**SCIENCE CHINA**
Information Sciences

• RESEARCH PAPER •

November 2022, Vol. 65 212204:1–212204:21
https://doi.org/10.1007/s11432-021-3458-9

# Finite-time stabilization of nonlocal Lipschitzian stochastic time-varying nonlinear systems with Markovian switching

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Received 2 November 2021/Revised 22 January 2022/Accepted 21 February 2022/Published online 21 October 2022

**Abstract** In this study, we investigate the stochastic finite-time stability theory and finite-time stabilization of stochastic time-varying nonlinear systems with Markovian switching in the sense of weak solutions. First, we present the stochastic finite-time stability theory under the framework of weak solutions: the existence of weak solutions and stochastic finite-time stability theorems for stochastic time-varying nonlinear systems with Markovian switching. Second, for a class of high-order stochastic nonlinear systems with Markovian switching, a finite-time stabilization controller is constructively designed, and the trivial weak solution of a closed-loop system is proven to be globally finite-time stable in probability. The effectiveness of the proposed finite-time control method is illustrated by a numerical example.

**Keywords** stochastic time-varying nonlinear systems, Markovian switching, weak solutions, stochastic finite-time stability, finite-time control

**Citation** Zhao G-H, Liu S-J. Finite-time stabilization of nonlocal Lipschitzian stochastic time-varying nonlinear systems with Markovian switching. Sci China Inf Sci, 2022, 65(11): 212204, https://doi.org/10.1007/s11432-021-3458-9

## 1 Introduction

Different from asymptotic stabilization (e.g., [1–6]), finite-time stabilized systems have a faster convergence rate and better disturbance rejection [7, 8]. Accordingly, finite-time stabilization has received intensive investigations (e.g., [8–18]), and finite-time control has been applied into practical systems, such as the spacecraft system (see [8, 19]), continuously stirred tank reactor system (see [11]), and mechanical system (see [20]). Systems with Markovian switching have also attracted considerable attention because it can model many practical systems with abrupt changes in their structure and parameters [5].

In recent years, for stochastic nonlinear systems with Markovian switching (SNSwMS), finite-time control problems (e.g., stability, stabilization and synchronization) have also attracted considerable attention (e.g., [21–29]). In [23, 24], coefficients were assumed to satisfy growth conditions (nonlocal Lipschitzian conditions) to guarantee the existence of a pathwise unique, strong solution. Moreover, the finite-time stability theorems of SNSwMS were obtained using the finite-time stability theorem [30, Theorem 3.1] of stochastic systems without Markovian switching. In [26], the finite-time synchronization and identification were investigated for two nonlinear coupled Markovian switching delayed networks with multiple weight uncertainties. In [27], the finite-time synchronization was studied for stochastic multilink dynamical networks with Markovian switching topologies. For discontinuous semi-Markovian switching neural networks with time delays and noise disturbance, the finite-time synchronization was considered in [21]. The finite-time synchronization was achieved for the master-slave Markovian jumping systems with mixed time delays in [28]. In [22], the finite-time stabilization was investigated for stochastic coupled systems on networks with Markovian switching. For complex delayed networks with Markovian jumping parameters and stochastic perturbations, the finite-time synchronization and identification were studied in [29].

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Among these existing studies, two types of studies on the existence of solutions are considered. (i) To guarantee the existence of solutions, a local Lipschitzian condition (e.g., [22, 25]) or some tough growth conditions (nonlocal Lipschitzian, e.g., [23, 24]) are imposed on the drift and diffusion terms. However, generally, the local Lipschitzian condition cannot be used in the finite-time stabilization of SNSwMS. Under the local Lipschitzian condition, almost all the sample paths of any solution starting from a nonzero state will never reach the origin [3]. Tough growth conditions make the extension of the finite-time control method to other control problems, such as stochastic finite-time adaptive control, difficult. (ii) The existence of solutions of a closed-loop system is directly ignored (e.g., [21, 22, 26–29]). For the finite-time stability analysis (of the closed-loop system), the expectation of stochastic integrals is directly taken for zero, and the theory of deterministic finite-time stability is used instead of stochastic techniques (e.g., [21, 26–29]). In fact, stochastic integrals in the generalized Itô formula are local martingales, and thus the expectations of stochastic integrals are not necessarily zero.

In this paper, we aim to establish a practical framework of stochastic finite-time control for SNSwMS, including the existence of solutions and stochastic finite-time stability analysis. Considering the complicated limitations on drift and diffusion terms in the case of strong solutions, the framework of weak solutions is a reasonable choice. This point is also mentioned in [31], which stated that for the bang-bang optimal control of nonlocal Lipschitzian systems, weak solutions are really needed (see [31, page 133]). According to [31, page 149], the concept of weak solutions is appropriate for control problems because one of the main uses of the theory of stochastic differential equations (SDEs) is to construct and study diffusions. Moreover, it is enough to find a solution to SDEs on some set-up $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geqslant 0}, \mathbb{P}, \xi, w(t))$, where $\{\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geqslant 0}, \mathbb{P}\}$ is a probability space with the filtration $\{\mathscr{F}_t\}_{t\geqslant 0}$, $\xi$ is an initial value, and $w(t)$ is a Brownian motion.

Although, the existence of weak solutions in stochastic nonlinear systems without Markovian switching (e.g., [18, 32–34]) has been considered. The analysis of the existence of weak solutions of SNSwMS is very different from that of stochastic nonlinear systems without Markovian switching because of the existence of Markovian switching. The differences lie in three aspects. (i) We cannot obtain the existence of weak solutions of SNSwMS from the existence of solutions of a martingale problem (e.g., [35]) because a constructed Markov chain should have a given transition function. To show that the constructed Markov chain has the given transition function, we need to directly investigate the existence of weak solutions for SNSwMS, instead of transforming the problem of the existence of weak solutions into the existence of solutions of the martingale problem (e.g., [32, 33]). (ii) In a construction method, an approximation cannot be generated by the Euler method, which is used to generate the approximation in the study of the existence of weak solutions of stochastic nonlinear systems without Markovian switching (e.g., [32–34]). In this method, we need to construct a Markov chain with the given transition function. (iii) In a Lyapunov method, to guarantee that the constructed Markov chain has the given transition function, we need to directly construct a global weak solution instead of constructing a local weak solution (e.g., [32, Theorem 4.2.3]). Because of the variable substitution used in [32, Theorem 4.2.3], the method of constructing a local weak solution of time-invariant stochastic nonlinear systems without Markovian switching in [32, Theorem 4.2.3] cannot be extended to time-varying stochastic nonlinear systems (with Markovian switching), for which the existence of weak solutions has not been studied. Hence, we need to study the existence of weak solutions of SNSwMS, which cannot be obtained by the existing results (e.g., [18, 32–34]).

In this paper, we first investigated the existence of a global weak solution of SNSwMS via the construction method (here, the requirements on coefficients are continuity and a linear growth condition). Then, we investigated the existence of a global weak solution of SNSwMS by replacing the linear growth condition with a suitable Lyapunov condition. Generally, a controller needs to be designed to make the closed-loop system satisfy the Lyapunov condition. Hence, the condition on the drift and diffusion terms to guarantee the existence of weak solutions is only continuity. This condition is practical (and simple) and makes it possible to extend the work on finite-time stabilization to other control problems, such as stochastic finite-time adaptive control.

Next, we developed the stochastic finite-time stability for SNSwMS in the sense of weak solutions. Because the requirement on coefficients is only continuity, an approximation generated by the Picard iteration may not be convergent (see [36]). Hence, the method of proving the finite-time stability in the sense of strong solutions (e.g., [22–24]) cannot be used to prove the finite-time stability in the sense of weak solutions. Thus, the analysis of stochastic finite-time stability in the sense of weak solutions is different from that in the sense of strong solutions.

Finally, we investigated the finite-time stabilization of a class of stochastic high-order nonlinear systems with Markovian switching under the framework of weak solutions, which has not been studied. A Lyapunov function was constructed by adding a power integrator technique, and then a finite-time controller was constructively designed. The finite-time stability theorems we developed rigorously proved the existence of a global weak solution and the finite-time stability of a closed-loop system. Different from existing studies on the finite-time control of SNSwMS (e.g., [21, 22, 26–29]), the existence of weak solutions and the finite-time stability of the closed-loop system were rigorously analyzed through our developed stochastic finite-time stability theory, instead of ignoring the existence of solutions or analyzing the finite-time stability using the deterministic finite-time stability theory and directly taking the expectations of the stochastic integrals for zero.

The main contributions of our work include the following. (1) The existence of weak solutions was investigated for the first time for SNSwMS. (2) The finite-time stability in probability for SNSwMS was studied in the sense of weak solutions, which allows extending the finite-time control method to other problems, such as stochastic finite-time adaptive control. (3) Our results hold for the time-varying stochastic nonlinear systems (with Markovian switching), which is different from the existing results about weak solutions obtained by Lyapunov functions (e.g., [18]). Based on this work, we can extend the results of [18] to time-varying stochastic nonlinear systems. Moreover, Skorokhod's prediction on the existence of weak solutions for SNSwMS in [37] is rigorously proven in the paper.

The rest of this work is organized as follows. The stochastic finite-time stability is presented in Section 2. Finite-time stabilization is provided in Section 3. A numerical example is given in Section 4. Concluding remarks are presented in Section 5.

## 2 Stochastic finite-time stability

The following notations will be used in this paper. $\mathbb{R}_+$ denotes the set of all nonnegative real numbers. $\mathbb{N}$ denotes the set $\{1, 2, \ldots\}$. $|\cdot|$ denotes both the Euclidean norm of a vector and the trace norm of a matrix. $a \vee b$ denotes the maximum of $a$ and $b$. $a \wedge b$ denotes the minimum of $a$ and $b$. For a given vector or a matrix $A$, $A^{\mathrm{T}}$ denotes its transpose, and $\mathrm{Tr}(A)$ denotes the trace of $A$ if $A$ is a square matrix. $\mathcal{K}$ denotes the set of all functions: $\mathbb{R}_+ \to \mathbb{R}_+$, which are continuous, strictly increasing and vanish at zero. $\mathcal{K}_\infty$ denotes the set of all functions, which are of class $\mathcal{K}$ and unbounded. $\mathbb{S}$ denotes the set $\{1, 2, \ldots, N\}$. $\mathcal{C}^2(\mathbb{R}^d \times \mathbb{S}; \mathbb{R}_+)$ denotes the set of all nonnegative functions $V(x, i)$ on $\mathbb{R}^d \times \mathbb{S}$, which are continuous and twice differentiable in $x$.

Consider the following system:

$$\mathrm{d}x(t) = f(t, x(t), r(t))\mathrm{d}t + g(t, x(t), r(t))\mathrm{d}w(t), \ t \geqslant t_0 \geqslant 0, \tag{1}$$

where $x(t) \in \mathbb{R}^d$, $r(t)$ is a right-continuous homogeneous Markov chain taking values in $\mathbb{S}$, $w(t)$ is an $m$-dimensional standard Brownian motion on some complete probability space, $f : \mathbb{R}_{t_0} \times \mathbb{R}^d \times \mathbb{S} \to \mathbb{R}^d$ and $g : \mathbb{R}_{t_0} \times \mathbb{R}^d \times \mathbb{S} \to \mathbb{R}^{d\times m}$ ($\mathbb{R}_{t_0} = [t_0, +\infty)$) are Borel measurable, $x(t_0)$ is a random variable, whose distribution has a compact support, $r(t_0) \in \mathbb{S}$ is a random variable, the transition function of $r(t)$ is standard, and the Markov chain $\{r(t), t \geqslant t_0\}$ is independent of the driven process $\{w(t), t \geqslant t_0\}$.

Inspired by the definition of a weak solution of stochastic systems without Markovian switching in [32], we present the following definition for system (1).

**Definition 1.** If there exist a continuous stochastic process $x(t)$ on a probability space ($\Omega^x$, $\mathscr{F}^x$, $\mathbb{P}^x$) with a filtration $\{\mathscr{F}^x_t\}_{t\geqslant 0}$ satisfying the usual conditions, an $m$-dimensional $\{\mathscr{F}^x_t\}$-adapted Brownian motion $w^x(t)$ with $\mathbb{P}^x\{w^x(0) = 0\} = 1$, and an $\{\mathscr{F}^x_t\}$-adapted Markov chain $r^x(t)$ on $[t_0, \infty)$ such that (i) $x(t_0)$ has the given distribution, (ii) $r^x(t)$ has the given initial distribution and the given transition function, (iii) $\forall t \geqslant t_0$, $\int_{t_0}^t |f(s, x(s), r^x(s))|\mathrm{d}s < \infty$ a.s., and $\int_{t_0}^t |g(s, x(s), r^x(s))|^2\mathrm{d}s < \infty$ a.s., (iv)

$$x(t) = x(t_0) + \int_{t_0}^t f(s, x(s), r^x(s))\mathrm{d}s + \int_{t_0}^t g(s, x(s), r^x(s))\mathrm{d}w^x(s) \ \text{ a.s.}, \quad t \geqslant t_0, \tag{2}$$

then the tuple ($x(t)$, $w^x(t)$, $r^x(t)$, $\Omega^x$, $\mathscr{F}^x$, $\{\mathscr{F}^x_t\}_{t\geqslant 0}$, $\mathbb{P}^x$), or simply, $x(t)$ is called a global weak solution to system (1).

Referring to the definition of finite-time stability of stochastic nonlinear systems (SNS) in the sense of weak solution in [18], we give the following notion for (1).

**Definition 2.** The trivial weak solution of system (1) is said to be globally finite-time stable in probability, if for any global weak solution $(x(t), w^x(t), r^x(t), \Omega^x, \mathscr{F}^x, \{\mathscr{F}^x_t\}_{t\geqslant 0}, \mathbb{P}^x)$ to system (1), simply denoted by $x(t)$, the following two properties are satisfied.

(i) Finite-time attractiveness in probability: the first hitting time $\varrho^x = \inf\{t \geqslant t_0 : x(t) = 0\}$ which is called the stochastic settling time, is finite a.s., that is $\mathbb{P}^x\{\varrho^x < \infty\} = 1$;

(ii) Stability in probability: $\forall \epsilon \in (0, 1)$, there exists a $\mathcal{K}$ class function $\beta$ such that $\mathbb{P}^x\big\{\sup_{t\geqslant t_0} |x(t)| \leqslant \beta(|x(t_0)|)\big\} \geqslant 1 - \epsilon$.

## 2.1 Existence of global weak solutions

Here we respectively present a construction method and a Lyapunov method to investigate the existence of global weak solutions of system (1).

### 2.1.1 *Construction method*

For the coefficients in system (1), we assume that there exists a function $c_0 : \mathbb{R}_{t_0} \to \mathbb{R}_+$ such that

$$\text{(Linear growth condition)} \quad |f(t, x, i)|^2 \vee |g(t, x, i)|^2 \leqslant c_0(t)(1 + |x|^2) \quad \forall (t, x, i) \in \mathbb{R}_{t_0} \times \mathbb{R}^d \times \mathbb{S}, \tag{3}$$

where $\int_{t_0}^T c_0^p(t)\mathrm{d}t < \infty$ for a constant $p > 2$ and any $T > t_0$.

For any given $n \in \mathbb{N}$, let $\Delta = 1/2^n$, $t_k^{(n)} = t_0 + k\Delta$, $k = 0, 1, \ldots$. We prepare an $m$-dimensional standard Brownian motion $w(t)$, a $d$-dimensional random variable $\varsigma$ with a given distribution $\mu$, and a right-continuous and homogeneous Markov chain $r(t)$ on $\mathbb{R}_{t_0}$ with $\mathbb{P}\{r(t + s) = j|r(s) = i\} = P_{ij}(t)$ for $s \geqslant t_0$, $t \geqslant t_0$ and $\mathbb{P}\{r(t_0) = i\} = p_i$ on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Construct

$$X_{k+1}^{(n)} = X_k^{(n)} + \int_{t_k^{(n)}}^{t_{k+1}^{(n)}} f(s, X_k^{(n)}, r(s))\mathrm{d}s + \int_{t_k^{(n)}}^{t_{k+1}^{(n)}} g(s, X_k^{(n)}, r(s))\mathrm{d}w(s) \text{ a.s.}, \tag{4}$$

$k = 1, 2, \ldots$, where $X_0^{(n)} = \varsigma$ a.s. The existence of stochastic integral $\int_{t_k^{(n)}}^{t_{k+1}^{(n)}} g(s, X_k^{(n)}, r(s))\mathrm{d}w(s)$ in (4) is guaranteed by linear growth condition (3), because we can prove $\mathbb{E}|X_k^{(n)}|^{2p} < \infty$, for $t_0 + (k + 1)\Delta \leqslant T$ $\forall T > t_0$, whose proof is similar to the proof of (6) in Lemma 1. Thus, $X_{k+1}^{(n)}$ in difference equation (4) exists a.s. under linear growth condition (3) and we can compute $X_k^{(n)}$ step by step. Then, let $Y_n(s) = X_k^{(n)}$ for $s \in [t_k^{(n)}, t_{k+1}^{(n)})$, and define a continuous extension of $X_k^{(n)}$:

$$X_n(t) = X_n(t_0) + \int_{t_0}^t f(s, Y_n(s), r(s))\mathrm{d}s + \int_{t_0}^t g(s, Y_n(s), r(s))\mathrm{d}w(s) \text{ a.s.}, \forall t \geqslant t_0, \ X_n(t_0) = \varsigma \text{ a.s.}, \tag{5}$$

**Remark 1.** Different from existing methods of constructing the approximation for studying the existence of weak solutions (e.g., [32–34, 38, 39]), we construct explicit difference equation (4) to generate approximation, because of the existence of Markovian switching and the lack of the Lipschitzian condition. The computation of the approximation $X_n(t)$ is more complicated than that in [32–34], but simpler than that in [38, 39]. However, the method in [32–34] cannot be used to generate the approximation, because we need to construct a Markov chain with the given transition function. With the help of the approximation $X_n(t)$, we can construct a weak solution to system (1).

According to [32, Theorem 1.4.3], the tightness of $\{\mathbb{P}X_n^{-1}\}$ ($\mathbb{P}X_n^{-1}(B) = \mathbb{P}\{X_n \in B\}$ $\forall B \in \mathscr{B}(\mathbb{R}^d)$, where $\mathscr{B}(\mathbb{R}^d)$ denotes the topological $\sigma$-field on $\mathbb{R}^d$) can be obtained by the following lemma, whose proof is given in Appendix A.

**Lemma 1.** If linear growth condition (3) holds, then for any $T > t_0$, there exist two positive constants $C_1 = C_1(p, t_0, T, \mu)$ and $C_2 = C_2(p, t_0, T, \mu)$ such that

$$\mathbb{E}\left(\sup_{t_0\leqslant t\leqslant T} \big|X_n(t)\big|^{2p}\right) \leqslant C_1, \tag{6}$$

$$\mathbb{E}|X_n(t) - X_n(s)|^{2p} \leqslant C_2|t - s|^{p-1}, \quad \forall\, t, s \in [t_0, T]. \tag{7}$$

Now, we present the result about the existence of global weak solutions of system (1). Different from the conditions of the existence of strong solutions (e.g., [3, 24]), there is no local Lipschitzian condition or some other restrictive conditions on coefficients in the following theorem, besides the continuity and linear growth condition (3).

**Theorem 1.** If $f(t,x,i)$ is continuous in $x \in \mathbb{R}^d$, $g(t,x,i)$ is jointly continuous in $(t,x) \in \mathbb{R}_{t_0} \times \mathbb{R}^d$ for every $i \in \mathbb{S}$, and linear growth condition (3) holds, then, for any given probability measure $\mu$ with compact support on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$, any given set $\{p_i \in [0,1] : i \in \mathbb{S}\}$ with $\sum_{i\in\mathbb{S}} p_i = 1$ and any given standard transition function $P_{ij}(t)$, there exists a global weak solution $(x(t), w^x(t), r^x(t), \Omega^x, \mathscr{F}^x, \{\mathscr{F}_t^x\}_{t\geqslant 0}, \mathbb{P}^x)$ to system (1) such that $x(t_0)$ has the given distribution (i.e., $\mathbb{P}^x\{x(t_0) \in B\} = \mu(B)$, $\forall B \in \mathscr{B}(\mathbb{R}^d)$), $r^x(t)$ has the given initial distribution (i.e., $\mathbb{P}^x\{r^x(t_0) = i\} = p_i$, $\forall i \in \mathbb{S}$) and the given transition function $P_{ij}(t)$ (i.e., $\mathbb{P}\{r^x(t+s) = j | r^x(s) = i\} = P_{ij}(t)$, $\forall i,j \in \mathbb{S}$, $t \geqslant t_0$, $s \geqslant t_0$).

*Proof.* **Tightness.** To show that the constructed Brownian motion $w^x(t)$ is standard, we first extend $X_n(t)$ and $r(t)$ in (5) to functions on $\mathbb{R}_+$. Let $X_n(t) = X_n(t_0)$ and $r(t) = r(t_0)$ for $t \in [0, t_0]$. Then, we show the tightness of the probability measures generated by $(X_n(t), w(t), r(t))$. $\{\mathbb{P}X_n^{-1}\}$ induced by stochastic process $X_n$ is a sequence of probability measures on $(\mathcal{C}(\mathbb{R}_+;\mathbb{R}^d), \mathscr{B}(\mathcal{C}(\mathbb{R}_+;\mathbb{R}^d)))$, where $\mathcal{C}(\mathbb{R}_+;\mathbb{R}^d)$ denotes the set of all continuous $\mathbb{R}^d$-valued functions defined on $\mathbb{R}_+$, and $\mathscr{B}(\mathcal{C}(\mathbb{R}_+;\mathbb{R}^d))$ is the topological $\sigma$-field on $\mathcal{C}(\mathbb{R}_+;\mathbb{R}^d)$. According to the proof of [32, Theorems 1.4.2 and 1.4.3], it follows from Lemma 1 that $\{\mathbb{P}X_n^{-1}\}$ is tight. Thus, $\forall \epsilon > 0$, there exists a compact set $K_\epsilon^X \subseteq \mathcal{C}(\mathbb{R}_+;\mathbb{R}^d)$ such that $\mathbb{P}\{X_n(t) \in K_\epsilon^X\} \geqslant 1 - \frac{\epsilon}{4}$. Similarly, it follows from $\mathbb{E}|w(t_2) - w(t_1)|^4 \leqslant 36m^2|t_2 - t_1|^2$ and $\mathbb{E}|w(0)|^4 = 0$ that there exists a compact set $K_\epsilon^w \subseteq \mathcal{C}(\mathbb{R}_+;\mathbb{R}^m)$ such that $\mathbb{P}\{w(t) \in K_\epsilon^w\} \geqslant 1 - \frac{\epsilon}{4}$. Additionally, referring to the proof of Lemma 6.1.11 in [40], we can show that $\lim_{\theta\to 0} W'_{N_1}(r,\theta) = 0$ a.s. for any positive integer $N_1$, where $W'_{N_1}(r,\theta) = \inf\{\max_{i\leqslant m_1} W(r;[\tau_{i-1},\tau_i)) : 0 < \tau_0 < \cdots < \tau_{m_1} = N_1, \inf_{i<m_1}(\tau_i - \tau_{i-1}) \geqslant \theta\}$ and $W(r;[\tau_{i-1},\tau_i)) = \sup_{s,t\in[\tau_{i-1},\tau_i)} |r(t) - r(s)|$. Then we have $\lim_{\theta\to 0} \mathbb{P}^x\{W'_{N_1}(r,\theta) \geqslant \eta\} = 0$ $\forall \eta > 0$. According to [40, Theorem 6.3.21], there exists a compact set $K_\epsilon^r \subseteq \mathbb{D}(\mathbb{R}_+;\mathbb{R})$ such that $\mathbb{P}\{r(t) \in K_\epsilon^r\} \geqslant 1 - \frac{\epsilon}{2}$, where $\mathbb{D}(\mathbb{R}_+;\mathbb{R})$ denotes the set of all càdlàg (right continuous and left limit) $\mathbb{R}$-valued functions defined on $\mathbb{R}_+$. Define $K_\epsilon = K_\epsilon^X \times K_\epsilon^w \times K_\epsilon^r$. Then we have

$$\mathbb{P}\{\Lambda_n(t) \in K_\epsilon\} \geqslant 1 - \epsilon, \qquad \Lambda_n(t) := (X_n(t), w(t), r(t)), \tag{8}$$

which implies that $\{\mathbb{P}\Lambda_n^{-1}\}$ is a tight set of probability measures on $(\mathcal{C}(\mathbb{R}_+;\mathbb{R}^{d+m})\times \mathbb{D}(\mathbb{R}_+;\mathbb{R}), \mathscr{B}(\mathcal{C}(\mathbb{R}_+;\mathbb{R}^{d+m}) \times \mathbb{D}(\mathbb{R}_+;\mathbb{R})))$.

**New stochastic processes.** Now, we show that there exists a sequence of stochastic processes having the same law with some subsequence of $\{\Lambda_n\}$ on a probability space. According to Prokhorov theorem, $\{\mathbb{P}\Lambda_n^{-1}\}$ is relatively compact, namely, $\{\mathbb{P}\Lambda_n^{-1}\}$ contains a weakly convergent subsequence. Assume that the subsequence $\{\mathbb{P}\Lambda_{n_l}^{-1}\}$ is weakly convergent ($\{\Lambda_{n_l}\}$ is a subsequence of $\{\Lambda_n\}$). Then, it follows from [32, Theorem 1.2.7] that there exist a probability space $(\Omega^x, \mathscr{F}^x, \mathbb{P}^x)$ and stochastic processes

$$\tilde{\Lambda}_{n_l}(\cdot) := (\tilde{X}_{n_l}(\cdot), \tilde{w}^{(n_l)}(\cdot), \tilde{r}^{(n_l)}(\cdot)) \ \text{ and } \ \tilde{\Lambda}^x(\cdot) := (x(\cdot), w^x(\cdot), r^x(\cdot)) \tag{9}$$

on $(\Omega^x, \mathscr{F}^x, \mathbb{P}^x)$ such that

$$\mathbb{P}^x\tilde{\Lambda}_{n_l}^{-1} = \mathbb{P}\Lambda_{n_l}^{-1}, \text{ i.e., law of } (\tilde{\Lambda}_{n_l}) = \text{law of } (\Lambda_{n_l}), \tag{10}$$

$$\tilde{\Lambda}_{n_l}(t) \to \tilde{\Lambda}^x(t) \text{ a.s., as } l \to \infty \text{ uniformly on } [0,T], \ \forall T > t_0. \tag{11}$$

Since $\mathbb{S}$ is a finite state space, Eq. (11) implies that there exists a random variable $\hat{l}$ on $\mathbb{N}$ such that

$$\mathbb{P}^x\{\tilde{r}^{(n_l)}(t) = r^x(t), \ l \geqslant \hat{l}\} = 1, \ \ \forall t \in [t_0, T], \ \forall T > t_0. \tag{12}$$

**Weak solution.** Let $\mathscr{F}_t^x \triangleq \sigma(\sigma(x(s), w^x(s), r^x(s) : 0 \leqslant s \leqslant t) \cup \mathscr{N})$, $\mathscr{F}_\infty^x \triangleq \sigma(\cup_{t\geqslant 0}\mathscr{F}_t^x)$ with $\mathscr{N} \triangleq \{A \in \mathscr{F}^x : \mathbb{P}^x(A) = 0\}$. In the rest of the proof, we show that $(x(t), w^x(t), r^x(t), \Omega^x, \mathscr{F}^x, \{\mathscr{F}_t^x\}_{t\geqslant 0}, \mathbb{P}^x)$ is a global weak solution to system (1). Since $\int_{t_k^{(n)}}^{t_{k+1}^{(n)}} g(s, X_k^{(n)}, r(s))\mathrm{d}w(s)$ in (4) exists a.s., $\int_{t_0}^t g(s, Y_{n_l}(s), r(s))\mathrm{d}w(s)$ exists a.s. This together with (10) implies that

$$\int_{t_0}^t g(s, \tilde{Y}_{n_l}(s), \tilde{r}^{(n_l)}(s))\mathrm{d}\tilde{w}^{(n_l)}(s) \text{ exists a.s.}, \forall t \geqslant t_0, \tag{13}$$

where $\tilde{Y}_{n_l}(s) = \tilde{X}_{n_l}(t_k^{(n_l)})$ for $s \in [t_k^{(n_l)}, t_{k+1}^{(n_l)})$. Let $F(X_{n_l}, w, r) = X_{n_l}(t) - X_{n_l}(t_0) - \int_{t_0}^t f(s, Y_{n_l}(s), r(s))\mathrm{d}s - \int_{t_0}^t g(s, Y_{n_l}(s), r(s))\mathrm{d}w(s)$. It follows from (5) that $\forall t \geqslant t_0$, $\mathbb{P}(F(X_{n_l}, w, r) = 0) = 1$. For any given $t \geqslant t_0$, Eq. (10) implies that random variables $F(X_{n_l}, w, r)$ and $F(\tilde{X}_{n_l}, \tilde{w}^{(n_l)}, \tilde{r}^{(n_l)})$ have the same law. Then we have that $\mathbb{P}^x(F(\tilde{X}_{n_l}, \tilde{w}^{(n_l)}, \tilde{r}^{(n_l)}) = 0) = 1$ $\forall t \geqslant t_0$, that is

$$\tilde{X}_{n_l}(t) = \tilde{X}_{n_l}(t_0) + \int_{t_0}^t f(s, \tilde{Y}_{n_l}(s), \tilde{r}^{(n_l)}(s))\mathrm{d}s + \int_{t_0}^t g(s, \tilde{Y}_{n_l}(s), \tilde{r}^{(n_l)}(s))\mathrm{d}\tilde{w}^{(n_l)}(s) \text{ a.s., } \forall t \geqslant t_0. \quad (14)$$

Because of (6) and (10), we have

$$\mathbb{E}^x \left( \sup_{t_0 \leqslant s \leqslant T} |\tilde{X}_{n_l}(s)|^{2p} \right) \leqslant C_1. \quad (15)$$

This implies that $\forall s \in [t_0, T]$, $|\tilde{X}_{n_l}(s)|^2 < \infty$ a.s. Thus, it follows from (11) that $\forall s \in [t_0, T]$, $|x(s)|^2 < \infty$ a.s. Then, by (3), we have $\forall\, T > t_0$, $\int_{t_0}^{\mathrm{T}} |g(s, x(s), r^x(s))|^2\mathrm{d}s \leqslant \int_{t_0}^{\mathrm{T}} c_0(s)(1+|x(s)|^2)\mathrm{d}s < \infty$ a.s. Similarly, $\int_{t_0}^{\mathrm{T}} |f(s, x(s), r^x(s))|\mathrm{d}s \leqslant (T_0 \int_{t_0}^{\mathrm{T}} |f(s, x(s), r^x(s))|^2\mathrm{d}s)^{1/2} < \infty$ a.s. According to [3, Definition 1.33], we obtain that

$$\int_{t_0}^t g(s, x(s), r^x(s))\mathrm{d}w^x(s) \text{ exists a.s., } \forall t \geqslant t_0. \quad (16)$$

The convergence of two integral terms of the right-side of (14) will be respectively analyzed.

**Step 1.** We analyze the convergence of $\int_{t_0}^t f(s, \tilde{Y}_{n_l}(s), \tilde{r}^{(n_l)}(s))\mathrm{d}s$. Note that for $s \in [t_0, t]$, there exists a non-negative integer $k$ such that $s \in [t_k^{(n_l)}, t_{k+1}^{(n_l)})$. Thus, $|\tilde{Y}_{n_l}(s) - x(s)| \leqslant |\tilde{X}_{n_l}(t_k^{(n_l)}) - \tilde{X}_{n_l}(s)| + |\tilde{X}_{n_l}(s) - x(s)|$. Note that Eq. (7) together with (10) implies $\mathbb{E}^x|\tilde{X}_{n_l}(t_k^{(n_l)}) - \tilde{X}_{n_l}(s)|^{2p} \leqslant C_2|t_k^{(n_l)} - s|^{p-1} \leqslant C_2/2^{n_l(p-1)}$. Thus, by (11), we have that

$$\tilde{Y}_{n_l}(s) \text{ converges to } x(s) \text{ in probability, as } l \to \infty. \quad (17)$$

Then, by (12), the dominated convergence theorem, and the continuity of $f(t, x, i)$ in $x \in \mathbb{R}^d$, we have that

$$\int_{t_0}^t f(s, \tilde{Y}_{n_l}(s), \tilde{r}^{(n_l)}(s))\mathrm{d}s \text{ converges to } \int_{t_0}^t f(s, x(s), r^x(s))\mathrm{d}s \text{ in probability, as } l \to \infty. \quad (18)$$

**Step 2.** Now, we prove the convergence of $\int_{t_0}^t g(s, \tilde{Y}_{n_l}(s), \tilde{r}^{(n_l)}(s))\mathrm{d}\tilde{w}^{(n_l)}(s)$. It follows from (15) and the Chebyshev's inequality that $\forall \epsilon_1 > 0$, there exists a constant $\lambda(p, t_0, t, \epsilon_1, \mu) > 0$ such that

$$\mathbb{P}^x \left\{ \sup_{t_0 \leqslant s \leqslant t} |\tilde{X}_{n_l}(s)| > \lambda(p, t_0, t, \epsilon_1, \mu) \right\} \leqslant \epsilon_1 \text{ for } t \in [t_0, T]. \quad (19)$$

Since $g(s, x, i)$ is jointly continuous in $(s, x)$, there exists a positive constant $M_1(t, \lambda)$ such that $M_1(t, \lambda) \geqslant \sup_{s \in [t_0, t], |x| \in [0, \lambda], r \in \mathbb{S}} |g_j(s, x, r)|$ for $j = 1, \ldots, m$ and it follows from (12) and (17) that

$$g(s, \tilde{Y}_{n_l}(s), \tilde{r}^{(n_l)}(s)) \text{ converges to } g(s, x(s), r^x(s)) \text{ in probability, as } l \to \infty. \quad (20)$$

Furthermore, by (19) and $\tilde{r}^{(n_l)}(s) \in \mathbb{S}$, we have that

$$\mathbb{P}^x \left\{ \sup_{t_0 \leqslant s \leqslant t} |g_j(s, \tilde{X}_{n_l}(s), \tilde{r}^{(n_l)}(s))| > M_1(t, \lambda) \right\} \leqslant \mathbb{P}^x \left\{ \sup_{t_0 \leqslant s \leqslant t} |\tilde{X}_{n_l}(s)| > \lambda \right\} \leqslant \epsilon_1. \quad (21)$$

Based on (13), (16), (20), (21), and Propositions A1 and A2 in Appendix A, according to [34, Theorem 2.6],

$$\int_{t_0}^t g(s, \tilde{Y}_{n_l}(s), \tilde{r}^{(n_l)}(s))\mathrm{d}\tilde{w}^{(n_l)}(s) \text{ converges to } \int_{t_0}^t g(s, x(s), r^x(s))\mathrm{d}w^x(s) \text{ in probability as } l \to \infty, \quad (22)$$

$\forall t \in [t_0, T]$. It follows from (18) and (22) that there exists a subsequence of

$$\left\{\int_{t_0}^{t} f(s, \tilde{Y}_{n_l}(s), \tilde{r}^{(n_l)}(s))\mathrm{d}s + \int_{t_0}^{t} g(s, \tilde{Y}_{n_l}(s), \tilde{r}^{(n_l)}(s))\mathrm{d}\tilde{w}^{(n_l)}(s)\right\}$$

converging to $\int_{t_0}^t f(s, x(s), r^x(s))\mathrm{d}s + \int_{t_0}^t g(s, x(s), r^x(s))\mathrm{d}w^x(s)$ a.s., as $l \to \infty$. Without loss of generality, assume that

$$\begin{aligned}
&\lim_{l\to\infty}\left(\int_{t_0}^{t} f(s, \tilde{Y}_{n_l}(s), \tilde{r}^{(n_l)}(s))\mathrm{d}s + \int_{t_0}^{t} g(s, \tilde{Y}_{n_l}(s), \tilde{r}^{(n_l)}(s))\mathrm{d}\tilde{w}^{(n_l)}(s)\right)\\
&= \int_{t_0}^{t} f(s, x(s), r^x(s))\mathrm{d}s + \int_{t_0}^{t} g(s, x(s), r^x(s))\mathrm{d}w^x(s) \ \text{ a.s., } \forall t \in [t_0, T].
\end{aligned} \tag{23}$$

Therefore, letting $l \to \infty$ in (14), we obtain that $\forall t \in [t_0, T]$, $\forall T > t_0$,

$$x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s), r^x(s))\mathrm{d}s + \int_{t_0}^{t} g(s, x(s), r^x(s))\mathrm{d}w^x(s) \ \text{ a.s.} \tag{24}$$

Since $\int_{t_0}^t f(s, x(s), r^x(s))\mathrm{d}s$ is continuous and $\int_{t_0}^t g(s, x(s), r^x(s))\mathrm{d}w^x(s)$ is a continuous local martingale, $x(t)$ is continuous. Furthermore, it follows from $X_{n_l}(t_0) = \varsigma$ a.s. for all $n_l$ and (10), that $\mathbb{P}^x\{\tilde{X}_{n_l}(t_0) \in B\} = \mu(B)\ \forall B \in \mathscr{B}(\mathbb{R}^d)$. Then, from (11) and the dominated convergence theorem, one can show that $\mathbb{P}^x\{x(t_0) \in B\} = \mu(B)$. Proposition A1 in Appendix A shows that $w^x(t)$ is an $\{\mathscr{F}_t^x\}$-adapted Brownian motion with $\mathbb{P}^x\{w^x(0) = 0\} = 1$, and $r^x(t)$ is an $\{\mathscr{F}_t^x\}$-adapted Markov chain with the given initial distribution and transition function. Hence, $(x(t), w^x(t), r^x(t), \Omega^x, \mathscr{F}^x, \{\mathscr{F}_t^x\}_{t\geqslant 0}, \mathbb{P}^x)$ is a global weak solution to system (1), if $\mathbb{P}^x\{\tau_\infty^x < \infty\} = 0$, where $\tau_k^x = \inf\{t \geqslant t_0 : |x(t)| \geqslant k\}$ and $\tau_\infty^x = \lim_{k\to\infty} \tau_k^x$.

Suppose that $\mathbb{P}^x\{\tau_\infty^x < \infty\} > 0$. Then there exist constants $\bar{T}^x > t_0$ and $\epsilon_2 \in (0, 1]$ such that $\mathbb{P}^x\{\tau_\infty^x \leqslant \bar{T}^x\} \geqslant \epsilon_2$. Note that $\mathbb{E}^x|x(\bar{T}^x \wedge \tau_k^x)|^{2p} \geqslant \mathbb{P}^x\{\tau_k^x \leqslant \bar{T}^x\}|k|^{2p}$, and Eq. (11) together with (15) implies that there exists a constant $\bar{C}(p, t_0, \bar{T}^x, \mu)$ such that $\mathbb{E}^x|x(\bar{T}^x \wedge \tau_k^x)|^{2p} \leqslant \bar{C}$. Thus $\mathbb{P}^x\{\tau_k^x \leqslant \bar{T}^x\}|k|^{2p} \leqslant \mathbb{E}^x|x(\bar{T}^x \wedge \tau_k^x)|^{2p} \leqslant \bar{C}$. Moreover, letting $k \to \infty$ in the above inequality, we have $\infty = \lim_{k\to\infty}(\mathbb{P}^x\{\tau_k^x \leqslant \bar{T}^x\}|k|^{2p}) \leqslant \bar{C}$, which is a contradiction. Therefore, $\mathbb{P}^x\{\tau_\infty^x < \infty\} = 0$.

**Remark 2.** Theorem 1 extends the existence of global weak solutions of SNS (e.g., [32, Theorem 4.2.2]) to that of SNSwMS. Due to the Markov chain, the existence of solutions of a martingale problem related to SNSwMS cannot directly imply the existence of weak solutions of SNSwMS, and so, we cannot obtain the results of the existence of weak solutions of SNSwMS from [35]. Different from [32, 33], we directly show that $x(t)$ is a global weak solution, instead of transforming the problem of the existence of global weak solutions into the existence of solutions of a martingale problem.

### 2.1.2 *Lyapunov method*

To analyze the existence of global weak solutions of the closed-loop system in control problems, the existence of weak solutions of system (1) is investigated by a Lyapunov function. We replace linear growth condition (3) in Theorem 1 by some suitable Lyapunov conditions.

First, we introduce some notions on Markov chain to give the generalized Itô formula on (1). Let $P(t) = (P_{ij}(t))_{N\times N}$. If all states in $\mathbb{S}$ are stable ($i \in \mathbb{S}$ is said to be stable, if $\lim_{t\to 0}\big((1 - P_{ii}(t))/t\big) < \infty$), then $Q = (q_{ij})_{N\times N}$ with $q_{ij} = P'_{ij}(0)$ is called the stable $q$-matrix of $P_{ij}(t)$. Furthermore, if $\sum_{j=1}^N q_{ij} = 0$ $\forall i \in \mathbb{S}$, then $Q$ is conservative (see [41]).

In the rest of the work, assume that the $q$-matrix $Q$ of $P_{ij}(t)$ is stable and conservative. Now, we introduce the generalized Itô formula for SNSwMS (1) (see [3]): for $V \in \mathcal{C}^2(\mathbb{R}^d \times \mathbb{S}; \mathbb{R}_+)$,

$$V(x(t), r(t)) = V(x(t_0), r(t_0)) + \int_{t_0}^{t} \mathscr{L}V(s, x(s), r(s))\mathrm{d}s + \bar{M}(t),\ \forall t \geqslant t_0, \tag{25}$$

where $\bar{M}(t) = \int_{t_0}^t V_x(x(s), r(s))g(s, x(s), r(s))\mathrm{d}w(s) + \int_{t_0}^t \int_{\mathbb{R}}(V(x(s), r(t_0) + h(r(s), l)) - V(x(s), r(s)))\bar{\mu}(\mathrm{d}s, \mathrm{d}l)$, $h : \mathbb{S} \times \mathbb{R} \to \mathbb{R}$, $\bar{\mu}(\mathrm{d}s, \mathrm{d}l)$ is a martingale measure, and

$$\mathscr{L}V(t, x, i) = V_x(x, i)f(t, x, i) + \frac{1}{2}\mathrm{Tr}[g^{\mathrm{T}}(t, x, i)V_{xx}(x, i)g(t, x, i)] + \sum_{j=1}^{N} q_{ij}V(x, j) \tag{26}$$

with $q_{i,j} \geqslant 0$ for $i \neq j$. Here $\sum_{j=1}^{N} q_{ij} V(x,j)$ is called the interconnected term in $\mathscr{L}V(t,x,i)$ (see [4]), and $\mathscr{L}V(t,x,i)$ can be seen as the "derivative" of $V(t,x,i)$ with respect to time.

Now, we have the following theorem about the existence of weak solutions.

**Theorem 2.** Consider system (1) with $f(t,x,i)$ and $g(t,x,i)$ being jointly continuous in $(t,x) \in \mathbb{R}_{t_0} \times \mathbb{R}^d$ for every $i \in \mathbb{S}$. If there exist a function $V \in C^2(\mathbb{R}^d \times \mathbb{S}; \mathbb{R}_+)$, two $\mathcal{K}_\infty$ class functions $\alpha_1$, $\alpha_2$ and constants $d_i > 0$, $c_{ij} \geqslant 1$ such that

$$V(x,i) \leqslant c_{ij} V(x,j),\ \forall x \in \mathbb{R}^d,\ \forall i,j \in \mathbb{S}, \tag{27}$$

$$\alpha_1(|x|) \leqslant V(x,i) \leqslant \alpha_2(|x|), \mathscr{L}V(t,x,i) \leqslant d_i V(x,i),\ \forall t \in \mathbb{R}_{t_0},\ \forall x \in \mathbb{R}^d,\ \forall i \in \mathbb{S}, \tag{28}$$

then, there exists a global weak solution $(x(t),\ w^x(t),\ r^x(t),\ \Omega^x,\ \mathscr{F}^x,\ \{\mathscr{F}_t^x\}_{t \geqslant 0},\ \mathbb{P}^x)$ to system (1).

*Proof.* **Approximation.** We construct an approximation for system (1). Let

$$f^{\bar{m}}(t,x,i) = \beta^2(t,x) f(t,x,i),\ g^{\bar{m}}(t,x,i) = \beta(t,x) g(t,x,i),\ \bar{m} \in \mathbb{N}_{t_0}, \tag{29}$$

where $\mathbb{N}_{t_0} = \{a : a \geqslant t_0 \text{ and } a \text{ is a positvie integer}\}$, $\beta(t,x) = 1$ for $t_0 \leqslant t \leqslant \bar{m}$ and $|x| \leqslant \bar{m}$, $\beta(t,x) = (2 - |x|/\bar{m})$ for $t_0 \leqslant t \leqslant \bar{m}$ and $\bar{m} < |x| \leqslant 2\bar{m}$, $\beta(t,x) = (2 - t/\bar{m})$ for $\bar{m} < t \leqslant 2\bar{m}$ and $|x| \leqslant \bar{m}$, $\beta(t,x) = (2 - |x|/\bar{m})(2 - t/\bar{m})$ for $\bar{m} < t \leqslant 2\bar{m}$ and $\bar{m} < |x| \leqslant 2\bar{m}$, and $\beta(t,x) = 0$ for $t > 2\bar{m}$ and $|x| > 2\bar{m}$. Clearly, $f^{\bar{m}}(t,x,i)$ and $g^{\bar{m}}(t,x,i)$ are jointly continuous in $(t,x) \in \mathbb{R}_{t_0} \times \mathbb{R}^d$ $\forall i \in \mathbb{S}$ and bounded $\forall (t,x,i) \in \mathbb{R}_{t_0} \times \mathbb{R}^d \times \mathbb{S}$. Then, according to Theorem 1, for $\bar{m} \in \mathbb{N}_{t_0}$, there exists a global weak solution $(x_{\bar{m}}(t),\ w^{x_{\bar{m}}}(t),\ r^{x_{\bar{m}}}(t),\ \Omega^{x_{\bar{m}}},\ \mathscr{F}^{x_{\bar{m}}},\ \{\mathscr{F}_t^{x_{\bar{m}}}\}_{t \geqslant 0},\ \mathbb{P}^{x_{\bar{m}}})$ to system

$$x_{\bar{m}}(t) = x_{\bar{m}}(t_0) + \int_{t_0}^{t} f^{\bar{m}}(s, x_{\bar{m}}(s), r^{x_{\bar{m}}}(s)) \mathrm{d}s + \int_{t_0}^{t} g^{\bar{m}}(s, x_{\bar{m}}(s), r^{x_{\bar{m}}}(s)) \mathrm{d}w^{x_{\bar{m}}}(t),\ t \geqslant t_0. \tag{30}$$

**Tightness.** We show that $\forall \epsilon > 0$, there exists a compact set $\bar{K}_\epsilon^X \subseteq \mathcal{C}(\mathbb{R}_+; \mathbb{R}^d)$ such that $\mathbb{P}^{x_{\bar{m}}}\{x_{\bar{m}} \in \bar{K}_\epsilon^X\} \geqslant 1 - \epsilon/4$, where $x_{\bar{m}}(t) = x_{\bar{m}}(t_0)$ for $t \in [0, t_0)$. According to (26), for $V$, we define an operator $\mathscr{L}_{\bar{m}}$ associated with (30) by $\mathscr{L}_{\bar{m}} V(t,x,i) = V_x(x,i) f^{\bar{m}}(t,x,\ i) + \frac{1}{2}\mathrm{Tr}[(g^{\bar{m}}(t,x,i))^{\mathrm{T}} V_{xx}(x,i) g^{\bar{m}}(t,x,i)] + \sum_{j=1}^{N} q_{ij} V(x,j)$. Then, by (27), (28), and the definitions of $f^{\bar{m}}$ and $g^{\bar{m}}$, we obtain that

$$\mathscr{L}_{\bar{m}} V(t,x,i) = \beta^2(t,x) \mathscr{L}V(t,x,i) + (1 - \beta^2(t,x)) \sum_{j=1}^{N} q_{ij} V(x,j) \leqslant \bar{c}_1 V(x,i), \tag{31}$$

where $\bar{c}_1 = \max\{\big((1-\bar{c})q_{ii}\big) \vee c_1 : i \in \mathbb{S}\}$, $c_1 = \max\{d_i : i \in \mathbb{S}\}$, and $\bar{c} = \max\{c_{ij} : i,j \in \mathbb{S}\}$. Let $\bar{\tau}_k^{\bar{m}} = \inf\{t \geqslant t_0 : |x_{\bar{m}}(t)| > k\}$ and $\bar{\tau}_\infty^{\bar{m}} = \lim_{k \to \infty} \bar{\tau}_k^{\bar{m}}$. Then, we have

$$\mathbb{P}^{x_{\bar{m}}}\{\bar{\tau}_\infty^{\bar{m}} = \infty\} = 1,\ \forall \bar{m} \in \mathbb{N}_{t_0} \tag{32}$$

by referring to the proof of [3, Theorem 3.19]. Let $U(t,x,r) = \mathrm{e}^{-\bar{c}_1(t-t_0)} V(x,r)$. Then, it follows from (31) that $\mathscr{L}U(t,x,i) = (\frac{\partial U}{\partial t} + \mathrm{e}^{-\bar{c}_1(t-t_0)} \mathscr{L}_{\bar{m}} V)(t,x,i) \leqslant 0$. By the generalized Itô formula, we have that

$$\begin{aligned} & U(t \wedge \bar{\tau}_k^{\bar{m}}, x_{\bar{m}}(t \wedge \bar{\tau}_k^{\bar{m}}), r^{x_{\bar{m}}}(t \wedge \bar{\tau}_k^{\bar{m}})) \\ & = U(t_0, x_{\bar{m}}(t_0), r^{x_{\bar{m}}}(t_0)) + \int_{t_0}^{t \wedge \bar{\tau}_k^{\bar{m}}} \mathscr{L}U(s, x_{\bar{m}}(s), r^{x_{\bar{m}}}(s)) \mathrm{d}s + \bar{M}_{\bar{m}}(t \wedge \bar{\tau}_k^{\bar{m}}), \end{aligned} \tag{33}$$

where $\bar{M}_{\bar{m}}(t \wedge \bar{\tau}_k^{\bar{m}})$ is a martingale and the definition of $\bar{M}_{\bar{m}}(t)$ is similar to that of $\bar{M}(t)$ in (25). Then, taking $k \to \infty$ in (33), we obtain that $U(t, x_{\bar{m}}(t), r^{x_{\bar{m}}}(t)) \leqslant U(t_0, x_{\bar{m}}(t_0), r^{x_{\bar{m}}}(t_0)) + \bar{M}_{\bar{m}}(t)$, which implies that $U(t, x_{\bar{m}}(t), r^{x_{\bar{m}}}(t))$ is a supermartingale. Then, it follows from the supermartingale inequality and (28) that for every $T > t_0$ and each $\lambda > 0$, $\mathbb{P}^{x_{\bar{m}}}\{\sup_{t_0 \leqslant s \leqslant T} U(s, x_{\bar{m}}(s), r^{x_{\bar{m}}}(s)) \geqslant \lambda\} \leqslant \mathbb{E}^{x_{\bar{m}}} \alpha_2(|x_{\bar{m}}(t_0)|)/\lambda$. Furthermore, by the definition of $U(t,x,r)$ and (28), one can show that $\mathbb{P}^{x_{\bar{m}}}\{\sup_{t_0 \leqslant s \leqslant T} |x_{\bar{m}}(s)| \geqslant \lambda\} \leqslant \mathrm{e}^{\bar{c}_1(T-t_0)} \mathbb{E}^{x_{\bar{m}}}(\alpha_2(|x_{\bar{m}}(t_0)|))/\alpha_1(\lambda)$. Since $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and the distribution of $x_{\bar{m}}(t_0)$ has a compact support, we obtain that there exists a $\lambda_0(\epsilon, t_0, T) > 0$ such that for each $\bar{m} \in \mathbb{N}_{t_0}$,

$$\mathbb{P}^{x_{\bar{m}}}\left\{\sup_{t_0 \leqslant s \leqslant T} |x_{\bar{m}}(s)| > \lambda_0(\epsilon, t_0, T)\right\} < \frac{\epsilon}{12}. \tag{34}$$

Due to the continuity of $f(t,x,i)$ and $g(t,x,i)$, there exists a constant $M(t_0,T,\lambda_0)>0$ such that $\sup_{t_0\leqslant s\leqslant T,|x|\leqslant\lambda_0,i\in\mathbb{S}}\big(|f(s,x,i)|\vee|g(s,x,i)|\big)\leqslant M$. Then, by the similar way of proving (7), we have that

$$\mathbb{E}^{x_{\bar m}}\{|x_{\bar m}(t\wedge\bar\tau^{\bar m}_{\lambda_0})-x_{\bar m}(s\wedge\bar\tau^{\bar m}_{\lambda_0})|^{2p}\}\leqslant 2^{2p-1}M^{2p}((T-t_0)^p+(p(2p-1))^{2p})|t-s|^p=:M_1|t-s|^p,\quad(35)$$

where $\bar\tau^{\bar m}_{\lambda_0}=\inf\{t\geqslant t_0:|x_{\bar m}(t)|>\lambda_0\}$. Note that the distribution of $x_{\bar m}(t_0)$ has a compact support. Thus, $\mathbb{E}^{x_{\bar m}}|x_{\bar m}(t_0)|^{2p}<\infty$ for $p>2$. And, referring to the proof of [32, Theorem 1.4.3] with $\gamma=\alpha=2p$ and $\beta=p-1$, we have that for each $\varepsilon>0$, $\lim_{h\downarrow 0}\sup_{\bar m\in\mathbb{N}_{t_0}}\mathbb{P}^{x_{\bar m}}\{\max_{t,s\in[t_0,T],|t-s|<h}|x_{\bar m}(t\wedge\bar\tau^{\bar m}_{\lambda_0})-x_{\bar m}(s\wedge\bar\tau^{\bar m}_{\lambda_0})|>\varepsilon\}=0$. Then, there exists a sequence $\{h_k\}$ such that $h_k>0$ and $h_k\downarrow 0$ as $k\to\infty$ and

$$\mathbb{P}^{x_{\bar m}}\Big\{\max_{t,s\in[t_0,T],|t-s|<h_k}\big|x_{\bar m}(t\wedge\bar\tau^{\bar m}_{\lambda_0})-x_{\bar m}(s\wedge\bar\tau^{\bar m}_{\lambda_0})\big|>1/k\Big\}<\epsilon/4^{k+1},\ \forall\bar m\in\mathbb{N}_{t_0}.\quad(36)$$

Note that $\big(\bigcup_{k=1}^{\infty}\{\max_{t,s\in[t_0,T],|t-s|\leqslant h_k}\big|x_{\bar m}(t)-x_{\bar m}(s)\big|>1/k\}\big)\subseteq\big((\bigcup_{k=1}^{\infty}\max_{t,s\in[t_0,T],|t-s|\leqslant h_k}\big|x_{\bar m}(t\wedge\bar\tau^{\bar m}_{\lambda_0})-x_{\bar m}(s\wedge\bar\tau^{\bar m}_{\lambda_0})\big|>1/k)\cap\{\bar\tau^{\bar m}_{\lambda_0}<T\}\big)$, and

$$\mathbb{P}^{x_{\bar m}}\{\bar\tau^{\bar m}_{\lambda_0}\leqslant T\}<\frac{\epsilon}{12},\quad(37)$$

which is implied by (34). Thus, we obtain that $\mathbb{P}^{x_{\bar m}}\{\bigcup_{k=1}^{\infty}(\max_{t,s\in[t_0,T],|t-s|\leqslant h_k}\big|x_{\bar m}(t)-x_{\bar m}(s)\big|>1/k)\}<\frac{\epsilon}{6}$ and $\mathbb{P}^{x_{\bar m}}\{|x_{\bar m}(t_0)|>\lambda_0\}\leqslant\frac{\epsilon}{12}$. Let $\bar K^X_\epsilon=\big(\bigcap_{k=1}^{\infty}\big\{v\in\mathcal{C}(\mathbb{R}_+;\mathbb{R}^d):\max_{|t-s|\leqslant h_k,0\leqslant s,t\leqslant T}|v(t)-v(s)|\leqslant 1/k\big\}\big)\cap\{v\in\mathcal{C}(\mathbb{R}_+;\mathbb{R}^d):|v(0)|\leqslant\lambda_0\}$. According to Ascoli-Arzelà Theorem, $\bar K^X_\epsilon$ is a compact set of $\mathcal{C}(\mathbb{R}_+;\mathbb{R}^d)$. Note that $x_{\bar m}(t)$ is a continuous stochastic process and $x_{\bar m}(t)=x_{\bar m}(t_0)$ for $t\in[0,t_0)$. Thus, we have $\mathbb{P}^{x_{\bar m}}\{x_{\bar m}\in\bar K^X_\epsilon\}\geqslant 1-\epsilon/4$.

**New stochastic processes.** We show that there exists a sequence of stochastic processes on a probability space, having the same law with some subsequence of $\{\bar\Lambda_{\bar m}\}$, where $\bar\Lambda_{\bar m}=(x_{\bar m},w^{x_{\bar m}},r^{x_{\bar m}})$ is defined in (30). Since homogeneous Markov chains $r^{x_{\bar m}}(t)$ and $r(t)$ have the same initial distribution and the same transition function, $r^{x_{\bar m}}(t)$ and $r(t)$ have the same law. Then, $\mathbb{P}^{x_{\bar m}}\{r^{x_{\bar m}}(t)\in K^r_\epsilon\}\geqslant 1-\frac{\epsilon}{2}$, where $K^r_\epsilon$ has been defined in (8). Then, similar to the proof of Theorem 1, on a probability space $\{\Omega^x$, $\mathscr{F}^x$, $\mathbb{P}^x\}$, there exist stochastic processes

$$\hat\Lambda_{\bar m_l}(\cdot):=(\hat x_{\bar m_l}(\cdot),\hat w^{\bar m_l}(\cdot),\hat r_{\bar m_l}(\cdot))\ \text{ and }\ \hat\Lambda^x(\cdot):=(x(\cdot),w^x(\cdot),r^x(\cdot))$$

such that

$$\mathbb{P}^x\hat\Lambda^{-1}_{\bar m_l}=\mathbb{P}^{x_{\bar m_l}}\bar\Lambda^{-1}_{\bar m_l},\ \text{i.e., law of }(\hat\Lambda_{\bar m_l})=\text{law of }(\bar\Lambda_{\bar m_l}),\quad(38)$$

$$\hat\Lambda_{\bar m_l}(t)\to\hat\Lambda^x(t)\ \text{a.s., as }l\to\infty,\ \text{uniformly on }[0,T],\ \forall T>t_0,\quad(39)$$

where $\{\bar\Lambda_{\bar m_l}\}$ is a subsequence of $\{\bar\Lambda_{\bar m}\}$. Similar to (12), there exists a random variable $\bar l\in\mathbb{N}$ such that

$$\mathbb{P}^x\{\hat r_{\bar m_l}(t)=r^x(t),\ l\geqslant\bar l\}=1.\quad(40)$$

**Weak solution.** Now, we show that $x(t)$ is a global weak solution to system (1). Let $\hat\tau^{\bar m_l}_k=\inf\{s\geqslant t_0:|\hat x_{\bar m_l}(s)|>k\}$, $\hat\tau^{\bar m_l}_\infty=\lim_{k\to\infty}\hat\tau^{\bar m_l}_k$, $\hat\tau_k=\inf\{s\geqslant t_0:|x(s)|>k\}$, $\hat\tau_\infty=\lim_{k\to\infty}\hat\tau_k$ and $\hat\tau^{\bar m_l}_{\lambda_0}=\inf\{s\geqslant t_0:|\hat x_{\bar m_l}(s)|>\lambda_0\}$. Then, by (32) and (38), we have $\mathbb{P}^x\{\hat\tau^{\bar m_l}_\infty=\infty\}=1$. Furthermore, it follows from (39) that $\mathbb{P}^x\{\hat\tau_\infty=\infty\}=1$. Then, by the continuity of $f(t,x,i)$ and $g(t,x,i)$, we have that $\forall T\geqslant t_0$, $\int_{t_0}^{\mathrm{T}}|f(s,x(s),r^x(s))|\mathrm{d}s<\infty$ a.s. and $\int_{t_0}^{\mathrm{T}}|g(s,x(s),r^x(s))|^2\mathrm{d}s<\infty$ a.s. which implies that $\int_{t_0}^{t}g(s,x(s),r^x(s))\mathrm{d}w^x(s)$ exists a.s. for $t\geqslant t_0$. It follows from (30) and (38) that $\forall t\geqslant t_0$,

$$\hat x_{\bar m_l}(t)=\hat x_{\bar m_l}(t_0)+\int_{t_0}^{t}f^{\bar m_l}(s,\hat x_{\bar m_l}(s),\hat r_{\bar m_l}(s))\mathrm{d}s+\int_{t_0}^{t}g^{\bar m_l}(s,\hat x_{\bar m_l}(s),\hat r_{\bar m_l}(s))\mathrm{d}\hat w^{\bar m_l}(s)\ \text{a.s.}\quad(41)$$

We show the convergence of the right-side terms of (41) by two steps.

**Step 1.** We prove the convergence of $\int_{t_0}^{t}f^{\bar m_l}(s,\hat x_{\bar m_l}(s),\hat r_{\bar m_l}(s))\mathrm{d}s$. Note that $\lim_{l\to\infty}f^{\bar m_l}(s,x,i)=f(s,x,i)\ \forall(s,x,i)\in\mathbb{R}_{t_0}\times\mathbb{R}^d\times\mathbb{S}$ and

$$f^{\bar m_l}(s,\hat x_{\bar m_l}(s),\hat r_{\bar m_l}(s))=f(s,x(s),r^x(s))+\big(f(s,\hat x_{\bar m_l}(s),\hat r_{\bar m_l}(s))-f(s,x(s),r^x(s))\big)$$

$$+ \left(f^{\bar{m}_l}(s, \hat{x}_{\bar{m}_l}(s), \hat{r}_{\bar{m}_l}(s)) - f(s, \hat{x}_{\bar{m}_l}(s), \hat{r}_{\bar{m}_l}(s))\right). \tag{42}$$

Thus, by the continuity of $f(t,x,i)$, (39) and (40), we have that $\forall s \in [t_0, T]$, $\forall T > 0$,

$$\lim_{l\to\infty} f^{\bar{m}_l}(s, \hat{x}_{\bar{m}_l}(s), \hat{r}_{\bar{m}_l}(s)) = f(s, x(s), r^x(s)) \ \text{ a.s.} \tag{43}$$

Then, it follows from the dominated convergence theorem that $\forall t \in [t_0, T],\ \forall T > t_0$,

$$\lim_{l\to\infty} \int_{t_0}^{t} f^{\bar{m}_l}(s, \hat{x}_{\bar{m}_l}(s), \hat{r}_{\bar{m}_l}(s))\mathrm{d}s = \int_{t_0}^{t} f(s, x(s), r^x(s))\mathrm{d}s \ \text{ a.s.} \tag{44}$$

**Step 2.** We show the convergence of $\int_{t_0}^{t} g^{\bar{m}_l}(s, \hat{x}_{\bar{m}_l}(s), \hat{r}_{\bar{m}_l}(s))\mathrm{d}\hat{w}^{\bar{m}_l}(s)$. By (35) and (38), we have

$$\mathbb{E}^x \left|\hat{x}_{\bar{m}_l}(t_2 \wedge \hat{\tau}_{\lambda_0}^{\bar{m}_l}) - \hat{x}_{\bar{m}_l}(t_1 \wedge \hat{\tau}_{\lambda_0}^{\bar{m}_l})\right|^{2p} \leqslant M_1(t_0, T, p, \lambda_0)|t_2 - t_1|^p,\ \forall t_1, t_2 \in [t_0, T]. \tag{45}$$

Then, according to the Chebyshev's inequality,

$$\lim_{h\to 0} \sup_{|t_1-t_2|<h, t_1,t_2\in[t_0,T]} \mathbb{P}^x\{|\hat{x}_{\bar{m}_l}(t_2 \wedge \hat{\tau}_{\lambda_0}^{\bar{m}_l}) - \hat{x}_{\bar{m}_l}(t_1 \wedge \hat{\tau}_{\lambda_0}^{\bar{m}_l})| > \delta\} = 0,\ \forall \delta > 0. \tag{46}$$

Note that $\{|\hat{x}_{\bar{m}_l}(t_2) - \hat{x}_{\bar{m}_l}(t_1)| > \delta\} \subseteq \left(\{|\hat{x}_{\bar{m}_l}(t_2 \wedge \hat{\tau}_{\lambda_0}^{\bar{m}_l}) - \hat{x}_{\bar{m}_l}(t_1 \wedge \hat{\tau}_{\lambda_0}^{\bar{m}_l})| > \delta\} \cup \{\hat{\tau}_{\lambda_0}^{\bar{m}_l} \leqslant T\}\right)$, and $\mathbb{P}^x\{\hat{\tau}_{\lambda_0}^{\bar{m}_l} \leqslant T\} < \epsilon/12$ which is implied by (37) and (38). Thus, $\lim_{h\to 0}\lim_{l\to\infty}\sup_{|t_1-t_2|<h,t_1,t_2\in[t_0,T]} \mathbb{P}^x\{|\hat{x}_{\bar{m}_l}(t_2) - \hat{x}_{\bar{m}_l}(t_1)| > \delta\} < \epsilon/12$. From the arbitrariness of $\epsilon$, we can let $\epsilon \to 0$ in the above inequality to obtain that

$$\lim_{h\to 0}\lim_{l\to\infty} \sup_{|t_1-t_2|<h, t_1,t_2\in[t_0,T]} \mathbb{P}^x\{|\hat{x}_{\bar{m}_l}(t_2) - \hat{x}_{\bar{m}_l}(t_1)| > \delta\} = 0,\ \forall \delta > 0. \tag{47}$$

Additionally, referring to the proof of Lemma A2 in Appendix A, we have

$$\lim_{h\to 0} \sup_{|t_1-t_2|<h, t_1,t_2\in[t_0,T]} \mathbb{P}^x\{|\hat{r}_{\bar{m}_l}(t_2) - \hat{r}_{\bar{m}_l}(t_1)| > \delta\} = 0,\ \forall \delta > 0. \tag{48}$$

Then, by a similar way as the proof of Proposition A2 in Appendix A, we obtain that for each $\epsilon_0 > 0$,

$$\lim_{h\to 0}\lim_{l\to\infty} \sup_{t_1,t_2\in[t_0,T], |t_1-t_2|<h} \mathbb{P}^x\{|g_j^{\bar{m}_l}(t_1, \hat{x}_{\bar{m}_l}(t_1), \hat{r}_{\bar{m}_l}(t_1)) - g_j^{\bar{m}_l}(t_2, \hat{x}_{\bar{m}_l}(t_2), \hat{r}_{\bar{m}_l}(t_2))| > \epsilon_0\} = 0. \tag{49}$$

The rest proof is similar to that of Theorem 1.

**Remark 3.** The existence of weak solutions of SNSwMS is studied by the Lyapunov method in Theorem 2, where the condition on both drift coefficient and diffusion coefficient is only continuity, and the studied system (1) is time-varying. Our requirement on coefficients to guarantee the existence of global weak solutions is only continuity, which is much simpler (more practical) than the growth conditions in [23,24], and then makes it possible to extend the finite-time control method to other control problems, such as stochastic finite-time adaptive control. To the best of the authors' knowledge, the existence of global weak solutions of time-varying stochastic nonlinear systems (with Markovian switching) has not been studied by the Lyapunov method.

### 2.2 Finite-time stability

Now, we consider the finite-time stability in probability for system (1) with $f(t,0,i) \equiv 0$ and $g(t,0,i) \equiv 0$ $\forall (t,i) \in \mathbb{R}_{t_0} \times \mathbb{S}$. We assume that $f(t,x,i)$ and $g(t,x,i)$ are jointly continuous in $(t,x) \in \mathbb{R}_{t_0} \times \mathbb{R}^d$ $\forall i \in \mathbb{S}$. First, for system (1), we present a lemma supplying a criterion on the stability in probability and an approximation of a global weak solution.

**Lemma 2.** If there exist a function $V \in C^2(\mathbb{R}^d \times \mathbb{S}; \mathbb{R}_+)$, two $\mathcal{K}_\infty$ class functions $\alpha_1$, $\alpha_2$ and constants $c_{ij} \geqslant 1$, $1 \leqslant i, j \leqslant N$ such that

$$\alpha_1(|x|) \leqslant V(x,i) \leqslant \alpha_2(|x|),\ V(x,i) \leqslant c_{ij}V(x,j),\ \forall x \in \mathbb{R}^d,\ \forall i,j \in \mathbb{S}, \tag{50}$$

$$\mathscr{L}V(t,x,i) \leqslant 0,\ \forall t \in \mathbb{R}_{t_0}, \quad \forall x \in \mathbb{R}^d,\ \forall i \in \mathbb{S}, \tag{51}$$

then, (i) there exists a global weak solution ($x(t)$, $w^x(t)$, $r^x(t)$, $\Omega^x$, $\mathscr{F}^x$, $\{\mathscr{F}_t^x\}_{t\geqslant 0}$, $\mathbb{P}^x$) to system (1); (ii) the trivial weak solution of system (1) is globally stable in probability; (iii) the following approximation

$$\check{Y}_n(t) = x\left(t_0+\frac{\iota}{n}\right) + \left(t-t_0-\frac{\iota}{n}\right) f\left(t_0+\frac{\iota}{n}, x\left(t_0+\frac{\iota}{n}\right), r^x\left(t_0+\frac{\iota}{n}\right)\right) + g\left(t_0+\frac{\iota}{n}, x\left(t_0+\frac{\iota}{n}\right),\right.$$
$$\left. r^x\left(t_0+\frac{\iota}{n}\right)\right)\left(w^x(t)-w^x\left(t_0+\frac{\iota}{n}\right)\right),\ t\in\left[t_0+\frac{\iota}{n}, t_0+\frac{\iota+1}{n}\right),\ \iota=0,1,\ldots \tag{52}$$

converges to $x(t)$ in probability, as $n\to\infty$, where $n$ is a positive integer.

*Proof.* (i) It follows from Theorem 2 that there exists a global weak solution ($x(t)$, $w^x(t)$, $r^x(t)$, $\Omega^x$, $\mathscr{F}^x$, $\{\mathscr{F}_t^x\}_{t\geqslant 0}$, $\mathbb{P}^x$) to system (1).

(ii) Let $\tau_k^x=\inf\{t\geqslant t_0: |x(t)|\geqslant k\}$ and $\tau_\infty^x=\lim_{k\to\infty}\tau_k^x$. By (50), (51) and the generalized Itô formula, we obtain $\mathbb{E}^x(\alpha_1(|x(t\wedge\tau_k^x)|))\leqslant\mathbb{E}^x(V(x(t\wedge\tau_k^x), r^x(t\wedge\tau_k^x)))\leqslant\mathbb{E}^x(\alpha_2(|x(t_0)|))$. Then, according to the Chebyshev's inequality, $\mathbb{P}^x\{\tau_k^x<t\}\alpha_1(k)\leqslant\mathbb{E}^x(\alpha_2(|x(t_0)|))$. This together with $\alpha_1\in\mathcal{K}_\infty$ leads to

$$\mathbb{P}^x\{\tau_\infty^x<\infty\}=0,\ \text{i.e., } \mathbb{P}^x\{\tau_\infty^x=\infty\}=1. \tag{53}$$

The rest proof of stability is respectively given by two cases: $\mathbb{P}^x(B^{\rm c})\neq 0$ and $\mathbb{P}^x(B)\neq 0$, where $B=\{x(t_0)\neq 0\}$, and $B^{\rm c}$ is its complementary set.

**Case 1.** $\mathbb{P}(B)\neq 0$. Referring to the proof of [42, Theorem 2], for each $\epsilon>0$, we can construct a $\mathcal{K}_\infty$ class function $\beta$ such that

$$\mathbb{P}^x\left\{I_B(\sup_{t\geqslant t_0}|x(t)|)>\beta(|x(t_0)|)\right\}\leqslant\epsilon. \tag{54}$$

**Case 2.** $\mathbb{P}^x(B^{\rm c})\neq 0$. It follows from the generalized Itô formula and (50) that $\mathbb{E}^x(I_{B^{\rm c}}\alpha_1(|x(t\wedge\tau_k^x)|))\leqslant\mathbb{E}^x(I_{B^{\rm c}}\alpha_2(|x(t_0)|))=0$. Then, taking $k\to\infty$, and using (53) and $\alpha_1\in\mathcal{K}_\infty$, we have that

$$\mathbb{P}^x\{I_{B^{\rm c}}|x(t)|>0\}=0 \text{ for any } t\geqslant t_0. \tag{55}$$

Combining (54) with (55) leads to $\mathbb{P}^x\big\{\sup_{t\geqslant t_0}|x(t)|>\beta(|x(t_0)|)\big\}\leqslant\epsilon$.

(iii) It follows from (1), (52), Hölder inequality, and [3, Theorem 2.11] that $\mathbb{E}^x\big|x(t\wedge\tau_k^x)-Y_n(t\wedge\tau_k^x)\big|^4\leqslant c_1(t_0,t,k)/n^2$, where $c_1(t_0,t,k)$ is a positive constant depending on $t_0$, $t$, and $x$. The rest of the proof is similar to that of [18, Lemma 2].

**Remark 4.** Different from existing theorems about the finite-time stability of SNSwMS (e.g., [22, Theorems 1 and 2], [23, Theorems 12 and 16], [24, Theorem 1]), the approximation generated by Picard iteration cannot be used to prove the finite-time attractiveness, because only the continuity on coefficients cannot guarantee the convergence of the approximation generated by Picard iteration (see [36]). Inspired by [18, Lemma 2], we construct the approximation $\{\check{Y}_n(t)\}$.

**Theorem 3.** Let condition (50) in Lemma 2 hold and Eq. (51) be replaced by the following: there exist constants $c_i>0$ and $0\leqslant\gamma<1$ such that

$$\mathscr{L}V(t,x,i)\leqslant -c_i(V(x,i))^\gamma,\ \forall t\in\mathbb{R}_{t_0},\ \forall x\in\mathbb{R}^d\setminus\{0\},\ \forall i\in\mathbb{S}. \tag{56}$$

Then, there exists a global weak solution ($x(t)$, $w^x(t)$, $r^x(t)$, $\Omega^x$, $\mathscr{F}^x$, $\{\mathscr{F}_t^x\}_{t\geqslant 0}$, $\mathbb{P}^x$) to system (1), and the trivial weak solution of system (1) is globally finite-time stable in probability.

*Proof.* According to Lemma 2, there exists a global weak solution ($x(t)$, $w^x(t)$, $r^x(t)$, $\Omega^x$, $\mathscr{F}^x$, $\{\mathscr{F}_t^x\}_{t\geqslant 0}$, $\mathbb{P}^x$) to system (1) and the trivial weak solution is globally stable in probability. We now pay our attention to proving the finite-time attractiveness in probability. It follows from (55) that

$$I_{B^{\rm c}}x(t)\equiv 0 \text{ a.s.}, \forall t\geqslant t_0. \tag{57}$$

**Case of $\gamma=0$.** By the generalized Itô formula, we have that $\mathbb{E}^x\big(I_BV(x(t\wedge\tau_k^x\wedge\varrho_\mu^x), r^x(t\wedge\tau_k^x\wedge\varrho_\mu^x))\big)=\mathbb{E}^x\big(I_BV(x(t_0), r^x(t_0))\big)+\mathbb{E}^x\int_{t_0}^{t\wedge\tau_k^x\wedge\varrho_\mu^x}I_B\mathscr{L}V(s,x(s),r^x(s)){\rm d}s$, which together with (50) and (56) leads to that

$$0\leqslant\mathbb{E}^x\big(I_B\alpha_1(|x(t\wedge\tau_k^x\wedge\varrho^x)|)\big)\leqslant\mathbb{E}^x\big(I_B\alpha_2(|x(t_0)|)\big)-c\mathbb{E}^x\big(I_B(t\wedge\tau_k^x\wedge\varrho^x-t_0)\big), \tag{58}$$

where $c := \min\{c_i, i \in \mathbb{S}\}$, and $\varrho^x$ is the settling time of $x(t)$. Then, taking $t \to \infty$ and $k \to \infty$ in (58) and applying the Fatou lemma and (53), we have $\mathbb{E}^x\big(I_B(\varrho^x - t_0)\big) \leqslant \mathbb{E}^x\big(I_B\alpha_2(|x(t_0)|)\big)/c$. Since the distribution of $x(t_0)$ is of compact support, we have $\mathbb{E}^x\big(I_B\alpha_2(|x(t_0)|)\big) < \infty$. If $\mathbb{P}^x\{I_B\varrho^x = \infty\} \neq 0$, then $\mathbb{E}^x\big(I_B(\varrho^x - t_0)\big) = \infty$, which contradicts with $\infty = \mathbb{E}^x\big(I_B(\varrho^x - t_0)\big) \leqslant \mathbb{E}^x\big(I_B\alpha_2(|x(t_0)|)\big)/c < \infty$. Hence, we have $\mathbb{P}^x\{I_B\varrho^x = \infty\} = 0$, which together with (57) leads to $\mathbb{P}^x\{\varrho^x < \infty\} = 1$ for the case of $\gamma = 0$.

**Case of $\gamma \in (0,1)$.** Since $x(t)$ is a stochastic process, we may write $x(t)$ as $x(t,\omega)$, $\omega \in \Omega^x$. Let $B_k = \{\frac{1}{k} < |x(t_0)| < k\} \cap B$, $\rho_k^x(\omega) = \inf\{t \geqslant t_0 : |x(t,\omega)| \notin (\frac{1}{k}, k)\}$ for $\omega \in B$, and $\rho_k^x(\omega) = t_0$ for $\omega \in B^{\rm c}$. It is clear that $\{\rho_k^x\}$ and $\{B_k\}$ are nondecreasing sequences. We now set $B_\infty = \lim_{k\to\infty} B_k$ and $\rho_\infty^x = \lim_{k\to\infty}\rho_k^x$. Since the initial distribution $\mu$ is of compact support, $B_\infty = B$. It follows from the generalised Itô formula and (56) that $\forall\ t_0 \leqslant \nu \leqslant t$,

$$\begin{aligned}&\mathbb{E}^x\big(I_{B_k}I_{\{t\leqslant\rho_k^x\}}V(x(t),r(t))\big) - \mathbb{E}^x\big(I_{B_k}I_{\{\nu\leqslant\rho_k^x\}}V(x(\nu),r(\nu))\big)\\&\leqslant -c\int_\nu^t \mathbb{E}^x\Big(I_{B_k}I_{\{s\leqslant\rho_k^x\}}\big(V(x(s),r(s))\big)^\gamma\Big)\mathrm{d}s,\end{aligned} \tag{59}$$

where $c$ has been defined in (58). By (50) and the definition of $\rho_k^x$, one can show that for each $s \in [t_0,\infty)$,

$$\big(\mathbb{E}^x(I_{B_k}I_{\{s\leqslant\rho_k^x\}}V(x(s),r(s)))\big)^\gamma \leqslant (\alpha_2(k))^\gamma\cdot(\mathbb{P}^x\{\omega\in B_k : s\leqslant\rho_k^x(\omega)\})^\gamma, \tag{60}$$

$$\mathbb{E}^x\big(I_{B_k}I_{\{s\leqslant\rho_k^x\}}\big(V(x(s),r(s))\big)^\gamma\big) \geqslant \left(\alpha_1\left(\frac{1}{k}\right)\right)^\gamma\cdot\mathbb{P}^x\{\omega\in B_k : s\leqslant\rho_k^x(\omega)\}. \tag{61}$$

Then, we have that

$$\begin{aligned}\mathbb{E}^x\big(I_{B_k}I_{\{s\leqslant\rho_k^x\}}\big(V(x(s),r(s))\big)^\gamma\big) \geqslant &\frac{(\alpha_1(\frac{1}{k}))^\gamma}{(\alpha_2(k))^\gamma}\cdot\big(\mathbb{E}^x(I_{B_k}I_{\{s\leqslant\rho_k^x\}}V(x(s),r(s)))\big)^\gamma\\&\cdot\big(\mathbb{P}^x\{\omega\in B_k : \rho_k^x(\omega) = \infty\}\big)^{1-\gamma},\end{aligned} \tag{62}$$

where $\mathbb{P}^x\{\omega\in B_k : \rho_k^x(\omega)=\infty\} \leqslant \mathbb{P}^x\{\omega\in B_k : s\leqslant\rho_k^x(\omega)\}$ is used.

Now, we consider two cases.

(i) If $\mathbb{P}^x\{\omega\in B_k : \rho_k^x(\omega)=\infty\} = 0\ \forall k\in\mathbb{N}$, then, by the dominated convergence theorem, we obtain

$$\mathbb{P}^x\{\omega\in B_\infty : \rho_\infty^x(\omega)=\infty\} = \mathbb{P}^x\{\omega\in B : |x(t,\omega)|>0,\ \text{for all}\ t\geqslant t_0\} = 0, \tag{63}$$

which together with (57) implies $\mathbb{P}^x(\varrho_\mu^x < \infty) = 1$. The required result is obtained.

(ii) If there exists a positive integer $k_0$ such that $\mathbb{P}^x\{\omega\in B_{k_0} : \rho_{k_0}^x(\omega)=\infty\}\neq 0$, then it follows from the increasement of $\{\rho_k^x\}$, (59) and (62) that $\forall k\geqslant k_0$, $\mathbb{E}^x(I_{B_k}I_{\{t\leqslant\rho_k^x\}}V(x(t),r(t))) - \mathbb{E}^x(I_{B_k}I_{\{\nu\leqslant\rho_k^x\}}V(x(\nu), r(\nu))) \leqslant -C_k\int_\nu^t(\mathbb{E}^x(I_{B_k}I_{\{s\leqslant\rho_k^x\}}V(x(s),r(s))))^\gamma\mathrm{d}s$, where $\tilde{c} = (\mathbb{P}^x\{\omega\in B_{k_0} : \rho_{k_0}^x(\omega)=\infty\})^{1-\gamma}$ and $C_k = c\tilde{c}(\alpha_1(\frac{1}{k}))^\gamma/(\alpha_2(k))^\gamma$. Then, we have $h(t)-h(\nu)\leqslant -C_k\int_\nu^t(h(s))^\gamma\mathrm{d}s$, where $h(t) = \mathbb{E}^x(I_{B_k}I_{\{t\leqslant\rho_k^x\}} V(x(t),r(t)))$. According to the proof of [30, Lemma 3.1], if $h(t)$ is continuous, then there exists a sequence $\{T_k : T_k = t_0 + (\mathbb{E}(I_{B_k}V(x(t_0),r(t_0))))^{1-\gamma}/C_k/(1-\gamma)\}$ such that $0 = h(T_k) = \mathbb{E}^x(I_{B_k}I_{\{T_k\leqslant\rho_k^x\}}V(x(T_k), r(T_k)))$. Since $V(x(T_k,\omega), r^x(T_k,\omega),\omega)\geqslant\alpha_1(1/k)\ \forall\omega\in\{\omega\in B_k : T_k\leqslant\rho_k^x(\omega)\}$, we have

$$\mathbb{P}^x\{\omega\in B_k : T_k\leqslant\rho_k^x(\omega)\} = 0. \tag{64}$$

Note that the definitions of $T_k$ and $C_k$ imply that $T_k\uparrow\infty$ as $k\to\infty$. Thus, taking $k\to\infty$ in (64) and applying the dominated convergence theorem, we obtain (63). Hence, the proof is complete, if the continuity of $h(t)$ is proven.

Since $x(t_0+\frac{\iota}{n}) + (t-t_0-\frac{\iota}{n})f(t_0+\frac{\iota}{n}, x(t_0+\frac{\iota}{n}), r^x(t_0+\frac{\iota}{n})) + g(t_0+\frac{\iota}{n}, x(t_0+\frac{\iota}{n}), r^x(t_0+\frac{\iota}{n}))(w^x(t) - w^x(t_0+\frac{\iota}{n}))$ obeys a multivariate normal distribution law for numbers $x(t_0+\frac{\iota}{n})\in\mathbb{R}^d$ and $r^x(t_0+\frac{\iota}{n})\in\mathbb{S}$, referring to the proof of [18, Theorem 1], we obtain $\mathbb{P}^x\{|x(t)|=k\}=0$ and $\mathbb{P}^x\{|x(t)|=1/k\}=0\ \forall t\geqslant t_0$ and $\forall k\geqslant k_0$, and then we can prove the continuity of $h(t)$.

**Corollary 1.** If conditions in Lemma 2 hold and $K(V(x,i))\big(cK(V(x,i)) + \mathscr{L}V(t,x,i)\big)\leqslant\frac{1}{2}K'(V(x,i)) \cdot|\frac{\partial V(x,i)}{\partial x}g(t,x,i)|^2 + \sum_{j=1}^N q_{ij}K^2(V(x,i))\big(V(x,j) - \int_{t_0}^{V(x,j)}\frac{1}{K(s)}\mathrm{d}s\big)$, $\forall(t,x,i)\in\mathbb{R}_{t_0}\times(\mathbb{R}^d\setminus\{0\})\times\mathbb{S}$, where $K:\mathbb{R}_+\to\mathbb{R}_+$ is a smooth function with $K(s)>0$ for $s>0$ and $\int_{t_0}^t\frac{\mathrm{d}s}{K^2(s)}<\infty\ \forall t>t_0$, and $K'(V) = \partial K(V)/\partial V$. Then, there exists a global weak solution to system (1) and the trivial weak solution is globally finite-time stable in probability with $t_0 = 0$.

*Proof.* Let $F(x(t),r(t))=\int_{t_0}^{V(x(t),r(t))}\frac{1}{K(s)}\mathrm{d}s$. Then we have $\mathscr{L}F(t,x,i)\leqslant -c$. Furthermore, using the similar proof of the case of $\gamma=0$ in Theorem 3, we can obtain the results.

**Remark 5.** Corollary 1 extends the finite-time stability result of SNS in [43] to SNSwMS. Different from SNS, Theorem 3 is not a special case of Corollary 1 with $K(s)=s^\gamma$ and $\gamma\in[0,1)$. Since $V(t_0,x(t_0))\geqslant t_0$ may not hold, Corollary 1 may not hold for $t_0>0$.

**Remark 6.** Theorem 3 extends the theorem about the finite-time stability of SNS in [18] to SNSwMS (1). Different from the existing theorems about the finite-time stability of SNSwMS (e.g., [21, Theorem 2.1], [22, Theorems 1 and 2], [23, Theorems 12 and 16], [24, Theorem 1], [25, Theorems 2.1 and 2.2]), there is no assumption about the existence of a pathwise unique strong solution or other restrictive condition on coefficients, besides continuity.

## 3 Finite-time stabilization

### 3.1 Problem formulation

We consider the following system:

$$\begin{cases}\mathrm{d}x_k(t)=\big(x_{k+1}^{p_k}(t)+f_k(t,x(t),r(t))\big)\mathrm{d}t+g_k(t,x(t),r(t))\mathrm{d}w(t),\ t\geqslant t_0\geqslant 0,\ k=1,\ldots,d-1,\\ \mathrm{d}x_d(t)=\big(u^{p_d}+f_d(t,x(t),r(t))\big)\mathrm{d}t+g_d(t,x(t),r(t))\mathrm{d}w(t),\end{cases}\tag{65}$$

where $x=(x_1,\ldots,x_d)^{\mathrm{T}}\in\mathbb{R}^d$ and $u\in\mathbb{R}$ respectively represent the state and the control input, $w(t)$ is a standard $m$-dimensional Brownian motion and $r(t)$ is a right-continuous homogeneous Markov chain with stable and conservative $q$-matrix $Q$ on some complete probability space, $x(t_0)$ is a random variable whose distribution is of compact support, both $f_k:\mathbb{R}_{t_0}\times\mathbb{R}^d\times\mathbb{S}\to\mathbb{R}$ and $g_k:\mathbb{R}_{t_0}\times\mathbb{R}^d\times\mathbb{S}\to\mathbb{R}^{1\times m}$ are continuous in $(t,x)\in\mathbb{R}_{t_0}\times\mathbb{R}^d$ and Borel measurable, and $p_k\geqslant 1$ is a ratio of odd integers with $p_d=1$.

We consider the following assumption on system (65), whose feasibility and reasonability have been verified in [13, 15, 18, 44].

**Assumption 1.** For each $i\in\mathbb{S}$ and $k=1,\ldots,d$, there exist known nonnegative smooth functions $\varphi_{k,i}$ and $\psi_{k,i}$ such that $\forall(t,x)\in\mathbb{R}_{t_0}\times\mathbb{R}^d$,

$$|f_k(t,x,i)|\leqslant(|x_1|+|x_2|+\cdots+|x_k|)\varphi_{k,i}(\bar{x}_k),\quad |g_k(t,x,i)|\leqslant(|x_1|+|x_2|+\cdots+|x_k|)\psi_{k,i}(\bar{x}_k),\tag{66}$$

where $\bar{x}_k=(x_1,\ldots,x_k)^{\mathrm{T}}$.

In our conference paper [44], for a special case ($p_k=1$, $k=1,\ldots,d$) of system (65), the outline of the controller design is presented, without the proof of the finite-time stability of the closed-loop system. Here, we consider the more general system with detailed analysis.

### 3.2 Finite-time controller design

Because of the especial change rate of the finite-time Lyapunov function, the existing methods addressing the interconnected term in the "derivative" of the Lyapunov function (e.g., [1, 4]) are inapplicable. Constructing a finite-time controller depending on Markov switching $r(t)$ is a challenging task. Thus, in this work, we construct a common finite-time controller. First, we introduce a common coordinate transformation

$$x_1^*=0,\ \ \xi_1=x_1^{\frac{1}{r_1}}-x_1^{*\frac{1}{r_1}},\ \ x_k^{*p_{k-1}}=-\xi_{k-1}^{r_{k-1}+\alpha}\chi_{k-1}(\bar{x}_{k-1}),\ \ \xi_k=x_k^{\frac{1}{r_k}}-x_k^{*\frac{1}{r_k}},\tag{67}$$

where $k=1,\ldots,d$, $\chi_k>0$ are smooth functions to be designed later, $r_1\in(0,1/2)$ is a ratio of odd integers, $\alpha\in(-r_1/(1+\sum_{i=1}^{d-1}p_1\cdots p_i),0)$ is a rational number with even numerator and odd denominator, and $r_k=(r_{k-1}+\alpha)/p_{k-1}>0$, $k=2,\ldots,d$.

**Step 1.** Let $V_1(x_1)=\frac{r_1}{4}x_1^{4/r_1}\leqslant 2\xi_1^4$. By (65) and (66), we have $\mathscr{L}V_1(t,x_1,i)=\xi_1^{4-r_1}\big(x_2^{p_1}+f_1(t,x,i)\big)+\frac{4-r_1}{2r_1}\xi_1^{4-2r_1}|g_1(t,x,i)|^2\leqslant\xi_1^{4-r_1}x_2^{*p_1}+\xi_1^{4-r_1}(x_2^{p_1}-x_2^{*p_1})+\xi_1^{4+\alpha}\rho_1(x_1)$, where $\rho_1(x_1)\geqslant\max\{\xi_1^{-\alpha}\varphi_{1,i}(x_1)+\frac{4-r_1}{2r_1}\xi_1^{-\alpha}\psi_{1,i}^2(x_1):i\in\mathbb{S}\}$ is a smooth function. Choose a virtual controller $x_2^{*p_1}=-\xi_1^{r_1+\alpha}(l_1+\rho_1(x_1)):\triangleq-\xi_1^{r_1+\alpha}\chi_1(x_1)$, where $l_1$ is any given positive constant. Then, taking $L_1=l_1$, we obtain that $\mathscr{L}V_1(t,x_1,i)\leqslant -L_1\xi_1^{4+\alpha}+\xi_1^{4-r_1}(x_2^{p_1}-x_2^{*p_1})$.

**Inductive step.** Suppose that at step $k-1$, there exist smooth functions $V_{k-1}(\bar{x}_{k-1})$ and $\chi_{k-1}(\bar{x}_{k-1})$ such that

$$\mathscr{L}V_{k-1}(t,\bar{x}_{k-1},i) \leqslant -L_{k-1}(\xi_1^{4+\alpha}+\cdots+\xi_{k-1}^{4+\alpha})+\xi_{k-1}^{4-r_{k-1}}(x_k^{p_{k-1}}-x_k^{*p_{k-1}}), \tag{68}$$

where $L_{k-1}$ is a positive constant, $x_k^{*p_{k-1}} = -\xi_{k-1}^{r_{k-1}+\alpha}\chi_{k-1}(\bar{x}_{k-1})$. By induction, we claim that Eq. (68) still holds at step $k$. To prove this claim, define $V_k(\bar{x}_k) = V_{k-1}(\bar{x}_{k-1})+W_k(\bar{x}_k)$ with $W_k(\bar{x}_k) \triangleq \int_{x_k^*}^{x_k}(s^{\frac{1}{r_k}} - x_k^{*\frac{1}{r_k}})^{4-r_k}\mathrm{d}s$. Then, we obtain that

$$\begin{aligned}\mathscr{L}V_k(t,\bar{x}_k,i) \leqslant & -L_{k-1}\left(\sum_{l=1}^{k-1}\xi_l^{4+\alpha}\right)+\frac{\partial W_k}{\partial x_k}x_{k+1}^{p_k}+\xi_{k-1}^{4-r_{k-1}}(x_k^{p_{k-1}}-x_k^{*p_{k-1}})+\frac{\partial W_k}{\partial x_k}f_k(t,x,i)\\ &+\sum_{l=1}^{k-1}\frac{\partial W_k}{\partial x_l}\left(x_{l+1}^{p_l}+f_l(t,x,i)\right)+\frac{1}{2}\sum_{l=1}^{k}\sum_{\sigma=1}^{k}\frac{\partial^2 W_k}{\partial x_l\partial x_\sigma}g_l(t,x,i)g_\sigma^{\mathrm{T}}(t,x,i),\end{aligned} \tag{69}$$

where $\frac{\partial W_k}{\partial x_k} = \xi_k^{4-r_k}$, $\frac{\partial^2 W_k}{\partial x_k^2} = \frac{4-r_k}{r_k}x_k^{1/r_k-1}\xi_k^{3-r_k}$, $\frac{\partial^2 W_k}{\partial x_l\partial x_k} = (r_k-4)\frac{\partial x_k^{*1/r_k}}{\partial x_l}\xi_k^{3-r_k}$, $\frac{\partial^2 W_k}{\partial x_l\partial x_\sigma} = \frac{\partial^2 W_k}{\partial x_\sigma\partial x_l} = (r_k-4)\frac{\partial^2 x_k^{*1/r_k}}{\partial x_l\partial x_\sigma}\int_{x_k^*}^{x_k}(s^{\frac{1}{r_k}}-x_k^{*\frac{1}{r_k}})^{3-r_k}\mathrm{d}s+(4-r_k)(3-r_k)\frac{\partial x_k^{*1/r_k}}{\partial x_l}\cdot\frac{\partial x_k^{*1/r_k}}{\partial x_\sigma}\int_{x_k^*}^{x_k}(s^{\frac{1}{r_k}}-x_k^{*\frac{1}{r_k}})^{2-r_k}\mathrm{d}s$ and $\frac{\partial W_k}{\partial x_l} = (r_k-4)\frac{\partial x_k^{*1/r_k}}{\partial x_l}\int_{x_k^*}^{x_k}(s^{\frac{1}{r_k}}-x_k^{*\frac{1}{r_k}})^{3-r_k}\mathrm{d}s$ for $l,\sigma = 1,2,\ldots,k-1$. Now, we estimate the terms on the right-side of (69). It follows from Lemmas B2 and B3 in Appendix B that for any given positive constant $\delta_{k,1} < L_{k-1}$, there exists a positive constant $\bar{l}_k$ such that

$$\left|\xi_{k-1}^{4-r_{k-1}}(x_k^{p_{k-1}}-x_k^{*p_{k-1}})\right| \leqslant 2^{1-p_{k-1}r_k}|\xi_{k-1}|^{4-r_{k-1}}|\xi_k|^{r_{k-1}+\alpha} \leqslant \delta_{k,1}\xi_{k-1}^{4+\alpha}+\bar{l}_k\xi_k^{4+\alpha}. \tag{70}$$

Submitting (70) and (C1)–(C3) in Appendix C into (69), we obtain that $\mathscr{L}V_k(t,\bar{x}_k,i) \leqslant -(L_{k-1}-\delta_k)(|\xi_1|^{4+\alpha}+\cdots+|\xi_{k-1}|^{4+\alpha})+\xi_k^{4-r_k}x_{k+1}^{*p_k}+\xi_k^{4-r_k}(x_{k+1}^{p_k}-x_{k+1}^{*p_k})+\tilde{\rho}_k(\bar{x}_k)|\xi_k|^{4+\alpha}$, where $\delta_k = \delta_{k,1}+\delta_{k,2}+\delta_{k,3}+\delta_{k,4} < L_{k-1}$ and $\tilde{\rho}_k = \bar{l}_k+\tilde{\rho}_{k,1}+\tilde{\rho}_{k,2}+\tilde{\rho}_{k,3}$. Take the virtual controller $x_{k+1}^{*p_k} = -\xi_k^{r_k+\alpha}(l_k+\tilde{\rho}_k) :\triangleq -\xi_k^{r_k+\alpha}\chi_k(\bar{x}_k)$, where $l_k$ is any given constant. Then, $\mathscr{L}V_k(t,\bar{x}_k,i) \leqslant -L_k(|\xi_1|^{4+\alpha}+\cdots+|\xi_k|^{4+\alpha})+\xi_k^{4-r_k}(x_{k+1}^{p_k}-x_{k+1}^{*p_k})$, where $L_k = \min\{L_{k-1}-\delta_k, l_k\} > 0$. This completes the proof of the inductive step.

**Step $d$.** By induction, we can construct a Lyapunov function

$$V_d(x) = \sum_{k=1}^{d}W_k = \sum_{k=1}^{d}\int_{x_k^*}^{x_k}(s^{1/r_k}-x_k^{*1/r_k})^{4-r_k}\mathrm{d}s, \tag{71}$$

and design a controller as

$$u = -\xi_d^{r_d+\alpha}\chi_d(x), \tag{72}$$

such that

$$\mathcal{L}V_d(t,x,i) \leqslant -L_d(\xi_1^{4+\alpha}+\cdots+\xi_d^{4+\alpha}), \tag{73}$$

where $L_d > 0$ is a constant.

### 3.3 Finite-time stability analysis

For the closed-loop system composed of (65) and (72), we have the following result.

**Theorem 4.** If Assumption 1 holds, then a closed-loop system composed of (65) and (72) has a global weak solution $(x(t), w^x(t), r^x(t), \Omega^x, \mathscr{F}^x, \{\mathscr{F}_t^x\}_{t\geqslant 0}, \mathbb{P}^x)$ and its trivial weak solution is globally finite-time stable in probability.

*Proof.* First, we prove the positive definiteness and radial unboundedness of $V_d$ defined in (71) by induction. It follows from $V_1 = \frac{r_1}{4}x_1^{\frac{4}{r_1}} \leqslant 2\xi_1^4$ that $V_1$ is positive definite and radially unbounded.

Inductive step. Suppose that $V_{k-1}$ is positive definite and radially unbounded. It follows from Lemma B2 in Appendix B that

$$V_{k-1}(\bar{x}_{k-1}) \leqslant 2\big(\xi_1^4+\xi_2^4+\cdots+\xi_{k-1}^4\big),\ k = 2,\ldots,d, \tag{74}$$

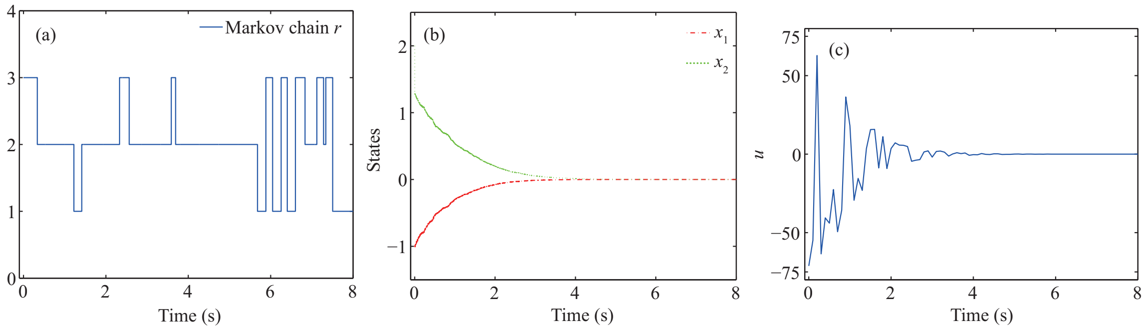


**Figure 1** (Color online) Trajectory of Markov chain (a), trajectories for states (b) and the control input (c) for the closed-loop system (76) and (77).

$$0 \leqslant V_{k-1} + \frac{r_k}{4} 2^{(4-r_k)(r_k-1)/r_k} |x_k - x_k^*|^{4/r_k} \leqslant V_k = V_{k-1} + W_k \leqslant 2(\xi_1^4 + \cdots + \xi_k^4). \tag{75}$$

Since Eq. (67) implies that $|\bar{x}_k| = 0 \Leftrightarrow \xi_1 = \cdots = \xi_k = 0$, and Eq. (74) and the positive definiteness of $V_{k-1}$ imply that $V_{k-1}(\bar{x}_{k-1}) = 0 \Leftrightarrow |\bar{x}_{k-1}| = 0$. Then, it follows from (75) that $V_k$ is positive definite.

When $|\bar{x}_k| \to \infty$, we discuss two cases. (i) If $|\bar{x}_{k-1}| \to \infty$, then the radial unboundedness of $V_{k-1}$ implies that $V_{k-1} \to \infty$, which together with Eq. (75) leads to $V_k \to \infty$. (ii) If $|\bar{x}_{k-1}|$ does not tend to infinity, then $|x_k| \to \infty$, which together with (75) implies that $V_k \to \infty$. Hence, we obtain that $V_k \to \infty$, as $|\bar{x}_k| \to \infty$.

Based on the above analysis, we have that the constructed Lyapunov function $V_d$ in (71) is positive definite and radially unbounded. According to [45, Lemma 4.3], there exist two $\mathcal{K}_\infty$ class functions $\alpha_1$ and $\alpha_2$ such that $\alpha_1(|x|) \leqslant V_d(x) \leqslant \alpha_1(|x|)$. Besides, it follows from Lemma B1 in Appendix B, (73) and (75) that $\mathscr{L} V_d \leqslant -\frac{L_d}{2} V_d^{\frac{4+\alpha}{4}}$. By Theorem 3, we obtain the results.

## 4 Numerical simulation

In this section, we give a numerical example to illustrate the efficiency of the proposed finite-time control method in Section 3. Consider

$$\mathrm{d}x_1(t) = (x_2^{p_1}(t) + f_1(x(t), r(t)))\mathrm{d}t + \frac{\sin(x_1(t))}{10+t}\mathrm{d}w(t), \quad \mathrm{d}x_2(t) = u\mathrm{d}t + \frac{\sin(x_1(t))\sin(x_2(t))}{2r(t)}\mathrm{d}w(t), \tag{76}$$

where $p_1 = 9/7$, $w(t)$ is a standard Brownian motion, $r(t)$ is a right-continuous homogeneous Markov chain with stable and conservative $q$-matrix $Q$ on some complete probability space, the state space of $r(t)$ is $\mathbb{S} = \{1, 2, 3\}$, $f_1(x, 1) = \sin(x_1)/5$, $f_1(x, 2) = \sin(x_1)\sin(x_2)/5$, and $f_1(x, 3) = x_1/5$.

Choose $r_1 = \frac{23}{49} \in (0, \frac{1}{2})$ being a ratio of odd integers, $\alpha = -\frac{2}{49} \in (-\frac{r_1}{1+p_1}, 0)$ being a rational number with even numerator and odd denominator. Then we have $r_2 = \frac{r_1+\alpha}{p_1} = \frac{1}{3}$. According to Subsection 3.2, we construct a Lyapunov function $V_2(x) = \frac{r_1}{4} x_1^{\frac{4}{r_1}} + \int_{x_2^*}^{x_2} (s^{\frac{1}{r_2}} - x_2^{*\frac{1}{r_2}})^{4-r_2} \mathrm{d}s$ and design a control law

$$\begin{aligned} u = -\xi_2^{r_2+\alpha}\Big( & 2.9042 + 11\bar{\psi}_2^2\xi_2^{-\alpha} + 1.3444(7.3333\hat{\rho}_{2,1})^{\frac{97}{84}} + 3.8342(9.7778\hat{\rho}_{2,1}^2\bar{\psi}_1^2\xi_1^{-\alpha})^{\frac{97}{49}} + 14.6208(\bar{\psi}_2^2\xi_1^{-\alpha})^{\frac{291}{245}} \\ & + 2.2719\big((7.3333\hat{\rho}_{2,1}\chi_1)^{\frac{194}{147}} + (7.3333\hat{\rho}_{2,1}\bar{\varphi}_1\xi_1^{-\alpha})^{\frac{194}{147}} + (3.6667\hat{\beta}_{2,1,1}\bar{\psi}_1^2\xi_1^{-\alpha})^{\frac{194}{147}}\big) \\ & + 2.2719(3.6667\hat{\rho}_{2,1}\bar{\psi}_1\bar{\psi}_2\xi_1^{-\alpha})^{\frac{194}{147}} + 4.1122(3.6667\hat{\rho}_{2,1}\bar{\psi}_1\bar{\psi}_2\xi_1^{-\alpha})^{\frac{291}{196}} + 124.0497(\bar{\psi}_2^2\chi_1^{\frac{14}{9}}\xi_1^{-\alpha})^{\frac{291}{196}}\Big), \end{aligned} \tag{77}$$

where $\chi_1(x_1) = 0.2376 x_1^{\frac{2}{23}} + \frac{45}{32}$, $\hat{\rho}_{2,1} = 2.1304\chi_1^{\frac{7}{3}} + 0.0483\chi_1^{\frac{4}{3}} x_1^{\frac{2}{23}}$, $\xi_1 = x_1^{\frac{1}{r_1}}$, $x_2^{*\frac{1}{r_2}} = -\xi_1\chi_1^{\frac{7}{3}}$, $\hat{\beta}_{2,1,1} = 2.4083\chi_1^{\frac{7}{3}} + 0.0013\chi_1^{\frac{1}{3}} x_1^{\frac{4}{23}} + 0.1617\chi_1^{\frac{4}{3}} x_1^{\frac{2}{23}}$, $\xi_2 = x_2^{\frac{1}{r_2}} - x_2^{*\frac{1}{r_2}}$, $\bar{\varphi}_1 = \bar{\varphi}_{1,i} = 0.2$, $\bar{\psi}_1 = \bar{\psi}_{1,i} = 0.1$, $\bar{\psi}_2 = \bar{\psi}_{2,i} = 0.5 x_1^{\frac{20}{69}}$ for $i = 1, 2, 3$. Then, we have $\mathscr{L} V_2 \leqslant -\frac{1}{128}(\xi_1^{4+\alpha} + \xi_2^{4+\alpha})$. Figures 1(a)–(c) provide the simulation results with initial values $r(0) = 3$, $x_1(0) = -1$, $x_2(0) = 2$, and $Q = (q_{ij})_{3\times 3}$ given by $q_{11} = -3$, $q_{12} = 1$, $q_{13} = 2$, $q_{21} = 1$, $q_{22} = -2$, $q_{23} = 1$, $q_{31} = 2$, $q_{32} = 2$, and $q_{33} = -4$. Figures 1(b) and (c) show that the states and the control input of the closed-loop system (76) and (77) converge to the origin in finite time.

## 5 Conclusion

We have established a finite-time control framework in the sense of weak solutions for the time-varying SNSwMS. The finite-time stabilization for a class of stochastic high-order nonlinear systems with Markovian switching was investigated. The finite-time Lyapunov function was constructed by adding a power integrator technique, and then a finite-time controller was systematically constructed. We have developed theorems on the existence of global weak solutions and established finite-time stability in probability theorem, by which the existence of global weak solutions and the finite-time stability in probability of the closed-loop system were rigorously analyzed.

**Acknowledgements** This work was supported by National Natural Science Foundation of China (Grant Nos. 61673284, 61807017).

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## Appendix A Proof of Lemma 1 and propositions used to prove Theorem 1

*Proof of Lemma 1.* Choose two positive integers $L$ and $\hat{L}$ such that $t_0 + L\Delta \leqslant t_0 + \hat{L}\Delta \leqslant T$. Using (4) and the method[1)] of estimating the 2$p$th moment, we have that $\sup_{0\leqslant L\leqslant \hat{L}} \mathbb{E}|X_L^{(n)}|^{2p} \leqslant 6^{p-1}\big(\mathbb{E}|X_0^{(n)}|^{2p} + I_1 + I_2 + I_3 + I_4 + I_5\big)$, where

$$\begin{aligned}
I_1 &= \mathbb{E}\sup_{0\leqslant L\leqslant \hat{L}}\left(\sum_{k=0}^{L-1}\left|\int_{t_k^{(n)}}^{t_{k+1}^{(n)}} f(s, X_k^{(n)}, r(s))\mathrm{d}s\right|^2\right)^p,\\
I_2 &= \mathbb{E}\sup_{0\leqslant L\leqslant \hat{L}}\left(\sum_{k=0}^{L-1} 2\left\langle X_k^{(n)}, \int_{t_k^{(n)}}^{t_{k+1}^{(n)}} f(s, X_k^{(n)}, r(s))\mathrm{d}s\right\rangle\right)^p,\\
I_3 &= \mathbb{E}\sup_{0\leqslant L\leqslant \hat{L}}\left(\sum_{k=0}^{L-1}\left|\int_{t_k^{(n)}}^{t_{k+1}^{(n)}} g(s, X_k^{(n)}, r(s))\mathrm{d}w(s)\right|^2\right)^p,\\
I_4 &= \mathbb{E}\sup_{0\leqslant L\leqslant \hat{L}}\left(\sum_{k=0}^{L-1} 2\left\langle X_k^{(n)}, \int_{t_k^{(n)}}^{t_{k+1}^{(n)}} g(s, X_k^{(n)}, r(s))\mathrm{d}w(s)\right\rangle\right)^p,\\
I_5 &= \mathbb{E}\sup_{0\leqslant L\leqslant \hat{L}}\left(\sum_{k=0}^{L-1} 2\left\langle \int_{t_k^{(n)}}^{t_{k+1}^{(n)}} f(s, X_k^{(n)}, r(s))\mathrm{d}s, \int_{t_k^{(n)}}^{t_{k+1}^{(n)}} g(s, X_k^{(n)}, r(s))\mathrm{d}w(s)\right\rangle\right)^p.
\end{aligned}$$

It follows from (3) and Hölder inequality that

$$\begin{aligned}
I_1 &\leqslant \hat{L}^{p-1}\Delta^{2p-1}\sum_{k=0}^{\hat{L}-1}\mathbb{E}\int_{t_k^{(n)}}^{t_{k+1}^{(n)}}\big|f(s, X_k^{(n)}, r(s))\big|^{2p}\mathrm{d}s \leqslant (2T_0)^{p-1}\int_{t_0}^{\mathrm{T}} c_0^p(s)\mathrm{d}s + \sum_{k=0}^{\hat{L}-1}\left((2T_0)^{p-1}\int_{t_k^{(n)}}^{t_{k+1}^{(n)}} c_0^p(s)\mathrm{d}s\right)\mathbb{E}|X_k^{(n)}|^{2p},\\
I_2 &\leqslant \hat{L}^{p-1}\sum_{k=0}^{\hat{L}-1}\mathbb{E}\left(\int_{t_k^{(n)}}^{t_{k+1}^{(n)}}\big|X_k^{(n)}\big|^2 + \big|f(s, X_k^{(n)}, r(s))\big|^2\mathrm{d}s\right)^p\\
&\leqslant (4T_0)^{p-1}\int_{t_0}^{\mathrm{T}} c_0^p(s)\mathrm{d}s + \sum_{k=0}^{\hat{L}-1}\left(\int_{t_k^{(n)}}^{t_{k+1}^{(n)}}(2T_0)^{p-1}\big(1 + 2^{p-1}c_0^p(s)\big)\mathrm{d}s\right)\mathbb{E}|X_k^{(n)}|^{2p},
\end{aligned}$$

where $T_0 = T - t_0$. By the proof of [3, Theorem 2.11] and linear growth condition (3), we have that

$$\begin{aligned}
I_3 &\leqslant \big(p(2p-1)\big)^p\hat{L}^{p-1}\Delta^{p-1}\sum_{k=0}^{\hat{L}-1}\mathbb{E}\int_{t_k^{(n)}}^{t_{k+1}^{(n)}}\big|g(s, X_k^{(n)}, r(s))\big|^{2p}\mathrm{d}s\\
&\leqslant (2T_0)^{p-1}\big(p(2p-1)\big)^p\left(\int_{t_0}^{\mathrm{T}} c_0^p(s)\mathrm{d}s + \sum_{k=0}^{\hat{L}-1}\left(\int_{t_k^{(n)}}^{t_{k+1}^{(n)}} c_0^p(s)\mathrm{d}s\right)\mathbb{E}|X_k^{(n)}|^{2p}\right).
\end{aligned}$$

1) Zhao G H, Liu M Z. Numerical methods for nonlinear stochastic delay differential equations with jumps. Appl Math Computation, 2014, 233: 222–231.

It follows from the proof of Burkholder-Davis-Gundy inequality (see [3]) that

$$I_4 \leqslant c_p \mathbb{E}\left(\sum_{k=0}^{\hat{L}-1}\int_{t_k^{(n)}}^{t_{k+1}^{(n)}} |X_k^{(n)}|^2 |g(s, X_k^{(n)}, r(s))|^2 \mathrm{d}s\right)^{\frac{p}{2}}$$

$$\leqslant 2(2T_0)^{\frac{p}{2}-1} c_p \int_{t_0}^{\mathrm{T}} c_0^{\frac{p}{2}}(s)\mathrm{d}s + \sum_{k=0}^{\hat{L}-1}\left(\int_{t_k^{(n)}}^{t_{k+1}^{(n)}} 2(2T_0)^{\frac{p}{2}-1} c_p c_0^{\frac{p}{2}}(s)\mathrm{d}s\right)\mathbb{E}|X_k^{(n)}|^{2p},$$

where $c_p = 2^p(p^{p+1}/(2(p-1)^{p-1}))^{p/2}$. Note that $I_5 \leqslant 2^{p-1}(I_1 + I_3)$. Thus, we have that

$$I_5 \leqslant (4T_0)^{p-1}\big(1 + (p(2p-1))^p\big)\left(\int_{t_0}^{T} c_0^p(s)\mathrm{d}s + \sum_{k=0}^{\hat{L}-1}\left(\int_{t_k^{(n)}}^{t_{k+1}^{(n)}} c_0^p(s)\mathrm{d}s\right)\mathbb{E}|X_k^{(n)}|^{2p}\right).$$

It follows from the above analysis that

$$\begin{aligned}\mathbb{E}\sup_{0\leqslant L\leqslant \hat{L}} |X_L^{(n)}|^{2p} \leqslant{}& 6^{p-1}\mathbb{E}|X_0^{(n)}|^{2p} + C_{00}\int_{t_0}^{\mathrm{T}} c_0^p(s)\mathrm{d}s + C_{01}\int_{t_0}^{\mathrm{T}} c_0^{\frac{p}{2}}(s)\mathrm{d}s\\ &+\sum_{k=0}^{\hat{L}-1}\left(\int_{t_k^{(n)}}^{t_{k+1}^{(n)}} C_{00}c_0^p(s) + C_{01}c_0^{\frac{p}{2}}(s) + (12T_0)^{p-1}\mathrm{d}s\right)\mathbb{E}\sup_{0\leqslant j\leqslant k}|X_j^{(n)}|^{2p},\end{aligned}\tag{A1}$$

where $C_{00}(p,t_0,T) = (12T_0)^{p-1} + (12T_0)^{p-1}\big(p(2p-1)\big)^p + (24T_0)^{p-1}\big(2 + (p(2p-1))^p\big)$ and $C_{01}(p,t_0,T) = 2(12T_0)^{\frac{p}{2}-1}6^{\frac{p}{2}}c_p$. According to the discrete Gronwall's inequality, we obtain

$$\begin{aligned}\mathbb{E}\sup_{0\leqslant L\leqslant \hat{L}} |X_L^{(n)}|^{2p} \leqslant{}& \left(6^{p-1}\mathbb{E}|X_0^{(n)}|^{2p} + C_{00}\int_{t_0}^{\mathrm{T}} c_0^p(s)\mathrm{d}s + C_{01}\int_{t_0}^{\mathrm{T}} c_0^{\frac{p}{2}}(s)\mathrm{d}s\right)\\ &\cdot\exp\left\{C_{00}\int_{t_0}^{\mathrm{T}} c_0^p(s)\mathrm{d}s + C_{01}\int_{t_0}^{\mathrm{T}} c_0^{\frac{p}{2}}(s)\mathrm{d}s + 12^{p-1}T_0^p\right\} := C_0(p,t_0,T,\mu).\end{aligned}\tag{A2}$$

Then, it follows from (5), Hölder inequality, and the proof of [3, Theorem 2.12] that

$$\begin{aligned}\mathbb{E}\left(\sup_{t_0\leqslant t\leqslant T}|X_n(t)|^{2p}\right) \leqslant{}& 3^{2p-1}\left(\mathbb{E}|X_n(t_0)|^{2p} + T_0^{2p-1}\mathbb{E}\int_{t_0}^{\mathrm{T}}\big|f(s,Y_n(s),r(s))\big|^{2p}\mathrm{d}s\right.\\ &\left.+T_0^{p-1}\left(\frac{(2p)^3}{2(2p-1)}\right)^p\cdot\mathbb{E}\int_{t_0}^{\mathrm{T}}\big|g(s,Y_n(s),r(s))\big|^{2p}\mathrm{d}s\right).\end{aligned}$$

This together with linear growth condition (3) and (A2) leads to (6).

It follows from (5), $(a+b)^{2q} \leqslant 2^{2q-1}(a^{2q}+b^{2q})$ $(a,b\in\mathbb{R},\ q>1)$, Hölder inequality, and the proof of [3, Theorem 2.11] that $\forall t,s\in[t_0,T], \mathbb{E}|X_n(t)-X_n(s)|^{2p}\leqslant 2^{2p-1}|t-s|^{2p-1}\mathbb{E}\int_s^t|f(\nu,Y_n(\nu),r(\nu))|^{2p}\mathrm{d}\nu + 2^{2p-1}(p(2p-1))^p|t-s|^{p-1}\mathbb{E}\int_s^t|g(\nu,Y_n(\nu),r(\nu))|^{2p}\mathrm{d}\nu$. Then, by (3) and (6), we can obtain (7).

**Proposition A1.** Set $\tilde{\mathscr{F}}_t^{n_l} = \sigma(\tilde{X}_{n_l}(s), \tilde{w}^{(n_l)}(s), \tilde{r}^{(n_l)}(s) : 0\leqslant s\leqslant t)$. Then, we have that $\tilde{r}^{(n_l)}(t)$ and $r^x(t)$ in (9) are Markov chains with the given initial distribution and transition function for $t\geqslant t_0$, $\tilde{w}^{(n_l)}(t)$ is an $\{\tilde{\mathscr{F}}_t^{n_l}\}$-adapted Brownian motion, and $w^x(t)$ in (9) is an $\{\mathscr{F}_t^x\}$-adapted Brownian motion with $w^x(0) = 0$ a.s.

*Proof.* Since $r(t)$ is a homogeneous Markov chain with $\mathbb{P}\{r(t_0) = i\} = p_i$ and $\mathbb{P}\{r(t+s) = j|r(s) = i\} = P_{ij}(t)$, $\forall i,j\in\mathbb{S}$, $t\geqslant t_0$, $s\geqslant t_0$, we have for any finite set $t_0\leqslant s_1 < s_2 < \cdots < s_k < s_{k+1}$ and corresponding set $i_1,\ldots,i_{k-1},i,j$ of states in $\mathbb{S}$,

$$\mathbb{P}\{r(s_{k+1}) = j|r(s_k) = i, r(s_{k-1}) = i_{k-1},\ldots,r(s_1) = i_1\} = \mathbb{P}\{r(s_{k+1}) = j|r(s_k) = i\}.$$

Then, by (10), we have that $\forall t\geqslant t_0$, $\forall s\geqslant t_0$,

$$\mathbb{P}^x\{\tilde{r}^{(n_l)}(s_{k+1}) = j|\tilde{r}^{(n_l)}(s_k) = i, \tilde{r}^{(n_l)}(s_{k-1}) = i_{k-1},\ldots,\tilde{r}^{(n_l)}(s_1) = i_1\} = \mathbb{P}^x\{\tilde{r}^{(n_l)}(s_{k+1}) = j|\tilde{r}^{(n_l)}(s_k) = i\},\tag{A3}$$

$$\mathbb{P}^x\{\tilde{r}^{(n_l)}(t_0) = i\} = p_i,\quad \mathbb{P}^x\{\tilde{r}^{(n_l)}(t+s) = j|\tilde{r}^{(n_l)}(s) = i\} = P_{ij}(t).\tag{A4}$$

Eq. (A3) implies that $\tilde{r}^{(n_l)}(t)$ is a Markov chain on $\mathbb{R}_{t_0}$, and Eq. (A4) implies that $\tilde{r}^{(n_l)}(t)$ is a homogeneous Markov chain with the given initial distribution and transition function for each $l$. Letting $l\to\infty$ in (A3) and (A4), and using (11), the dominated convergence theorem, and the definition of $\mathscr{F}_t^x$, we can obtain that $r^x(t)$ is an $\{\mathscr{F}_t^x\}$-adapted homogeneous Markov chain with the given initial distribution and transition function.

Since $\{w(t), t\geqslant 0\}$ is a $\sigma(X_{n_l}(s), w(s), r(s) : 0\leqslant s\leqslant t)$ Brownian motion with $w(0) = 0$ a.s., we have that for any $0\leqslant s\leqslant t\leqslant T$ and $0\leqslant t_1\leqslant t_2\leqslant\cdots\leqslant t_n\leqslant s$, $\mathbb{E}\big(\psi(Z^{(n_l)})(w(t)-w(s))\big) = 0$ and $\mathbb{E}\big(\psi(Z^{(n_l)})((w(t)-w(s))(w(t)-w(s))^{\mathrm{T}} - (t-s)I)\big) = 0$, where $Z^{(n_l)} = \{X_{n_l}(t_1),\ldots,X_{n_l}(t_n), w(t_1),\ldots,w(t_n), r(t_1),\ldots,r(t_n)\}$, $\psi(\Xi,\Theta)$ is any given bounded, measurable and continuous (with respect to $\Xi\in\mathbb{R}^{(d+m)n}$) function on $\mathbb{R}^{(d+m)n}\times\underbrace{\mathbb{S}\times\cdots\times\mathbb{S}}_{n\ \text{terms}}$ and $I$ is an $m\times m$ unit matrix. Let $\tilde{Z}^{(n_l)} = \{\tilde{X}_{n_l}(t_1),\ldots,\tilde{X}_{n_l}(t_n),$ $\tilde{w}^{(n_l)}(t_1),\ldots,\tilde{w}^{(n_l)}(t_n), \tilde{r}^{(n_l)}(t_1),\ldots,\tilde{r}^{(n_l)}(t_n)\}$. Then, by (10), we have $\tilde{w}^{(n_l)}(0) = 0$ a.s., $\mathbb{E}^x\big(\psi(\tilde{Z}^{(n_l)})(\tilde{w}^{(n_l)}(t)-\tilde{w}^{(n_l)}(s))\big) = 0$ and $\mathbb{E}^x\big(\psi(\tilde{Z}^{(n_l)})((\tilde{w}^{(n_l)}(t)-\tilde{w}^{(n_l)}(s))(\tilde{w}^{(n_l)}(t)-\tilde{w}^{(n_l)}(s))^{\mathrm{T}} - (t-s)I)\big) = 0$. Then, according to [39], we obtain

$$\mathbb{E}^x\big(\tilde{w}^{(n_l)}(t)\big|\tilde{\mathscr{F}}_s^{n_l}\big) = \tilde{w}^{(n_l)}(s),\quad \mathbb{E}^x\big((\tilde{w}^{(n_l)}(t)-\tilde{w}^{n_l}(s))(\tilde{w}^{(n_l)}(t)-\tilde{w}^{(n_l)}(s))^{\mathrm{T}}\big|\tilde{\mathscr{F}}_s^{n_l}\big) = (t-s)I.\tag{A5}$$

It follows from [32, Theorem 2.6.1] that $\tilde{w}^{(n_l)}(t)$ is an $\{\tilde{\mathscr{F}}_t^{n_l}\}$-adapted Brownian motion. Taking $l$ into (A5), using (11), (12), the continuity (with respect to $\Xi$) of $\psi(\Xi,\Theta)$, and dominated convergence theorem, we can obtain that $w^x(t)$ is an $\{\mathscr{F}_t^x\}$-adapted Brownian motion with $w^x(0) = 0$ a.s.

**Proposition A2.** If conditions in Theorem 1 hold, then $\forall\epsilon_0 > 0$ and $\forall T > t_0$,

$$\lim_{h\to 0}\lim_{l\to\infty}\sup_{|t_1-t_2|<h,\ t_1,t_2\in[t_0,T]}\mathbb{P}^x\{|g_j(t_1,\tilde{Y}_{n_l}(t_1),\tilde{r}^{(n_l)}(t_1)) - g_j(t_2,\tilde{Y}_{n_l}(t_2),\tilde{r}^{(n_l)}(t_2))| > \epsilon_0\} = 0. \tag{A6}$$

To prove Proposition A2, we prepare two lemmas as follows.

**Lemma A1.** If conditions in Theorem 1 hold, then

$$\lim_{h\to 0}\lim_{l\to\infty}\sup_{|t_1-t_2|<h,\ t_1,t_2\in[t_0,T]}\mathbb{P}^x\{|\tilde{Y}_{n_l}(t_1) - \tilde{Y}_{n_l}(t_2)| > \delta\} = 0, \quad \forall T > t_0,\ \delta > 0. \tag{A7}$$

*Proof.* The definition of $Y_{n_l}(t)$ in (5) implies that there exist two times $t_{k_1}^{(n_l)}$ and $t_{k_2}^{(n_l)}$ such that $t_1 \in [t_{k_1}^{(n_l)}, t_{k_1+1}^{(n_l)})$, $t_2 \in [t_{k_2}^{(n_l)}, t_{k_2+1}^{(n_l)})$, $Y_{n_l}(t_1) = X_{n_l}(t_{k_1}^{(n_l)})$ and $Y_{n_l}(t_2) = X_{n_l}(t_{k_2}^{(n_l)})$. Then, by (7), we have $\mathbb{E}|Y_{n_l}(t_2)-Y_{n_l}(t_1)|^{2p} = \mathbb{E}|X_{n_l}(t_{k_2}^{(n_l)})-X_{n_l}(t_{k_1}^{(n_l)})|^{2p} \leqslant C_1(|t_2-t_1|+2^{-n_l})^{p-1}$. Then, by the Chebyshev's inequality, we have $\lim_{h\to 0}\lim_{l\to\infty}\sup_{|t_1-t_2|<h,\ t_1,t_2\in[t_0,T]}\mathbb{P}\{|Y_{n_l}(t_1) - Y_{n_l}(t_2)| > \delta\} = 0$. Furthermore, by (10), we have (A7).

**Lemma A2.** For the right-continuous homogeneous Markov chain $\tilde{r}^{(n_l)}(t) : \mathbb{R}_{t_0} \to \mathbb{S}$ with the standard transition function $P_{ij}(t)$. Then for each $\delta > 0$, each $T > t_0$,

$$\lim_{h\to 0}\sup_{|t_1-t_2|<h,\ t_1,t_2\in[t_0,T]}\mathbb{P}^x\{|\tilde{r}^{(n_l)}(t_1) - \tilde{r}^{(n_l)}(t_2)| > \delta\} = 0. \tag{A8}$$

*Proof.* Without loss of generality, we assume that $t_0 \leqslant t_1 \leqslant t_2 \leqslant T$. Note that $\tilde{r}^{(n_l)}(t)$ is a homogeneous Markov chain. Thus, by the total probability theorem, we have

$$\begin{aligned}\mathbb{P}^x\{|\tilde{r}^{(n_l)}(t_1) - \tilde{r}^{(n_l)}(t_2)| > \delta\} &= \sum_{i=1}^{N}\mathbb{P}^x\{|\tilde{r}^{(n_l)}(t_1) - \tilde{r}^{(n_l)}(t_2)| > \delta \big| \tilde{r}^{(n_l)}(t_1) = i\}\mathbb{P}^x\{\tilde{r}^{(n_l)}(t_1) = i\} \\ &= \sum_{i=1}^{N}\sum_{j\in\mathbb{S},j\notin[i-\delta,i+\delta]} P_{ij}(t_2-t_1)\mathbb{P}^x\{\tilde{r}^{(n_l)}(t_1) = i\}.\end{aligned} \tag{A9}$$

Note that the standard transition function $P_{ij}(t)$ satisfies $\lim_{t\to 0} P_{ij}(t) = 0$ for $i \neq j$. Thus we can obtain (A8).

*Proof of Proposition A2.* Since $\forall\epsilon_0 > 0$, $\{|g_j(t_2,\tilde{Y}_{n_l}(t_2),\tilde{r}^{(n_l)}(t_1)) - g_j(t_2,\tilde{Y}_{n_l}(t_2),\tilde{r}^{(n_l)}(t_2))| > \frac{\epsilon_0}{2}\} \subseteq \{|\tilde{r}^{(n_l)}(t_1) - \tilde{r}^{(n_l)}(t_2)| > \frac{1}{2}\}$, by (A8), we obtain $\lim_{h\to 0}\sup_{|t_1-t_2|<h,\ t_1,t_2\in[t_0,T]}\mathbb{P}^x\{|g_j(t_2,\tilde{Y}_{n_l}(t_2),\tilde{r}^{(n_l)}(t_1)) - g_j(t_2,\tilde{Y}_{n_l}(t_2),\tilde{r}^{(n_l)}(t_2))| > \epsilon_0/2\} = 0$. Additionally, note that $\{|g_j(t_1,\tilde{Y}_{n_l}(t_1),\tilde{r}^{(n_l)}(t_1)) - g_j(t_2,\tilde{Y}_{n_l}(t_2),\tilde{r}^{(n_l)}(t_2))| > \epsilon_0\} \subset (\{|g_j(t_1,\tilde{Y}_{n_l}(t_1),\tilde{r}^{(n_l)}(t_1)) - g_j(t_2,\tilde{Y}_{n_l}(t_2),\tilde{r}^{(n_l)}(t_1))| > \epsilon_0/2\} \cup \{|g_j(t_2,\tilde{Y}_{n_l}(t_2),\tilde{r}^{(n_l)}(t_1)) - g_j(t_2,\tilde{Y}_{n_l}(t_2),\tilde{r}^{(n_l)}(t_2))| > \epsilon_0/2\})$. Thus, Proposition A2 is proven, if

$$\lim_{h\to 0}\lim_{l\to\infty}\sup_{|t_1-t_2|<h,\ t_1,t_2\in[t_0,T]}\mathbb{P}^x\left\{|g_j(t_1,\tilde{Y}_{n_l}(t_1),\tilde{r}^{(n_l)}(t_1)) - g_j(t_2,\tilde{Y}_{n_l}(t_2),\tilde{r}^{(n_l)}(t_1))| > \frac{\epsilon_0}{2}\right\} = 0. \tag{A10}$$

Now, we prove (A10). By (6), (10) and the Chebyshev's inequality, we have that for each $\varepsilon_1 > 0$, there exists a constant $\tilde{M}(t_0,\mu,p,T,\varepsilon_1) > 0$ such that $\mathbb{P}^x\{\sup_{t_0\leqslant t\leqslant T}|\tilde{Y}_{n_l}(t)| > \tilde{M}(t_0,\mu,p,T,\varepsilon_1)\} < \varepsilon_1$. Let $\tau_{n_l\tilde{M}} = \inf\{t \geqslant t_0 : |\tilde{Y}_{n_l}(t)| \geqslant \tilde{M}\}$. Note that the continuity of $g_j(t,x,i)$ implies that $g_j(t,x,i)$ is continuous uniformly with respect to $(t,x)$ on $[t_0,T]\times\{x\in\mathbb{R}^d : |x| \in [-\tilde{M},\tilde{M}]\}$ for each $i \in \mathbb{S}$. Thus, there exists a constant $\sigma_1(\frac{\epsilon_0}{2}) > 0$ such that

$$\begin{aligned}&\left\{|g_j(t_1,\tilde{Y}_{n_l}(t_1),\tilde{r}^{(n_l)}(t_1)) - g_j(t_2,\tilde{Y}_{n_l}(t_2),\tilde{r}^{(n_l)}(t_1))| > \frac{\epsilon_0}{2}\right\} \\ \subseteq &\left(\{\tau_{n_l\tilde{M}} \leqslant T\} \cup \left\{|g_j(t_1\wedge\tau_{n_l\tilde{M}},\tilde{Y}_{n_l}(t_1\wedge\tau_{n_l\tilde{M}}),\tilde{r}^{(n_l)}(t_1\wedge\tau_{n_l\tilde{M}})) - g_j(t_2\wedge\tau_{n_l\tilde{M}},\tilde{Y}_{n_l}(t_2\wedge\tau_{n_l\tilde{M}}),\tilde{r}^{(n_l)}(t_1\wedge\tau_{n_l\tilde{M}}))| > \frac{\epsilon_0}{2}\right\}\right) \\ \subseteq &\left(\{\tau_{n_l\tilde{M}} \leqslant T\} \cup \left\{|\tilde{Y}_{n_l}(t_1\wedge\tau_{n_l\tilde{M}}) - \tilde{Y}_{n_l}(t_2\wedge\tau_{n_l\tilde{M}})| \geqslant \frac{\sqrt{2}}{2}\sigma_1\left(\frac{\epsilon_0}{2}\right)\right\} \cup \left\{|t_1\wedge\tau_{n_l\tilde{M}} - t_2\wedge\tau_{n_l\tilde{M}}| \geqslant \frac{\sqrt{2}}{2}\sigma_1\left(\frac{\epsilon_0}{2}\right)\right\}\right) \\ \subseteq &\left(\{\tau_{n_l\tilde{M}} \leqslant T\} \cup \left\{|\tilde{Y}_{n_l}(t_1) - \tilde{Y}_{n_l}(t_2)| \geqslant \frac{\sqrt{2}}{2}\sigma_1\left(\frac{\epsilon_0}{2}\right)\right\} \cup \left\{|t_1 - t_2| \geqslant \frac{\sqrt{2}}{2}\sigma_1\left(\frac{\epsilon_0}{2}\right)\right\}\right).\end{aligned} \tag{A11}$$

Then, it follows from (A7) and $\mathbb{P}^x\{\tau_{n_l\tilde{M}} \leqslant T\} \leqslant \varepsilon_1$ that

$$\lim_{h\to 0}\lim_{l\to\infty}\sup_{|t_1-t_2|<h,\ t_1,t_2\in[t_0,T]}\mathbb{P}^x\left\{|g_j(t_1,\tilde{Y}_{n_l}(t_1),\tilde{r}^{(n_l)}(t_1)) - g_j(t_2,\tilde{Y}_{n_l}(t_2),\tilde{r}^{(n_l)}(t_1))| \geqslant \frac{\epsilon_0}{2}\right\} \leqslant \varepsilon_1. \tag{A12}$$

Taking $\varepsilon_1 \to 0$ into (A12), we obtain (A10).

## Appendix B Lemmas used in Section 3

**Lemma B1** ([12])**.** Let $y_1,\ldots,y_d$ and $0 < q \leqslant 1$ be real numbers. Then, $(|y_1|+|y_2|+\cdots+|y_d|)^q \leqslant |y_1|^q + |y_2|^q + \cdots + |y_d|^q$.

**Lemma B2.** Let $q \geqslant 1$ be an odd integer or a ratio of odd integers. Then $|x-y|^q \leqslant 2^{q-1}|x^q - y^q|$, $\forall\ x,\ y\ \in \mathbb{R}$ [2].

**Lemma B3.** Suppose $n$ and $\delta$ are positive real numbers, and $a \geqslant 0$, $b \geqslant 0$, and $\pi > 0$ are continuous functions. Then for any constant $c > 0$, $a^n b^\delta \pi \leqslant c\cdot a^{n+\delta} + \frac{\delta}{n+\delta}[\frac{n}{c(n+\delta)}]^{\frac{n}{\delta}} b^{n+\delta}\pi^{\frac{n+\delta}{\delta}}$ [2].

2) Qian C J, Lin W. Non-Lipschitz continuous stabilizers for nonlinear systems with uncontrollable unstable linearization. Syst Control Lett, 2001, 42: 185–200.

## Appendix C Propositions used in Subsection 3.2

**Proposition C1.** For any given positive constant $\delta_{k,2} < L_{k-1} - \delta_{k,1}$, there exists a smooth function $\tilde{\rho}_{k,1}(\bar{x}_k) \geqslant 0$ such that

$$\sum_{l=1}^{k-1}\left(\frac{\partial W_k}{\partial x_l}(x_{l+1}^{p_l} + f_l(t,x,i))\right) \leqslant \delta_{k,2}(|\xi_1|^{4+\alpha} + \cdots + |\xi_{k-1}|^{4+\alpha}) + \tilde{\rho}_{k,1}|\xi_k|^{4+\alpha}, \forall i \in \mathbb{S}. \tag{C1}$$

**Proposition C2.** For any given positive constant $\delta_{k,3} < L_{k-1} - \delta_{k,1} - \delta_{k,2}$, there exists a smooth function $\tilde{\rho}_{k,2}(\bar{x}_k) \geqslant 0$ such that

$$\left|\frac{\partial W_k}{\partial x_k} f_k(t,x,i)\right| \leqslant \delta_{k,3}(|\xi_1|^{4+\alpha} + \cdots + |\xi_{k-1}|^{4+\alpha}) + \tilde{\rho}_{k,2}|\xi_k|^{4+\alpha}, \forall i \in \mathbb{S}. \tag{C2}$$

**Proposition C3.** For any given positive constant $\delta_{k,4} < L_{k-1} - \delta_{k,1} - \delta_{k,2} - \delta_{k,3}$, there exists a smooth function $\tilde{\rho}_{k,3}(\bar{x}_k) \geqslant 0$ such that $\forall i \in \mathbb{S}$,

$$\left|\frac{1}{2}\sum_{l,\sigma=1}^{k}\frac{\partial^2 W_k}{\partial x_l \partial x_\sigma} g_l(t,x,i) g_\sigma^{\mathrm{T}}(t,x,i)\right| \leqslant \delta_{k,4}\sum_{h=1}^{k-1}|\xi_h|^{4+\alpha} + \tilde{\rho}_{k,3}|\xi_k|^{4+\alpha}. \tag{C3}$$

*Proof of Proposition C1.* It follows from Lemma B1 and (67) that for $\sigma = 2, \ldots, k$,

$$|x_\sigma| \leqslant |\xi_\sigma|^{r_\sigma} + |\xi_{\sigma-1}|^{r_\sigma}\chi_{\sigma-1}^{1/p_{\sigma-1}}(\bar{x}_{\sigma-1}). \tag{C4}$$

Then, we obtain that $\sum_{\sigma=1}^{l}|x_\sigma| \leqslant \sum_{\sigma=1}^{l-1}|\xi_\sigma|^{r_l}(|\xi_\sigma|^{r_\sigma - r_l} + |\xi_\sigma|^{r_{\sigma+1} - r_l}\chi_\sigma^{1/p_\sigma}(\bar{x}_\sigma)) + |\xi_l|^{r_l}$, which together with (66) leads to

$$|f_l(t,x,i)| \leqslant (|\xi_1|^{r_l} + |\xi_2|^{r_l} + \cdots + |\xi_l|^{r_l})\bar{\varphi}_{l,i}(\bar{x}_l), \tag{C5}$$

where $\bar{\varphi}_{l,i} \geqslant 0$ is a smooth function $\forall i \in \mathbb{S}$. Referring to the proof of [12, Proposition 4], we obtain that for all $l = 1, 2, \ldots, k-1$, there exist smooth functions $\hat{\rho}_{k,l}(\bar{x}_{k-1}) \geqslant 0$ such that

$$\frac{\partial x_k^{*\frac{1}{r_k}}}{\partial x_l} \leqslant (|\xi_1|^{1-r_l} + \cdots + |\xi_{k-1}|^{1-r_l})\hat{\rho}_{k,l}(\bar{x}_{k-1}). \tag{C6}$$

Then, it follows from (C4)–(C6) that

$$\left|\frac{\partial W_k}{\partial x_l}(x_{l+1}^{p_l} + f_l(t,x,i))\right| \leqslant 2(4-r_k)\hat{\rho}_{k,l}\left(\sum_{h=1}^{k-1}|\xi_h|^{1-r_l}\right)|\xi_k|^3\left(2^{p_l-1}(|\xi_{l+1}|^{r_l+\alpha} + |\xi_l|^{r_l+\alpha}\chi_l(\bar{x}_l)) + \left(\sum_{j=1}^{l}|\xi_j|^{r_l}\right)\bar{\varphi}_{l,i}\right). \tag{C7}$$

By Lemma B3, we obtain that

$$\begin{aligned}
&2^{p_l}(4-r_k)\hat{\rho}_{k,l}|\xi_h|^{1-r_l}|\xi_k|^3|\xi_{l+1}|^{r_l+\alpha} = 2^{p_l}(4-r_k)\hat{\rho}_{k,l}|\xi_h|^{1-r_l}|\xi_k|^{3+r_l+\alpha}\\
&\quad\leqslant c_1^1\xi_h^{4+\alpha} + \frac{3+r_l+\alpha}{4+\alpha}\left(\frac{1-r_l}{c_1^1(4+\alpha)}\right)^{\frac{1-r_l}{3+r_l+\alpha}}\left(2^{p_l}(4-r_k)\hat{\rho}_{k,l}\right)^{\frac{4+\alpha}{3+r_l+\alpha}}\xi_k^{4+\alpha},\ \text{when } l = k-1,
\end{aligned} \tag{C8}$$

$$\begin{aligned}
&2^{p_l}(4-r_k)\hat{\rho}_{k,l}|\xi_h|^{1-r_l}|\xi_k|^3|\xi_{l+1}|^{r_l+\alpha} \leqslant 2^{p_l}(4-r_k)\hat{\rho}_{k,l}|\xi_k|^3\left(c_1^2\xi_h^{1+\alpha} + \frac{r_l+\alpha}{1+\alpha}\left(\frac{1-r_l}{c_1^2(1+\alpha)}\right)^{\frac{1-r_l}{r_l+\alpha}}\xi_{l+1}^{1+\alpha}\right)\\
&\quad\leqslant c_1^3\xi_h^{4+\alpha} + \frac{3}{4+\alpha}\left(\frac{1+\alpha}{c_1^3(4+\alpha)}\right)^{\frac{1+\alpha}{3}}\left(2^{p_l}(4-r_k)\hat{\rho}_{k,l}c_1^2\right)^{\frac{4+\alpha}{3}}\xi_k^{4+\alpha}\\
&\qquad + c_1^4\xi_{l+1}^{4+\alpha} + \frac{3}{4+\alpha}\left(\frac{1+\alpha}{c_1^4(4+\alpha)}\right)^{\frac{1+\alpha}{3}}\left(2^{p_l}(4-r_k)\hat{\rho}_{k,l}\frac{r_l+\alpha}{1+\alpha}\left(\frac{1-r_l}{c_1^2(1+\alpha)}\right)^{\frac{1-r_l}{r_l+\alpha}}\right)^{\frac{4+\alpha}{3}}\xi_k^{4+\alpha},\ \text{when } l \neq k-1,
\end{aligned} \tag{C9}$$

and estimations of $2^{p_l}(4-r_k)\hat{\rho}_{k,l}\chi_l(\bar{x}_l)|\xi_h|^{1-r_l}|\xi_k|^3|\xi_l|^{r_l+\alpha}$ and $2(4-r_k)\hat{\rho}_{k,l}\bar{\varphi}_{l,i}|\xi_h|^{1-r_l}|\xi_k|^3|\xi_j|^{r_l}$ are similar to that of (C9), where $c_1^1, c_1^2, c_1^3, c_1^4$ are any given positive real numbers. Thus, for any given positive constants $\delta_{k,2,1}, \delta_{k,2,2}, \ldots, \delta_{k,2,k-1}$ satisfying $\delta_{k,1} + \delta_{k,2,1} + \cdots + \delta_{k,2,k-1} < L_{k-1}$, there exist smooth functions $\tilde{\rho}_{k,1,l,i}(\bar{x}_k) \geqslant 0$ $(l = 1, 2, \ldots, k-1)$ such that

$$\left|\frac{\partial W_k}{\partial x_l}(x_{l+1}^{p_l} + f_l(t,x,i))\right| \leqslant \delta_{k,2,l}(|\xi_1|^{4+\alpha} + \cdots + |\xi_{k-1}|^{4+\alpha}) + \tilde{\rho}_{k,1,l,i}|\xi_k|^{4+\alpha},\quad \forall i \in \mathbb{S}. \tag{C10}$$

By summing the above inequality for $l$ from 1 to $k-1$, we obtain (C1) with $\delta_{k,2} = \sum_{l=1}^{k-1}\delta_{k,2,l}$ and smooth function $\tilde{\rho}_{k,1} \geqslant \max\{\sum_{l=1}^{k-1}\tilde{\rho}_{k,1,l,i} : i \in \mathbb{S}\}$.

*Proof of Proposition C2.* The proof of Proposition C2 is similar to that of Proposition C1.

*Proof of Proposition C3.* Referring to the proof of [13, Proposition 12], we have that for all $l, \sigma = 1, 2, \ldots, k-1$, there exist smooth functions $\hat{\beta}_{k,l,\sigma}(\bar{x}_{k-1}) \geqslant 0$ such that

$$\left|\frac{\partial^2 x_k^{*\frac{1}{r_k}}}{\partial x_l \partial x_\sigma}\right| \leqslant \left(\sum_{h=1}^{k-1}|\xi_h|^{1-2r_{l\vee\sigma}}\right)\hat{\beta}_{k,l,\sigma}(\bar{x}_{k-1}). \tag{C11}$$

Similar to (C5), we obtain that there exists a smooth function $\bar{\psi}_{l,i}(\bar{x}_l) \geqslant 0$ such that $|g_l(t,x,i)| \leqslant \sum_{j=1}^{l} |\xi_j|^{r_l} \bar{\psi}_{l,i}(\bar{x}_l)$ $\forall i \in \mathbb{S}$. Then, it follows from Lemma B3, (C6), and (C11) that for any given positive constants $\delta_{k,4,1,l,\sigma}$ $(l, \sigma = 1, 2, \ldots, k-1)$ with $\delta_{k,1} + \delta_{k,2} + \delta_{k,3} + \sum_{l,\sigma=1}^{k-1} \delta_{k,4,1,l,\sigma} < L_{k-1}$, there exist smooth functions $\tilde{\rho}_{k,3,1,l,\sigma,i}(\bar{x}_k) \geqslant 0$ such that

$$\begin{aligned}
&\left|\frac{1}{2}\frac{\partial^2 W_k}{\partial x_l \partial x_\sigma} g_l(t,x,i) g_\sigma^{\mathrm{T}}(t,x,i)\right| \\
&\leqslant \left((4-r_k)|\xi_k|^3 \hat{\beta}_{k,l,\sigma}\left(\sum_{h=1}^{k-1}|\xi_h|^{1-2r_{l\vee\sigma}}\right) + (4-r_k)\cdot(3-r_k)|\xi_k|^2\left(\sum_{h=1}^{k-1}|\xi_h|^{1-r_l}\right)\hat{\rho}_{k,l}\left(\sum_{h=1}^{k-1}|\xi_h|^{1-r_\sigma}\right)\hat{\rho}_{k,\sigma}\right) \\
&\quad \cdot \left(\sum_{h=1}^{l}|\xi_h|^{r_l}\right)\bar{\psi}_{l,i}\left(\sum_{h=1}^{\sigma}|\xi_h|^{r_\sigma}\right)\bar{\psi}_{\sigma,i} \leqslant \delta_{k,4,1,l,\sigma}\sum_{h=1}^{k-1}|\xi_h|^{4+\alpha} + \tilde{\rho}_{k,3,1,l,\sigma,i}|\xi_k|^{4+\alpha}.
\end{aligned} \quad \text{(C12)}$$

Then, by summing this inequality for $l$ from 1 to $k-1$ and $\sigma$ from 1 to $k-1$, we obtain that

$$\left|\frac{1}{2}\sum_{l,\sigma=1}^{k-1}\frac{\partial^2 W_k}{\partial x_l \partial x_\sigma} g_l(t,x,i) g_\sigma^{\mathrm{T}}(t,x,i)\right| \leqslant \delta_{k,4,1}\sum_{j=1}^{k-1}|\xi_j|^{4+\alpha} + \tilde{\rho}_{k,3,1}|\xi_k|^{4+\alpha}, \quad \forall i \in \mathbb{S}, \quad \text{(C13)}$$

where $\delta_{k,4,1} = \sum_{l,\sigma=1}^{k-1} \delta_{k,4,1,l,\sigma}$ and smooth function $\tilde{\rho}_{k,3,1} \geqslant \max\{\sum_{l,\sigma=1}^{k-1} \tilde{\rho}_{k,3,1,l,\sigma,i}(\bar{x}_k) : i \in \mathbb{S}\}$. Similarly, we have that $\left|\sum_{l=1}^{k-1} \frac{\partial^2 W_k}{\partial x_l \partial x_k} \cdot g_l(t,x,i) g_k^{\mathrm{T}}(t,x,i)\right| \leqslant \delta_{k,4,2} \sum_{j=1}^{k-1} |\xi_j|^{4+\alpha} + \tilde{\rho}_{k,3,2}|\xi_k|^{4+\alpha}$ and $\left|\frac{1}{2}\frac{\partial^2 W_k}{\partial x_k^2} g_k(t,x,i) g_k^{\mathrm{T}}(t,x,i)\right| \leqslant \delta_{k,4,3} \sum_{j=1}^{k-1} |\xi_j|^{4+\alpha}$ $+\tilde{\rho}_{k,3,3}|\xi_k|^{4+\alpha}$, $\forall i \in \mathbb{S}$, where $\tilde{\rho}_{k,3,2}(\bar{x}_k) \geqslant 0$ and $\tilde{\rho}_{k,3,3}(\bar{x}_k) \geqslant 0$ are smooth functions, and positive constants $\delta_{k,4,2}$ and $\delta_{k,4,3}$ satisfy $\delta_{k,1} + \delta_{k,2} + \delta_{k,3} + \delta_{k,4,1} + \delta_{k,4,2} + \delta_{k,4,3} < L_{k-1}$. These together with (C13) lead to (C3) with $\delta_{k,4} = \delta_{k,4,1} + \delta_{k,4,2} + \delta_{k,4,3}$ and $\tilde{\rho}_{k,3} = \tilde{\rho}_{k,3,1} + \tilde{\rho}_{k,3,2} + \tilde{\rho}_{k,3,3}$.