufficient criteria have been obtained. For systems with bounded time delays, exponential stability has been analyzed. For systems with infinite time delays, asymptotical stability problems have been solved. An example has been presented to illustrate the effectiveness of the proposed methods. Future work will focus on nonlinear positive singular systems and stochastic singular systems.

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