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Static Output Feedback Control for Uncertain Roesser-Type Continuous-Time Two-Dimensional Piecewise Affine Systems

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Appendix A The Analysis of Regularity and Non-impulsiveness

The closed loop system (3) is in a descriptor form and its regularity and non-impulsiveness can be guaranteed by finding two nonsingular matrices $\mathcal{M}_i$ and $\mathcal{N}_i$ such that [1]:

$$\mathcal{M}_i E_i \mathcal{N}_i = \mathsf{diag}\{\mathbf{I}_{n_{\bar{x}}}, \mathbf{0}_{n_u\times n_u}\},\ \mathcal{M}_i \bar{A}_i \mathcal{N}_i = \mathsf{diag}\{\mathbf{A}_i,\ \mathbf{I}_{n_u}\}, \tag{A1}$$

which are respectively equivalent to

$$\mathcal{M}_i^{-1}\mathsf{diag}\{\mathbf{I}_{n_{\bar{x}}}, \mathbf{0}_{n_u\times n_u}\}\mathcal{N}_i^{-1} = E_i,\ \mathcal{M}_i^{-1}\mathsf{diag}\{\mathbf{A}_i,\ \mathbf{I}_{n_u}\}\mathcal{N}_i^{-1} = \bar{A}_i. \tag{A2}$$

Define

$$\mathcal{M}_i^{-1}:=\begin{cases}\begin{bmatrix}\mathbf{I}_{n_x} & B_i+\Delta B_i\\ \mathbf{0}_{n_u\times n_x} & -\mathbf{I}_{n_u}\end{bmatrix}, & i\in\mathcal{I}_0,\\ \begin{bmatrix}1 & \mathbf{0}_{1\times n_x} & \mathbf{0}_{1\times n_u}\\ \mathbf{0}_{n_x\times 1} & \mathbf{I}_{n_x} & B_i+\Delta B_i\\ \mathbf{0}_{n_u\times 1} & \mathbf{0}_{n_u\times n_x} & -\mathbf{I}_{n_u}\end{bmatrix}, & i\in\mathcal{I}_1,\end{cases}\quad \mathcal{N}_i^{-1}:=\begin{cases}\begin{bmatrix}\mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_u}\\ -K_i(C_i+\Delta C_i) & \mathbf{I}_{n_u}\end{bmatrix}, & i\in\mathcal{I}_0,\\ \begin{bmatrix}1 & \mathbf{0}_{1\times n_x} & \mathbf{0}_{1\times n_u}\\ \mathbf{0}_{n_x\times 1} & \mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_u}\\ -k_i & -K_i(C_i+\Delta C_i) & \mathbf{I}_{n_u}\end{bmatrix}, & i\in\mathcal{I}_1,\end{cases} \tag{A3}$$

and

$$\mathbf{A}_i := \begin{cases}A_i+\Delta A_i+(B_i+\Delta B_i)K_i(C_i+\Delta C_i), & i\in\mathcal{I}_0,\\ \begin{bmatrix}0 & \mathbf{0}_{1\times n_x}\\ a_i+\Delta a_i+(B_i+\Delta B_i)k_i & A_i+\Delta A_i+(B_i+\Delta B_i)K_i(C_i+\Delta C_i)\end{bmatrix}, & i\in\mathcal{I}_1.\end{cases} \tag{A4}$$

Then, it can be verified that the equations in (A2) hold with the matrices defined in (A4), and thus the regularity and non-impulsiveness are ensured.

Appendix B The Proof of Theorem 1

Choose the following piecewise Lyapunov function candidate,

$$V(t_1,t_2) = \bar{x}^\top(t_1,t_2)E_i\mathcal{F}_i^\top\mathcal{P}_i\mathcal{F}_i\bar{x}(t_1,t_2),\ x(t_1,t_2)\in\mathcal{S}_i,\ i\in\mathcal{I} \tag{B1}$$

where $\mathcal{F}_i := \mathsf{diag}\left\{\bar{F}_i,\ \mathbf{I}_{n_u}\right\}$ with $\bar{F}_i$ and $\mathcal{P}_i$ are defined as in (5) and (7), respectively. Note that $V(t_1,t_2)$ can be rewritten as

$$V(t_1,t_2) = V^h(t_1,t_2)+V^v(t_1,t_2),\ x(t_1,t_2)\in\mathcal{S}_i,\ i\in\mathcal{I}, \tag{B2}$$

where

$$V^h(t_1,t_2):=\begin{cases}x^{h\top}(t_1,t_1)F_i^{h\top}P_hF_i^hx^h(t_1,t_1), & i\in\mathcal{I}_0,\\ \begin{bmatrix}1\\ x^h(t_1,t_1)\end{bmatrix}^\top\begin{bmatrix}f_i^{h\top}\\ F_i^{h\top}\end{bmatrix}P_h\begin{bmatrix}f_i^{h\top}\\ F_i^{h\top}\end{bmatrix}^\top\begin{bmatrix}1\\ x^h(t_1,t_1)\end{bmatrix}, & i\in\mathcal{I}_1,\end{cases} \tag{B3}$$

$$V^v(t_1,t_2):=\begin{cases}x^{v\top}(t_1,t_1)F_i^{v\top}P_vF_i^vx^v(t_1,t_1), & i\in\mathcal{I}_0,\\ \begin{bmatrix}1\\ x^v(t_1,t_1)\end{bmatrix}^\top\begin{bmatrix}f_i^{v\top}\\ F_i^{v\top}\end{bmatrix}P_v\begin{bmatrix}f_i^{v\top}\\ F_i^{v\top}\end{bmatrix}^\top\begin{bmatrix}1\\ x^v(t_1,t_1)\end{bmatrix}, & i\in\mathcal{I}_1.\end{cases} \tag{B4}$$

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For the case of $i \in \mathcal{I}_0$, there exists a constant $\lambda_h > 0$ guaranteeing

$$V^h(t_1,t_2) \leqslant \lambda_h \|x^h(t_1,t_2)\|^2. \tag{B5}$$

On the other hand, for the case of $i \in \mathcal{I}_1$, it follows that

$$V^h(t_1,t_2) \leqslant \lambda_h \left(\|x^h(t_1,t_2)\|^2 + 1\right) \leqslant \lambda_h \left(\|x^h(t_1,t_2)\|^2 + \frac{\|x^h(t_1,t_2)\|^2}{\tau}\right) = \frac{\lambda_h(1+\tau)}{\tau}\|x^h(t_1,t_2)\|^2, \tag{B6}$$

where $\tau := \min_{x(t_1,t_2)\in\mathcal{S}_i, i\in\mathcal{I}_1} \|x^h(t_1,t_2)\|^2 > 0$, since $\|x^h(t_1,t_2)\| \neq 0$ in the case of $x(t_1,t_2) \in \mathcal{S}_i$, $i \in \mathcal{I}_1$. Taking the inequalities (B5) and (B6) into consideration, one can conclude that

$$V^h(t_1,t_2) \leqslant \beta_h \|x^h(t_1,t_2)\|^2, \tag{B7}$$

where $\beta_h > 0$. In addition, a constant $\alpha_h > 0$ exists such that

$$\alpha_h \|x^h(t_1,t_2)\|^2 \leqslant V^h(t_1,t_2). \tag{B8}$$

Then, one has

$$\alpha_h \|x^h(t_1,t_2)\|^2 \leqslant V^h(t_1,t_2) \leqslant \beta_h \|x^h(t_1,t_2)\|^2, \tag{B9}$$

where $\alpha_h > 0$ and $\beta_h > 0$.

Similarly, there also exist two constants α_v and β_v such that

$$\alpha_v \|x^v(t_1,t_2)\|^2 \leqslant V^v(t_1,t_2) \leqslant \beta_v \|x^v(t_1,t_2)\|^2 \tag{B10}$$

where $\alpha_v > 0$ and $\beta_v > 0$. Define

$$\mathtt{div}V(t_1,t_2) := \frac{\partial V^h(t_1,t_2)}{\partial t_1} + \frac{\partial V^v(t_1,t_2)}{\partial t_2}. \tag{B11}$$

If the following inequality holds, the claimed results will follow

$$\mathtt{div}V(t_1,t_2) + z^\top(t_1,t_2)z(t_1,t_2) - \gamma^2 w^\top(t_1,t_2)w(t_1,t_2) < 0. \tag{B12}$$

Applying S-procedure yields

$$\text{LHS}(B12) \leqslant \begin{bmatrix} \bar{x}(t_1,t_2) \\ w(t_1,t_2) \end{bmatrix}^\top \begin{bmatrix} \mathsf{Sym}\left\{\bar{A}_i^\top \mathcal{F}_i^\top \mathcal{P}_i \mathcal{F}_i\right\} + \bar{L}_i^\top \bar{L}_i + \mathcal{G}_i^\top N_i \mathcal{G}_i & \mathcal{F}_i^\top \mathcal{P}_i^\top \mathcal{F}_i \bar{D}_i \\ \bar{D}_i^\top \mathcal{F}_i^\top \mathcal{P}_i \mathcal{F}_i & -\gamma^2 \mathbf{I}_{n_w} \end{bmatrix} \begin{bmatrix} \bar{x}(t_1,t_2) \\ w(t_1,t_2) \end{bmatrix}. \tag{B13}$$

Thus, if the inequalities (6) in Theorem 1 hold, the claimed results follow.

Appendix C The Proof of Theorem 2

The LMI conditions (6) in Theorem 2 for $i \in \mathcal{I}_0$ is a special case for $i \in \mathcal{I}_1$ in the context of $a_i + \Delta a_i = 0$. The proof for the special case $i \in \mathcal{I}_0$ is omitted and only the derivations of the more complex LMI conditions (8) in Theorem 2 for $i \in \mathcal{I}_1$ will be presented.

By observing the inner structure of the closed loop system matrices $\bar{A}_i$ and $\bar{D}_i$ in (4), it can be found that the controller gains only reside in the second row (for $i \in \mathcal{I}_0$) and the third row (for $i \in \mathcal{I}_1$) of the system matrices. Therefore, the Lyapunov matrices R_{i1} can be prescribed as

$$R_{i1} := R_{i2}H, \ i \in \mathcal{I}, \tag{C1}$$

where $H := \begin{bmatrix} \mathbf{I}_{n_u} & \mathbf{0}_{n_u \times (n_f^h + n_f^v - n_u)} \end{bmatrix}$, and $R_{i2} \in \mathbb{R}^{n_u \times n_u}$ are matrices to be designed.

Note that by the construction of the matrices R_{i1} in the form of (C1), the matrices R_{i2} can be absorbed by the controller gains K_i and k_i by defining

$$\bar{K}_i = R_{i2}^\top K_i, \ i \in \mathcal{I}, \ \bar{k}_i = R_{i2}^\top k_i, \ i \in \mathcal{I}_1. \tag{C2}$$

Substituting (C1) into (6) and taking the system matrices uncertainties into consideration yields

$$\begin{bmatrix} \mathsf{Sym}\left\{\mathcal{A}_i^\top \Pi_{i1} + \mathcal{K}_i^\top \Pi_{i2}\right\} + \bar{L}^\top \bar{L} + \mathcal{G}_i^\top N_i \mathcal{G}_i + \mathsf{Sym}\left\{\bar{\Pi}_i^\top \mathcal{U}_i \boldsymbol{\Delta}_i \mathcal{W}_i\right\} & \star \\ D_{i1}^\top \Pi_{i1} + D_{i2}^\top \bar{K}_i \Pi_{i2} & -\gamma^2 \mathbf{I}_{n_w} \end{bmatrix} < 0, \tag{C3}$$

where

$$\begin{cases} \mathcal{A}_i := \begin{bmatrix} a_i & A_i & B_i \end{bmatrix}, \ \Pi_{i1} := \begin{bmatrix} F_i^\top P f_i & F_i^\top P F_i & \mathbf{0}_{(n_x^h+n_x^v)\times n_u} \end{bmatrix}, \ \Pi_{i2} := \begin{bmatrix} H f_i & H F_i & \mathbf{I}_{n_u} \end{bmatrix}, \ \mathcal{K}_i := \begin{bmatrix} \bar{k}_i & \bar{K}_i C_i & -R_{i2}^\top \end{bmatrix}, \\ \mathcal{U}_i := \begin{bmatrix} U_{i1} & \mathbf{0}_{n_x \times p_1} \\ \mathbf{0}_{n_u \times p_1} & \bar{K}_i U_{i2} \end{bmatrix}, \ \boldsymbol{\Delta}_i := \begin{bmatrix} \Delta_i & \mathbf{0} \\ \mathbf{0} & \Delta_i \end{bmatrix}, \ \mathcal{W}_i := \begin{bmatrix} W_{i2} & W_{i1} & W_{i3} \\ \mathbf{0}_{p_2\times 1} & W_{i1} & \mathbf{0}_{p_2 \times n_u} \end{bmatrix}, \ \bar{\Pi}_i := \begin{bmatrix} \Pi_{i1} \\ \Pi_{i2} \end{bmatrix}. \end{cases} \tag{C4}$$

Thus, by Lemma A (shown below to this proof) and Schur complement with the introduction of scalars $\epsilon_i > 0$, $i \in \mathcal{I}$, one has that (8) implies (C3).

Moreover, the inequalities in (8) imply that $-R_{i2} - R_{i2}^\top < 0$, which shows that R_{i2} are nonsingular matrices. It follows from $K_i = R_{i2}^{-\top} \bar{K}_i$, $i \in \mathcal{I}$, $k_i = R_{i2}^{-\top} \bar{k}_i$, $i \in \mathcal{I}_1$ that one has the controller gains. The proof is completed.

Lemma A. [2]: Given symmetric matrx Λ, and matrices U, W, and Δ, the inequality

$$\Lambda + U\Delta W + W^\top \Delta^\top U^\top < 0$$

is solvable subject to $\Delta^\top \Delta \leqslant \mathbf{I}$, if and only if

$$\Lambda + \epsilon^{-1} W^\top W + \epsilon U U^\top < 0$$

holds for some positive scalar ϵ.

Appendix D Simulation Studies

Consider a modified nonlinear Darboux equation [3], which is usually used to describe the water stream heating processes,

$$\frac{\partial^2 f(s,t)}{\partial s \partial t} = (a_1 + \arctan(f(s,t)))\frac{\partial f(s,t)}{\partial t} + a_2 \frac{\partial f(s,t)}{\partial s} + a_0 f(s,t) + b_0 u(s,t) + d_0 w(s,t), \tag{D1}$$

where $f(s,t)$ denotes an unknown function at $s \in [0, s_f]$ (space) and $t \in [0,\infty)$ (time), $w(s,t)$ denotes the disturbance input, a_0, a_1, a_2, b_0, and d_0 denote real constant coefficients. Choose the system states as $x^h(s,t) = \frac{\partial f(s,t)}{\partial t} - a_2 f(s,t)$, $x^v(s,t) = f(s,t)$. Choose the parameters as $a_0 = 0.2$, $a_1 = -3$, $a_2 = -1$, $b_0 = 1$, and $d_0 = 1$. Then, the following 2-D nonlinear system can be obtained,

$$\begin{bmatrix} \frac{\partial x^h(s,t)}{\partial x} \\ \frac{\partial x^v(s,t)}{\partial t} \end{bmatrix} = \begin{bmatrix} a_1 + \arctan(x^v(s,t)) & a_2 \arctan(x^v(s,t)) + a_1 a_2 + a_0 \\ 1 & a_2 \end{bmatrix} \begin{bmatrix} x^h(s,t) \\ x^v(s,t) \end{bmatrix} + \begin{bmatrix} b_0 \\ 0 \end{bmatrix} u(s,t) + \begin{bmatrix} d_0 \\ 0 \end{bmatrix} w(s,t). \tag{D2}$$

Linearizing (D2) at $x^* = (0,-10)$, $x^* = (0,0)$, and $x^* = (0,10)$ yields a 2-D PWA system (1) with systems matrices as follows,

$$\begin{cases} A_1 = \begin{bmatrix} -4.4711 & 4.7701 \\ 1.0000 & -1.0000 \end{bmatrix}, \ A_2 = \begin{bmatrix} -3.0000 & 3.2000 \\ 1.0000 & -1.0000 \end{bmatrix}, \ A_3 = \begin{bmatrix} -1.5289 & 1.6299 \\ 1.0000 & -1.0000 \end{bmatrix}, \ a_1 = a_3 = \begin{bmatrix} 0.9901 \\ 0 \end{bmatrix}, \ a_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ B_1 = B_2 = B_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ D_{11} = D_{21} = D_{31} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ C_i = \begin{bmatrix} 0 & 1 \end{bmatrix}, \ D_{i2} = 1, \ L_i = \begin{bmatrix} 1 & 0 \end{bmatrix}, \ i \in \mathcal{I} = \{1,2,3\}. \end{cases}$$

The differences between the original nonlinear system and the 2-D PWA model are considered as norm-bound uncertainties with matrices given by

$$U_{i1} = \begin{bmatrix} 0.2 & 0.2 \end{bmatrix}^\top, \ U_{i2} = 0.3, \ W_{i1} = \begin{bmatrix} 0.2 & 0.1 \end{bmatrix}, \ W_{12} = W_{32} = 0.1, \ W_{22} = 0, \ W_{i3} = 0.2, \ i \in \mathcal{I}.$$

The concerned state space is divided into the subspaces $\{\mathcal{S}_1, \mathcal{S}_3\} \in \mathcal{I}_1$, and subspace $\{\mathcal{S}_2\} \in \mathcal{I}_0$, where

$$\mathcal{S}_1 = \{x(s,t) \mid -d_2 \leqslant x^v(s,t) < -d_1\}, \ \mathcal{S}_2 = \{x(s,t) \mid -d_1 \leqslant x^v(s,t) < d_1\}, \ \mathcal{S}_3 = \{x(s,t) \mid d_1 \leqslant x^v(s,t) < d_2\}$$

with $d_1 = 4$ and $d_2 = 15$.

In order to facilitate the proposed controller design, the boundary continuity matrices and characteristic matrices F_i^h, F_i^v, G_i, $i \in \mathcal{I} = \{1,2,3\}$, and f_i^h, f_i^v, g_i, $i \in \mathcal{I}_1 = \{1,3\}$ are constructed as follows,

$$\left[\begin{array}{c|c|c|c|c} F_1^h & F_2^h & F_3^h & f_1^h & f_3^h \end{array}\right] = \left[\begin{array}{c|c|c|c|c} 1 & 1 & 1 & 0 & 0 \end{array}\right], \ \left[\begin{array}{c|c|c|c|c} F_1^v & F_2^v & F_3^v & f_1^v & f_3^v \end{array}\right] = \left[\begin{array}{c|c|c|c|c} 0 & 0 & 1 & 0 & -d_1 \\ 1 & 0 & 0 & d_1 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{array}\right], \ \left[\begin{array}{c|c|c|c|c} G_1 & G_2 & G_3 & g_1 & g_3 \end{array}\right] = \left[\begin{array}{c|c|c|c|c} \eta & \mathbf{0} & \eta & d_2 & -d_1 \\ -\eta & \mathbf{0} & -\eta & -d_1 & d_2 \end{array}\right],$$

where $\eta := \begin{bmatrix} 0 & 1 \end{bmatrix}$.

The aim is to design a PWA SOF controller (2) such that the closed loop system (3) is asymptotically stable with $\mathcal{H}_\infty$ performance $\gamma_{\min}$.

By Algorithm 1 (Theorem 2), a feasible solution is obtained with the following controller gains,

$$\left[\begin{array}{c|c|c|c|c} K_1 & K_2 & K_3 & k_1 & k_3 \end{array}\right] = \left[\begin{array}{c|c|c|c|c} -0.9759 & -1.0157 & -0.9907 & -0.5050 & -0.2925 \end{array}\right]$$

with the optimal $\mathcal{H}_\infty$ performance $\gamma_{\min} = 2.5032$.

In comparison, by Algorithm 1 (Theorem 2 with $F_i^h = \mathbf{I}_{n_x^h}$, $F_i^v = \mathbf{I}_{n_x^v}$, $f_i^h = 0$, and $f_i^v = 0$), the following common Lyapunov function based controller gains are obtained,

$$\left[\begin{array}{c|c|c|c|c} K_1 & K_2 & K_3 & k_1 & k_3 \end{array}\right] = \left[\begin{array}{c|c|c|c|c} -0.8282 & -0.9897 & -1.3089 & -0.0596 & -0.7343 \end{array}\right]$$

with the optimal $\mathcal{H}_\infty$ performance $\gamma_{\min} = 3.6466$. By observing the obtained $\mathcal{H}_\infty$ performance indexes, one can find that the proposed piecewise Lyapunov function based method tends to be less conservative in comparison with the common Lyapunov function based method.

In this simulation, choose $w(s,t) = 10\exp(-0.2(s+t))\cos(2s)\cos(t)$. Figs. D1 (a)-(c) depict the responses of the system states $x^h(s,t)$, $x^v(s,t)$, and the control input $u(s,t)$, respectively, under the boundary condition $x^h(0,t) = 0$, $0 < t \leqslant 40$, $x^v(s,0) = -12$, $0 \leqslant s \leqslant 30$. It is calculated that the maximum ratio $\sqrt{\int_{t_1=0}^{T_{f1}} \int_{t_2=0}^{T_{f2}} \|z(t_1,t_2)\|^2 \mathrm{d}t_1 \mathrm{d}t_2} \Big/ \sqrt{\int_{t_1=0}^{T_{f1}} \int_{t_2=0}^{T_{f2}} \|w(t_1,t_2)\|^2 \mathrm{d}t_1 \mathrm{d}t_2}$ is 0.0242, from which the effectiveness of the proposed method is shown.

References

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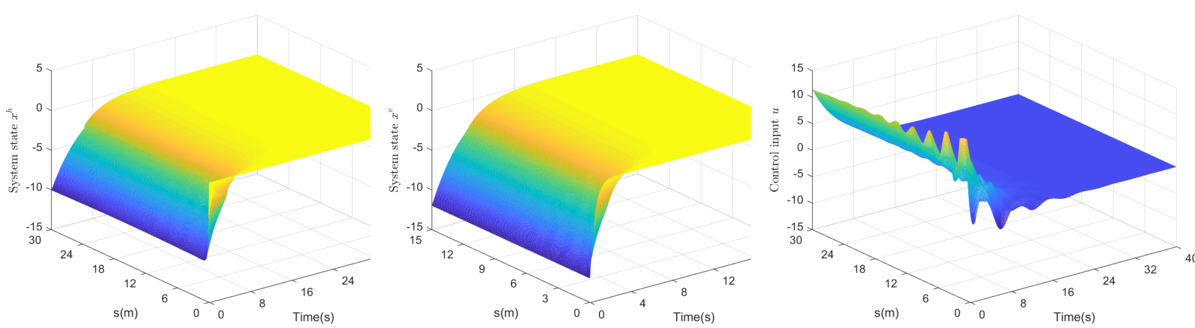

Figure D1 (a) Response of the horizontal state $x^h(s,t)$; (b) Response of the vertical state $x^v(s,t)$; (c) Response of the control input $u(s,t)$