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# Group consensus of multi-agent systems with additive noises

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**Abstract** This paper is concerned with group consensus of multi-agent systems (MASs) that consist of two groups in additive noise environments. First, a control protocol is proposed based on the state information of each agent's neighbors corrupted by additive noises. Second, some sufficient conditions and necessary conditions are obtained for the following two types of group consensus behaviors. (1) Pure group consensus: agents in both groups have the same behavior (weak consensus or strong consensus); (2) hybrid group consensus: agents in different groups achieve different consensus behaviors. It is revealed that the influence between the two groups should be attenuated such that the MASs can achieve group consensus in additive noise environments, i.e., the affected group must fight against the influence that comes from another group. Finally, some simulation examples are given to illustrate the feasibility of the theoretical results.

**Keywords** group consensus, multi-agent systems, additive noises, pure group consensus, hybrid group consensus

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## 1 Introduction

With the development of automation and robotics technology, the coordination control of multi-agent systems (MASs) has become an important topic in the control community, with the consensus problem being the basic problem. In recent years, consensus problems of MASs have received wide attention and have obtained many promising research findings. Vicsek et al. [1] proposed a discrete-time multi-agent model and developed consensus issues. Based on [1], Jadbabaie et al. [2] investigated linear MASs and gave the consensus conditions. For the problem on how time-delays affect consensus behaviors, many important and interesting research results have been obtained, as shown in [3–10].

The above studies all focused on the global consensus problem that all agents converge to a common state under the given control protocol. Note that the agents may be assigned different tasks, and different groups are then required to achieve different consensus behaviors. Some agents are required to converge to one common state, and some converge to another common state. This behavior is called group consensus and plays an important role in group formation control [11] and distributed cluster control for multi-microgrids [12]. In fact, group consensus problems have been studied extensively. Yu and Wang [13] studied the group average consensus of MASs under ideal environments. Further, they discussed the case with communication time-delays in [14]. An et al. [15] and Oyedeji and Mahmoud [16] also made efforts to study how time-delays affect group consensus behaviors. However, there is a common assumption in [13, 14] that for each node in the graph, the in-degree from other groups is zero at any time. The assumption mentioned above is called the balance assumption, which may largely limit its application. The assumption was then relaxed in [17, 18] when considering group consensus of MASs. Huang et al. [19] studied a class of cluster consensus by a tracking approach. Here, the models mentioned above were assumed to be deterministic.

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In real MASs, measurement noises inevitably exist in the communication between any two agents, in which additive and multiplicative noises are dominant. So far, fruitful achievements have been made in consensus problems of MASs with the noises mentioned above. Li and Zhang [20] gave the average consensus conditions of first-order MASs with generally directed topologies concerning additive noises and introduced a control gain function. Wang and Zhang [21] investigated the stochastic strong consensus of MASs with general directed topology and additive noises in detail. Research results obtained in [21] further extended the results in [20]. In the case in which additive and multiplicative measurement noises coexist in MASs, Zong et al. [22] explored sufficient conditions and necessary conditions for achieving a consensus and revealed some relationships between the mean square consensus and almost sure consensus. For complex environments with both time-delays and noises, some conditions for achieving a consensus were given in [23]. Other research findings of the coordination control of MASs under measurement uncertain environments have also been obtained, such as in [24–28].

Note that the above consensus results under measurement noises are related to the global consensus. However, group consensus of MASs under measurement noise environments has not been studied systematically. For the case with both multiplicative noises and time-delays, Shang [29] studied group consensus problems of MASs in directed topologies with the required weak balance assumption that for all nodes in the graph, the in-degrees from the other groups are equal at any time. Song et al. [30] sought an asymptotic group consensus condition under a special kind of network, i.e., the Boolean network, in which additive noises were concerned and one group needed to satisfy the zero in-degree condition. Note that almost all efforts about group consensus under noisy environments mentioned above are based on some balance assumptions. However, how to investigate the group consensus under measurement noises without these balance assumptions remains an open problem.

Motivated by the above discussion, this paper investigates the group consensus problem of MASs in additive noise environments. This paper assumes that the MASs consist of two groups, and there is communication between the two groups. Due to the presence of communication between the two groups, the existing conditions on control gain functions for the single group system are not sufficient to guarantee the group consensus under additive noise environments. Moreover, we do not need these balance assumptions to hold; i.e., the in-degree mentioned above can be unequal, which leads to the failure of the protocol in [29,30] and makes it more difficult to derive the conditions for achieving group consensus behaviors. Based on the state information of each agent's neighbors corrupted by additive noises, this paper proposes a new type of control protocol. By the semi-decoupled skill and some estimation methods, this paper obtains some new conditions about the control gain functions for achieving different group consensus behaviors and reveals the effect of the communication between the two groups on the group consensus under additive noise environments.

The main contribution of this paper can be concluded as follows.

(1) Concepts of pure group consensus and hybrid group consensus are proposed. The following four types of group consensus are established under additive noise environments: (a) agents in both groups achieve stochastic weak consensus; (b) agents in the unaffected group and affected group achieve stochastic weak consensus and strong consensus, respectively; (c) agents in the unaffected group and affected group achieve stochastic strong consensus and weak consensus, respectively; (d) agents in both groups achieve stochastic strong consensus. (a) and (d) are called pure group consensus, while (b) and (c) are named hybrid group consensus. These studies are more comprehensive and helpful to solve some practical problems, such as group formation control of multi-unmanned aerial vehicle systems and distributed cluster control for multi-microgrids.

(2) Under additive noise environments, it is revealed that the influence between the two groups should be attenuated for achieving group consensus, and its attenuation rate can be related to control gain functions. Based on these findings, this paper develops a condition that can reveal the attenuation rate of the influence between the two groups and obtains some conditions such that the MASs can achieve different group consensus behaviors.

The rest of the paper is organized as follows. Section 2 introduces the problem formulation. Section 3 discusses the main results and establishes some new conditions for different group consensus behaviors. Section 4 explores some simulation results. Finally, Section 5 gives the conclusion and enumerates some topics for future research.

**Notations**. The following notations are used in the analysis process. $\mathbf{1}_n$ and $\vartheta_{n,i}$ are two $n$-dimensional column vectors, where each element is one for $\mathbf{1}_n$, and the $i$th element being one and others being zero for $\vartheta_{n,i}$. Let $I_m$ denote the $m$-dimensional identity matrix. For any complex number $\lambda$ in the complex space

$\mathbb{C}$, let $\mathrm{Re}(\lambda)$ denote its real part. For the matrix or vector $P$, let $P^{\mathrm{T}}$ and $\|P\|$ denote its transpose and Euclidean norm, respectively. For a given random variable or vector $X$, let $\mathbb{E}X$ denote its mathematical expectation. For a (local) continuous martingale $R(t)$, let $\langle R\rangle(t)$ denote its quadratic variation.

## 2 Problem formulation

Let $\mathcal{G}=(\mathcal{V},\mathcal{E},\mathcal{A})$ be a weighted directed graph, where $\mathcal{V}=\{1,\ldots,n\}$ is the set of nodes with $i$ representing the $i$th agent, $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ denotes the set of edges and $(i,j)\in\mathcal{E}$ is an edge of $\mathcal{G}$, $\mathcal{A}=[a_{ij}]\in\mathbb{R}^{n\times n}$ is the weighted adjacency matrix with real adjacency elements $a_{ij}=1$ or 0 indicating whether or not there is an information flow from agent $j$ to agent $i$ directly. Moreover, we assume $a_{ii}=0$ for all $i\in\mathcal{V}$. Let $N_i=\{j\in\mathcal{V}|(i,j)\in\mathcal{E}\}$ denote the set of neighbors of node $i$ and $\mathcal{L}$ denote the Laplacian matrix of $\mathcal{G}$. A directed path is a sequence of edges with the form $(i,i+1),(i+1,i+2),\ldots$. A directed graph contains a directed spanning tree if there exists at least one node that has a directed path to all other nodes.

In this paper, we consider a continuous-time multi-agent system (MAS) consisting of $n+m$ agents with directed communication graph. We assume that the topology graph $\mathcal{G}=(\mathcal{V},\mathcal{E},\mathcal{A})$ contains two subnetworks $\mathcal{G}_1=(\mathcal{V}_1,\mathcal{E}_1,\mathcal{A}_1)$ and $\mathcal{G}_2=(\mathcal{V}_2,\mathcal{E}_2,\mathcal{A}_2)$, where $\mathcal{E}_1=\mathcal{V}_1\times\mathcal{V}_1$, $\mathcal{E}_2=\mathcal{V}_2\times\mathcal{V}_2$ and $\mathcal{V}_1=\{1,\ldots,n\}$, $\mathcal{V}_2=\{n+1,\ldots,n+m\}$, that is, the first $n$ agents belong to one group, which is defined as $\Omega_1$, and the last $m$ agents belong to another group, which is defined as $\Omega_2$. Moreover, $N_i=N_{1i}\cup N_{2i}$, where $N_{1i}=\{j\in\mathcal{V}_1|(i,j)\in\mathcal{E}\}$ and $N_{2i}=\{j\in\mathcal{V}_2|(i,j)\in\mathcal{E}\}$ denote the set of neighbors belonging to $\Omega_1$ and $\Omega_2$ of agent $i$, respectively. Here, we give the dynamics of each agent as follows:

$$\dot{x}_i(t)=u_i(t),\quad i=1,\ldots,n,n+1,\ldots,n+m, \tag{1}$$

where $x_i(t)\in\mathbb{R}$ and $u_i(t)\in\mathbb{R}$ denote the state and the control input of the $i$th agent, respectively. Let $X_1(t)=[x_1(t),\ldots,x_n(t)]^{\mathrm{T}}$, $X_2(t)=[x_{n+1}(t),\ldots,\ x_{n+m}(t)]^{\mathrm{T}}$ and initial states $X_1(0)=[x_1(0),\ldots,x_n(0)]^{\mathrm{T}}$, $X_2(0)=[x_{n+1}(0),\ldots,\ x_{n+m}(0)]^{\mathrm{T}}$. For the directed graph $\mathcal{G}$ and the communication between the two groups, we give the following assumption.

**Assumption 1.** There are information flows from the nodes in group $\Omega_1$ to the nodes in group $\Omega_2$, and not vice versa. That is, the communication between the two groups is unidirectional and the agents in group $\Omega_2$ can receive the information from the agents in group $\Omega_1$. Meanwhile, $\mathcal{G}_1$ and $\mathcal{G}_2$ contain a spanning tree, respectively.

**Remark 1.** In previous study [29], the weak balance assumption that the in-degree from other groups is identical at every time for each node is required to examine the group consensus. In this paper, we remove the assumption and require the condition that both $\mathcal{G}_1$ and $\mathcal{G}_2$ have spanning trees. In fact, if $\mathcal{G}_1$ or $\mathcal{G}_2$ does not contain a spanning tree and the whole topology graph $\mathcal{G}$ contains a spanning tree, the group consensus will not be achieved, which is revealed in the simulation example in Section 4.

When additive noises exist in the MAS (1), the information exchange between agents cannot be performed accurately. Here, in additive noise environments, it is often assumed that the $i$th agent can receive the information from its neighbors as follows:

$$y_{ji}(t)=x_j(t)+\sigma_{ji}\eta_{ji}(t),\quad j\in N_i,$$

where $y_{ji}(t)$ denotes the measured value of the $j$th agent's state by the $i$th agent, $\eta_{ji}(t)\in\mathbb{R}$ denotes the measurement noise, and $\sigma_{ji}$ is the corresponding intensity function. Then, for each agent in different groups, the measurement information from its neighbors has the following form:

$$\begin{cases}\bar{y}_{ji}(t)=x_j(t)+\sigma_{ji}\eta_{ji}(t), & i\in\mathcal{V}_1,\ j\in N_i,\\ \bar{y}_{ji}(t)=\bar{\bar{y}}_{ji1}(t)+\kappa(t)\bar{\bar{y}}_{ji2}(t), & i\in\mathcal{V}_2,\ j\in N_i,\end{cases}$$

where $\bar{\bar{y}}_{ji1}(t)=x_j(t)+\sigma_{ji}\eta_{ji}(t), i\in\mathcal{V}_2, j\in N_{2i}$ is the measured value of agent $i$ to its neighbor $j$ belonging to group $\Omega_2$ and $\bar{\bar{y}}_{ji2}(t)=x_j(t)+\sigma_{ji}\eta_{ji}(t), i\in\mathcal{V}_2, j\in N_{1i}$ is the measured value of agent $i$ to its neighbor $j$ belonging to group $\Omega_1$. $\kappa(t)$ is a time-varying function, which reflects the communication intensity between the two groups.

**Remark 2.** Under the balance condition, Yu and Wang [13] proposed a new group control protocol and studied group average consensus of MASs under ideal environments, where $\sigma_{ji}=0$ and $\kappa(t)=1$. In this paper, we will explore general conditions on $\kappa(t)$ for achieving different group consensus behaviors under the additive noise environments.

Based on the measurement information above, we consider the following control protocol:

$$u_i(t)=\begin{cases}g_1(t)\sum_{j=1}^{n}a_{ij}(\bar{y}_{ji}(t)-x_i(t)),\ \forall i\in\mathcal{V}_1,\\ g_2(t)\sum_{j=n+1}^{n+m}a_{ij}(\bar{\bar{y}}_{ji1}(t)-x_i(t))+g_2(t)\sum_{j=1}^{n}a_{ij}\bar{\bar{y}}_{ji2}(t),\ \forall i\in\mathcal{V}_2,\end{cases}\tag{2}$$

where $g_1(t)$ and $g_2(t)\in C((0,\infty);[0,\infty))$ are two time-varying control gain functions.

**Remark 3.** Obviously, the communication between agents in both the same group and different groups is considered in this protocol. Under this control protocol, we can study the above four types of group consensus behaviors flexibly and obtain some new conditions for achieving group consensus under additive noise environments. Inspired by [20], for the two groups, we introduce time-varying consensus gain functions $g_1(t)$ and $g_2(t)$ in our control protocol (2) respectively to attenuate additive noises. It can be seen from (2) that there are some coupling relationships between $\kappa(t)$, $g_1(t)$, and $g_2(t)$, which may make it more difficult to analyze group consensus problems. In this paper, we will obtain some conditions on $g_1(t)$, $g_2(t)$, and $\kappa(t)$ for achieving group consensus.

In general, independent Gaussian white noises are used to model measurement noises, which satisfy the following assumption.

**Assumption 2.** The stochastic process $\eta_{ji}(t)$ satisfies $\int_0^t\eta_{ji}(s)\mathrm{d}s=\omega_{ji}(t),t\geqslant 0,i,j=1,2,\ldots,n+m$, where $\{\omega_{ji}(t),i,j=1,2,\ldots,n+m\}$ are independent Brownian motions.

In this paper, the MAS (1) under (2) is a stochastic system rather than a deterministic system. Here, we introduce global consensus definitions in mean square and almost sure sense.

**Definition 1** ([22])**.** For all agents $i,j$ that belong to the same group, if $\lim_{t\to\infty}\mathbb{E}\|x_i(t)-x_j(t)\|^2=0$ (or $\lim_{t\to\infty}\|x_i(t)-x_j(t)\|=0$, a.s.), then the agents in this group are said to reach mean square weak consensus (MSWC) (or almost sure weak consensus (ASWC)); i.e., the agents in this group achieve stochastic weak consensus.

**Definition 2** ([22])**.** For all agents $i,j$ that belong to the same group, if there is a random variable $\bar{x}^*$, such that $\mathbb{E}\|\bar{x}^*\|<\infty$, $\mathbb{P}\{\|\bar{x}^*\|<\infty\}=1$, $\lim_{t\to\infty}\mathbb{E}\|x_i(t)-\bar{x}^*\|^2=0$ (or $\lim_{t\to\infty}\|x_i(t)-\bar{x}^*\|=0$, a.s.), then the agents in this group are said to reach mean square strong consensus (MSSC) (or almost sure strong consensus (ASSC)); i.e., the agents in this group achieve stochastic strong consensus.

Then, based on above global consensus definitions, in this paper, we will discuss group consensus problems involving the following four types of group consensus behaviors: (a) agents in both groups achieve stochastic weak consensus; (b) agents in groups $\Omega_1$ and $\Omega_2$ achieve stochastic weak consensus and strong consensus, respectively; (c) agents in groups $\Omega_1$ and $\Omega_2$ achieve stochastic strong consensus and weak consensus, respectively; (d) agents in both groups achieve stochastic strong consensus. (a) and (d) are called pure group consensus, while (b) and (c) are named hybrid group consensus. These discussions are more comprehensive and helpful to solve some practical problems, such as group formation control [11] and distributed cluster control for multi-microgrids [12].

**Remark 4.** For the group consensus in the mean square sense, it is easy to see from the definitions of different group consensuses that the strong group consensus implies weak group consensus. In fact, if agents in any group achieve strong consensus, all agents in this group will move together, which satisfies the definition of weak consensus. Similarly, considering two groups, the strong group consensus implies the hybrid group consensus and the weak group consensus. Moreover, the hybrid group consensus implies the weak group consensus. However, there is no direct relationship between the two classes of the hybrid group consensus. The results also hold for the group consensus in the almost sure sense.

Here, we first introduce the following lemma.

**Lemma 1** ([22])**.** For the Laplacian matrix $\mathcal{L}'$ of directed graph $\mathcal{G}'$, we have the following assertions:

(1) There exists a probability measure $\pi$ such that $\pi^{\mathrm{T}}\mathcal{L}'=0$.

(2) There exist a nonsingular matrix $Q=(\frac{1}{\sqrt{n}}\mathbf{1}_n,\widetilde{Q})$ and

$$Q^{-1}=\begin{pmatrix}\upsilon^{\mathrm{T}}\\ \overline{Q}\end{pmatrix},\quad Q^{-1}\mathcal{L}'Q=\begin{pmatrix}0&0\\0&\widetilde{\mathcal{L}}'\end{pmatrix},\tag{3}$$

where $n$ is the number of nodes, $\overline{Q}\in\mathbb{R}^{(n-1)\times n}$, $\widetilde{\mathcal{L}}'\in\mathbb{R}^{(n-1)\times(n-1)}$, and $\upsilon$ is a left eigenvector of $\mathcal{L}'$ such that $\upsilon^{\mathrm{T}}\mathcal{L}'=0$ and $\frac{1}{\sqrt{n}}\upsilon^{\mathrm{T}}\mathbf{1}_n=1$.

(3) The directed graph contains a spanning tree if and only if each eigenvalue of $\widetilde{\mathcal{L}}'$ has positive real part. Moreover, if the directed graph contains a spanning tree, then the probability measure $\pi$ is unique and $\upsilon = \sqrt{n}\pi$.

Under protocol (2), from (1) we can obtain

$$\dot{X}_1(t) = -g_1(t)\mathcal{L}_{11}X_1(t) + g_1(t)\sum_{i,j=1}^{n} a_{ij}\sigma_{ji}\vartheta_{n,i}\eta_{ji}(t) \tag{4}$$

and

$$\begin{aligned}\dot{X}_2(t) =& -g_2(t)\mathcal{L}_{22}X_2(t) - g_2(t)\kappa(t)\mathcal{L}_{21}X_1(t) + g_2(t)\sum_{i,j=n+1}^{n+m} a_{ij}\sigma_{ji}\vartheta_{m,i-n}\eta_{ji}(t) \\ &+ g_2(t)\kappa(t)\sum_{i=n+1}^{n+m}\sum_{j=1}^{n} a_{ij}\sigma_{ji}\vartheta_{m,i-n}\eta_{ji}(t),\end{aligned} \tag{5}$$

where $\mathcal{L}_{11} = [l_{ij}]$ is defined as

$$l_{ij} = \begin{cases} -a_{ij}, & j \neq i,\ i \in \mathcal{V}_1,\ j \in N_{1i}, \\ \sum_{k=1,k\neq i}^{n} a_{ik}, & j = i,\ i \in \mathcal{V}_1,\ j \in N_{1i}, \end{cases}$$

$\mathcal{L}_{22} = \left[\bar{l}_{ij}\right]$ is defined as

$$\bar{l}_{ij} = \begin{cases} -a_{ij}, & j \neq i,\ i \in \mathcal{V}_2,\ j \in N_{2i}, \\ \sum_{k=n+1,k\neq i}^{n+m} a_{ik}, & j = i,\ i \in \mathcal{V}_2,\ j \in N_{2i}, \end{cases}$$

and $\mathcal{L}_{21} = [l_{ij}]$ is defined as $l_{ij} = -a_{ij}, i \in \mathcal{V}_2, j \in N_{1i}$. Here, $\mathcal{L}_{11}$ and $\mathcal{L}_{22}$ are the Laplacian matrices of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively.

## 3 Main results

In this section, we will give sufficient conditions and some necessary conditions of pure group consensus and hybrid group consensus for MASs in additive noise environments, respectively.

### 3.1 Stochastic weak group consensus

Firstly, we study stochastic weak group consensus, that is, agents in both groups achieve stochastic weak consensus.

In fact, for agents in the first group, since each agent only exchanges information with its neighbors in its own group, consensus problems of the first group can be considered as general global consensus problems. For the global consensus under additive noise environments, here, the following conditions on the time-varying control gain function $g_1(t)$ were proposed to solve the stochastic weak consensus [20,22]:

(C1) $\int_0^\infty g_1(t)\mathrm{d}t = \infty$;

(C2) $\lim_{t\to\infty} g_1(t) = 0$;

(C3) $\lim_{t\to\infty} g_1(t)\log\int_0^t g_1(s)\mathrm{d}s = 0$.

**Remark 5.** Condition (C1) is called convergence condition. With this condition, all agents' states in the first group can reach a common value with a proper rate under additive noise environments. Conditions (C1) and (C2) are sufficient conditions of MSWC, and (C1) and (C3) are sufficient conditions of ASWC [22]. At the same time, it can be seen that the form of condition (C3) is more complex, but it can reflect the decay rate of the control gain function accurately.

However, in this paper, there is communication between agents in different groups. Due to the presence of the communication between two groups, the existing findings on the control gain function $g_2(t)$ for general global stochastic consensus are not sufficient to guarantee the consensus of the second group. Here, we will redesign the two control gain functions, and consider the following conditions on $g_1(t)$, $g_2(t)$, and $\kappa(t)$:

(C4) $\int_0^\infty g_2(t)\mathrm{d}t = \infty$;
(C5) $\lim_{t\to\infty} g_2(t) = 0$;
(C6) $\lim_{t\to\infty} \kappa^2(t) \int_0^t g_1^2(s)\mathrm{d}s = 0$.

**Theorem 1.** For the MAS (1), suppose that Assumption 1 holds. Then, under protocol (2), the agents in the first group can achieve MSWC if conditions (C1) and (C2) hold. In the meantime, if conditions (C4)–(C6) hold, then the agents in the second group can also achieve MSWC.

*Proof.* First, we consider the first group. Let $Q_1 = (\frac{1}{\sqrt{n}}\mathbf{1}_n, \widetilde{Q}_1)$ and $Q_1^{-1} = (\begin{smallmatrix}\bar{v}^{\mathrm{T}}\\ \overline{Q}_1\end{smallmatrix}), Q_1^{-1}\mathcal{L}_{11}Q_1 = (\begin{smallmatrix}0 & 0\\ 0 & \widetilde{\mathcal{L}}_{11}\end{smallmatrix})$. From Lemma 1 we know $\bar{v} = \sqrt{n}\bar{\pi}$, $\bar{\pi}^{\mathrm{T}}\mathcal{L}_{11} = 0$. Let $J_n = \frac{1}{\sqrt{n}}\mathbf{1}_n\bar{v}^{\mathrm{T}}$. By the properties of matrix $\mathcal{L}_{11}$ and Lemma 1, we have $\mathcal{L}_{11}\mathbf{1}_n = 0$ and $\bar{v}^{\mathrm{T}}\mathcal{L}_{11} = 0$, and then $(I_n - J_n)\mathcal{L}_{11} = \mathcal{L}_{11} = \mathcal{L}_{11}(I_n - J_n)$. Let $\delta(t) = (I_n - J_n)X_1(t) = [\delta_1(t), \delta_2(t), \ldots, \delta_n(t)]^{\mathrm{T}}$, where $\delta_i(t) \in \mathbb{R}, i = 1, \ldots, n$. Let $\tilde{\delta}(t) = Q_1^{-1}\delta(t) = [\tilde{\delta}_1(t), \tilde{\delta}_2(t), \ldots, \tilde{\delta}_n(t)]^{\mathrm{T}}$, $\bar{\delta}(t) = [\tilde{\delta}_2(t), \ldots, \tilde{\delta}_n(t)]^{\mathrm{T}}$, $\tilde{\delta}_i(t) \in \mathbb{R}$. By the definition of $Q_1^{-1}$ in Lemma 1, we can get $\tilde{\delta}_1(t) = \bar{v}^{\mathrm{T}}\delta(t) = \bar{v}^{\mathrm{T}}(I_n - J_n)X_1(t) = 0$ and

$$\mathrm{d}\bar{\delta}(t) = -g_1(t)\widetilde{\mathcal{L}}_{11}\bar{\delta}(t)\mathrm{d}t + \mathrm{d}M(t),$$

where $M(t) = \sum_{i,j=1}^n a_{ij}\sigma_{ji}\overline{Q}_1(I_n - J_n)\vartheta_{n,i}\int_0^t g_1(s)\mathrm{d}\omega_{ji}(s)$. From [22], we need to prove that $\lim_{t\to\infty} \mathbb{E}\|\bar{\delta}(t)\|^2 = 0$ for any initial state. Then, by the semi-decoupled skill, variation of constants formula, and some estimation methods, we can get $\lim_{t\to\infty} \mathbb{E}\|\bar{\delta}(t)\|^2 = 0$ from (C1) and (C2). Then we have the conclusion that the first part of Theorem 1 is true. Then we consider the second group. Let $\gamma(t) = Q_1^{-1}X_1(t) = [\gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t)]^{\mathrm{T}}$, $\bar{\gamma}(t) = [\gamma_2(t), \ldots, \gamma_n(t)]^{\mathrm{T}}$, where $\gamma_i(t) \in \mathbb{R}, i = 1, \ldots, n$. Then, we can obtain $\mathrm{d}\gamma_1(t) = \bar{v}^{\mathrm{T}}g_1(t)\sum_{i,j=1}^n a_{ij}\sigma_{ji}\vartheta_{n,i}\mathrm{d}\omega_{ji}(t)$. Then $\gamma_1(t) = \gamma_1(0) + \bar{v}^{\mathrm{T}}\sum_{i,j=1}^n a_{ij}\sigma_{ji}\vartheta_{n,i}\int_0^t g_1(s)\mathrm{d}\omega_{ji}(s)$, where $\gamma_1(0) = \bar{v}^{\mathrm{T}}X_1(0)$. We also have

$$\mathrm{d}\bar{\gamma}(t) = -g_1(t)\widetilde{\mathcal{L}}_{11}\bar{\gamma}(t)\mathrm{d}t + \mathrm{d}M'(t), \tag{6}$$

where $M'(t) = \sum_{i,j=1}^n a_{ij}\sigma_{ji}\overline{Q}_1\vartheta_{n,i}\int_0^t g_1(s)\mathrm{d}\omega_{ji}(s)$. From [23] we can obtain $\lim_{t\to\infty}\bar{\gamma}(t) = [0, \ldots, 0]^{\mathrm{T}}$. Then, from the above analysis and (5) we have

$$\begin{aligned}\mathrm{d}X_2(t) = &- g_2(t)\mathcal{L}_{22}X_2(t)\mathrm{d}t - g_2(t)\kappa(t)\mathcal{L}_{21}Q_1\gamma(t)\mathrm{d}t\\ &+ g_2(t)\sum_{i,j=n+1}^{n+m} a_{ij}\sigma_{ji}\vartheta_{m,i-n}\mathrm{d}\omega_{ji}(t)\\ &+ g_2(t)\kappa(t)\sum_{i=n+1}^{n+m}\sum_{j=1}^{n} a_{ij}\sigma_{ji}\vartheta_{m,i-n}\mathrm{d}\omega_{ji}(t).\end{aligned} \tag{7}$$

Let $Q_2 = (\frac{1}{\sqrt{m}}\mathbf{1}_m, \widetilde{Q}_2)$ and $Q_2^{-1} = (\begin{smallmatrix}\bar{\bar{v}}^{\mathrm{T}}\\ \overline{Q}_2\end{smallmatrix}), Q_2^{-1}\mathcal{L}_{22}Q_2 = (\begin{smallmatrix}0 & 0\\ 0 & \widetilde{\mathcal{L}}_{22}\end{smallmatrix})$, $\bar{\bar{v}} = \sqrt{m}\bar{\bar{\pi}}$, $\bar{\bar{\pi}}^{\mathrm{T}}\mathcal{L}_{22} = 0$. Let $J_m = \frac{1}{\sqrt{m}}\mathbf{1}_m\bar{\bar{v}}^{\mathrm{T}}$. By the properties of matrix $\mathcal{L}_{22}$ and Lemma 1, we have $\mathcal{L}_{22}\mathbf{1}_m = 0$ and $\bar{\bar{v}}^{\mathrm{T}}\mathcal{L}_{22} = 0$, and then $(I_m - J_m)\mathcal{L}_{22} = \mathcal{L}_{22} = \mathcal{L}_{22}(I_m - J_m)$. Let $\xi(t) = (I_m - J_m)X_2(t) = [\xi_1(t), \xi_2(t), \ldots, \xi_m(t)]^{\mathrm{T}}$, where $\xi_i(t) \in \mathbb{R}, i = 1, \ldots, m$. Let $\tilde{\xi}(t) = Q_2^{-1}\xi(t) = [\tilde{\xi}_1(t), \tilde{\xi}_2(t), \ldots, \tilde{\xi}_m(t)]^{\mathrm{T}}$, $\bar{\xi}(t) = [\tilde{\xi}_2(t), \ldots, \tilde{\xi}_m(t)]^{\mathrm{T}}$, $\tilde{\xi}_i(t) \in \mathbb{R}$. By the definition of $Q_2^{-1}$ in Lemma 1, we can get $\tilde{\xi}_1(t) = \bar{\bar{v}}^{\mathrm{T}}\xi(t) = \bar{\bar{v}}^{\mathrm{T}}(I_m - J_m)X_2(t) = 0$ and

$$\mathrm{d}\bar{\xi}(t) = -g_2(t)\widetilde{\mathcal{L}}_{22}\bar{\xi}(t)\mathrm{d}t - g_2(t)\kappa(t)P\gamma(t)\mathrm{d}t + \mathrm{d}\bar{M}(t) + \mathrm{d}\bar{M}'(t), \tag{8}$$

where $P = \overline{Q}_2(I_m - J_m)\mathcal{L}_{21}Q_1$, $\bar{M}(t) = \sum_{i,j=n+1}^{n+m} a_{ij}\sigma_{ji}\overline{Q}_2(I_m - J_m)\vartheta_{m,i-n}\int_0^t g_2(s)\mathrm{d}\omega_{ji}(s)$, $\bar{M}'(t) = \sum_{i=n+1}^{n+m}\sum_{j=1}^{n} a_{ij}\sigma_{ji}\overline{Q}_2(I_m - J_m)\vartheta_{m,i-n}\int_0^t g_2(s)\kappa(s)\mathrm{d}\omega_{ji}(s)$. From [23], we now need to prove that $\lim_{t\to\infty}\mathbb{E}\left\|\bar{\xi}(t)\right\|^2 = 0$ for any initial state. Then, according to the matrix theorem, we have $H\widetilde{\mathcal{L}}_{22}H^{-1} = J$, where $H$ is a complex invertible matrix, $J$ is the Jordan normal form of $\widetilde{\mathcal{L}}_{22}$. And we know $J = \mathrm{diag}(J_{\lambda_2,n_2}, \ldots, J_{\lambda_l,n_l})$, $\sum_{k=2}^{l} n_k = m - 1$, where $\lambda_2, \ldots, \lambda_l$ are all the eigenvalues of $\widetilde{\mathcal{L}}_{22}$ and $J_{\lambda_k,n_k}$ is the Jordan block corresponding to eigenvalue $\lambda_k$, whose dimension is $n_k$. Let $Y(t) = H\bar{\xi}(t) = [Y_2(t), \ldots, Y_m(t)]^{\mathrm{T}}$ with $Y_j(t) \in \mathbb{R}$, and then we have $\mathrm{d}Y(t) = -g_2(t)JY(t)\mathrm{d}t - g_2(t)\kappa(t)D\gamma(t)\mathrm{d}t + H\mathrm{d}\bar{M}(t) + H\mathrm{d}\bar{M}'(t)$, where $D = HP$. Here, we first consider the $k$th Jordan block, and let $\zeta_k(t) = [\zeta_{k,1}(t), \ldots, \zeta_{k,n_k}(t)]^{\mathrm{T}}$, $D(k) = [D_{k,1}^{\mathrm{T}}, \ldots, D_{k,n_k}^{\mathrm{T}}]^{\mathrm{T}}$, $H(k) = [H_{k,1}^{\mathrm{T}}, \ldots, H_{k,n_k}^{\mathrm{T}}]^{\mathrm{T}}$, where $\zeta_{k,j}(t) = Y_{k_j}(t)$,

$D_{k,j}$ is the $k_j$th row of $D$ with $k_j = \sum_{l=2}^{k-1} n_l + j$, $D_{k,n_k} = [d_1, \ldots, d_n]$, and $H_{k,j} = H_{k_j}$ is the $k_j$th row of $H$ with $k_j = \sum_{l=2}^{k-1} n_l + j$. Then we have

$$\begin{aligned} \mathrm{d}\zeta_k(t) = &- g_2(t) J_{\lambda_k, n_k} \zeta_k(t)\mathrm{d}t - g_2(t)\kappa(t) D(k)\gamma(t)\mathrm{d}t \\ &+ H(k)\mathrm{d}\bar{M}(t) + H(k)\mathrm{d}\bar{M}'(t). \end{aligned} \tag{9}$$

Then we have the following semi-decoupled equations:

$$\begin{aligned} \mathrm{d}\zeta_{k,n_k}(t) = &- g_2(t)\lambda_k \zeta_{k,n_k}(t)\mathrm{d}t - g_2(t)\kappa(t) D_{k,n_k}\gamma(t)\mathrm{d}t \\ &+ \mathrm{d}\bar{M}_{k,n_k}(t) + \mathrm{d}\bar{M}'_{k,n_k}(t), \end{aligned} \tag{10}$$

and

$$\begin{aligned} \mathrm{d}\zeta_{k,j}(t) = &- g_2(t)\lambda_k \zeta_{k,j}(t)\mathrm{d}t - g_2(t)\zeta_{k,j+1}(t)\mathrm{d}t \\ &- g_2(t)\kappa(t) D_{k,j}\gamma(t)\mathrm{d}t + \mathrm{d}\bar{M}_{k,j}(t) + \mathrm{d}\bar{M}'_{k,j}(t), \end{aligned} \tag{11}$$

where $\bar{M}_{k,j}(t) = \sum_{i,q=n+1}^{n+m} r_{k_j,i} a_{iq}\sigma_{qi} \int_0^t g_2(s)\mathrm{d}\omega_{qi}(s)$, $\bar{M}'_{k,j}(t) = \sum_{i=n+1}^{n+m} r_{k_j,i} \sum_{q=1}^{n} a_{iq}\sigma_{qi} \int_0^t g_2(s) \times\kappa(s)\mathrm{d}\omega_{qi}(s)$, and $r_{k_j,i} = H_{k_j}\overline{Q}_2(I_m - J_m)\vartheta_{m,i-n}$, $j = 1, \ldots, n_k - 1$. By means of a variation of constants formula for (10), we can get

$$\begin{aligned} \zeta_{k,n_k}(t) = &\mathrm{e}^{-\lambda_k \int_0^t g_2(u)\mathrm{d}u} \zeta_{k,n_k}(0) \\ &- \int_0^t \mathrm{e}^{-\lambda_k \int_s^t g_2(u)\mathrm{d}u} g_2(s)\kappa(s) d_1\gamma_1(s)\mathrm{d}s \\ &- \sum_{p=2}^{n} \int_0^t \mathrm{e}^{-\lambda_k \int_s^t g_2(u)\mathrm{d}u} g_2(s)\kappa(s) d_p\gamma_p(s)\mathrm{d}s \\ &+ Z_{k,n_k}(t) + Z'_{k,n_k}(t), \end{aligned} \tag{12}$$

where $Z_{k,n_k}(t) = \int_0^t \mathrm{e}^{-\lambda_k \int_s^t g_2(u)\mathrm{d}u}\mathrm{d}\bar{M}_{k,n_k}(s)$ and $Z'_{k,n_k}(t) = \int_0^t \mathrm{e}^{-\lambda_k \int_s^t g_2(u)\mathrm{d}u}\mathrm{d}\bar{M}'_{k,n_k}(s)$. By taking the Euclidean norm of (12), we have

$$\begin{aligned} \|\zeta_{k,n_k}(t)\| \leqslant &\mathrm{e}^{-\lambda_k \int_0^t g_2(u)\mathrm{d}u} \|\zeta_{k,n_k}(0)\| \\ &+ \left\| \int_0^t \mathrm{e}^{-\lambda_k \int_s^t g_2(u)\mathrm{d}u} g_2(s)\kappa(s) d_1\gamma_1(s)\mathrm{d}s \right\| \\ &+ \left\| \sum_{p=2}^{n} \int_0^t \mathrm{e}^{-\lambda_k \int_s^t g_2(u)\mathrm{d}u} g_2(s)\kappa(s) d_p\gamma_p(s)\mathrm{d}s \right\| \\ &+ \|Z_{k,n_k}(t)\| + \|Z'_{k,n_k}(t)\|. \end{aligned}$$

Then, by Jensen inequality and taking expectation of the above equation, we can obtain

$$\begin{aligned} \mathbb{E}\|\zeta_{k,n_k}(t)\|^2 \leqslant &5\mathrm{e}^{-2\mathrm{Re}(\lambda_k) \int_0^t g_2(u)\mathrm{d}u} \|\zeta_{k,n_k}(0)\|^2 \\ &+ 5\mathbb{E}\left| \int_0^t \mathrm{e}^{-\lambda_k \int_s^t g_2(u)\mathrm{d}u} g_2(s)\kappa(s) d_1\gamma_1(s)\mathrm{d}s \right|^2 \\ &+ 5\mathbb{E}\sum_{p=2}^{n} \left| \int_0^t \mathrm{e}^{-\lambda_k \int_s^t g_2(u)\mathrm{d}u} g_2(s)\kappa(s) d_p\gamma_p(s)\mathrm{d}s \right|^2 \\ &+ C_{k,n_k} \int_0^t \mathrm{e}^{-2\mathrm{Re}(\lambda_k) \int_s^t g_2(u)\mathrm{d}u} g_2^2(s)\mathrm{d}s \\ &+ C'_{k,n_k} \int_0^t \mathrm{e}^{-2\mathrm{Re}(\lambda_k) \int_s^t g_2(u)\mathrm{d}u} g_2^2(s)\kappa^2(s)\mathrm{d}s, \end{aligned} \tag{13}$$

where $C_{k,j} = \sum_{i,q=n+1}^{n+m} |r_{k_j,i}|^2 a_{iq}^2\sigma_{qi}^2$ and $C'_{k,j} = \sum_{i=n+1}^{n+m}\sum_{q=1}^{n} |r_{k_j,i}|^2 a_{iq}^2\sigma_{qi}^2$. Now, our task is to prove that the second term, the third term, and the last term on the right hand side of (13) vanish at infinite time because other terms tend to zero, the proof can be found in [20, 23]. Let $k$ be fixed and write $S_{k,n_k}(t) = \int_0^t \mathrm{e}^{-\lambda_k \int_s^t g_2(u)\mathrm{d}u} g_2(s)\kappa(s)d_1\gamma_1(s)\mathrm{d}s$. We have $\lim_{t\to\infty}\mathbb{E}\|S_{k,n_k}(t)\|^2 \leqslant \lim_{t\to\infty}\mathbb{E}(\int_0^t \mathrm{e}^{-\mathrm{Re}(\lambda_k)\int_s^t g_2(u)\mathrm{d}u} g_2(s)\kappa(s)|d_1\gamma_1(s)|\mathrm{d}s)^2$. Let $U(t) = \int_0^t \mathrm{e}^{-\mathrm{Re}(\lambda_k)\int_s^t g_2(u)\mathrm{d}u} g_2(s)\kappa(s)|d_1\gamma_1(s)|\mathrm{d}s$, we have

$$\sqrt{\mathbb{E}(U(t))^2} \leqslant \int_0^t \mathrm{e}^{-\mathrm{Re}(\lambda_k)\int_s^t g_2(u)\mathrm{d}u} g_2(s)\kappa(s)\sqrt{\mathbb{E}|d_1\gamma_1(s)|^2}\mathrm{d}s.$$

By means of L'Hôpital's rule we can get

$$\lim_{t\to\infty}\sqrt{\mathbb{E}(U(t))^2} \leqslant \lim_{t\to\infty}\frac{\sqrt{\mathbb{E}|d_1\gamma_1(t)|^2}\kappa(t)}{\mathrm{Re}(\lambda_k)} \leqslant \lim_{t\to\infty}\frac{\sqrt{c_3\int_0^t g_1^2(s)\mathrm{d}s + c_4\mathbb{E}|X_1(0)|^2}\kappa(t)}{\mathrm{Re}(\lambda_k)},$$

where $c_3 = d_1^2\sum_{k=1}^n \bar{v}_k^2 \sum_{i,j=1}^n (a_{ij}\sigma_{ji})^2$ and $c_4$ is a constant. From (C6) we have $\lim_{t\to\infty}\kappa(t) = 0$, and then we have $\lim_{t\to\infty}\mathbb{E}(U(t))^2 = 0$. Then $\lim_{t\to\infty}\mathbb{E}\|S_{k,n_k}(t)\|^2 = 0$. Note that $\lim_{t\to\infty}\sum_{p=2}^n d_p\gamma_p(t) = 0$; then we can also get $\lim_{t\to\infty}\mathbb{E}\sum_{p=2}^n |\int_0^t \mathrm{e}^{-\lambda_k\int_s^t g_2(u)\mathrm{d}u} g_2(s)\kappa(s)d_p\gamma_p(t)\mathrm{d}s|^2 = 0$. Similarly, from (C5), (C6), and L'Hôpital's rule, we can get

$$\begin{aligned}
&\lim_{t\to\infty}\int_0^t \mathrm{e}^{-2\mathrm{Re}(\lambda_k)\int_s^t g_2(u)\mathrm{d}u} g_2^2(s)\kappa^2(s)\mathrm{d}s \\
&= \lim_{t\to\infty}\frac{\int_0^t \mathrm{e}^{2\mathrm{Re}(\lambda_k)\int_0^s g_2(u)\mathrm{d}u} g_2^2(s)\kappa^2(s)\mathrm{d}s}{\mathrm{e}^{2\mathrm{Re}(\lambda_k)\int_0^t g_2(u)\mathrm{d}u}} \\
&= \lim_{t\to\infty}\frac{g_2(t)\kappa^2(t)}{2\mathrm{Re}(\lambda_k)} = 0.
\end{aligned} \tag{14}$$

Then we obtain $\lim_{t\to\infty}\mathbb{E}\|\zeta_{k,n_k}(t)\|^2 = 0$ for the given $k$. Similarly, we can obtain that $\mathbb{E}\|\zeta_{k,j}(t)\|^2 = 0$ for the given $k$ and all $j = 1,\ldots,n_k - 1$. Repeating the above process, similar induction yields $\lim_{t\to\infty}\mathbb{E}\|\zeta_k(t)\|^2 = 0$ for all $k = 1,\ldots,l$. Hence,

$$\lim_{t\to\infty}\mathbb{E}\|\bar{\xi}(t)\|^2 = 0, \tag{15}$$

that is, the second part of Theorem 1 is true. This together with the above analysis guarantees that the agents in both groups achieve MSWC.

Theorem 1 proves that under conditions (C1), (C2), and (C4)–(C6), agents in two groups can achieve the MSWC. Then, based on above results, we examine the ASWC for both groups. Here, the following condition on the control gain $g_2(t)$ is required:

(C7) $\lim_{t\to\infty} g_2(t)\log\int_0^t g_2(s)\mathrm{d}s = 0$.

**Theorem 2.** For the MAS (1), suppose that Assumption 1 holds. Then, under protocol (2), the agents in the first group can achieve ASWC if conditions (C1) and (C3) hold. In the meantime, if conditions (C4)–(C7) hold, then the agents in the second group can also achieve ASWC.

*Proof.* Similarly, we can easily prove that the first "if" part is true under (C1) and (C3). Then we prove the second "if" part. According to the above analysis and [23], we now need to examine that for any initial state, $\lim_{t\to\infty}\|\zeta_k(t)\| = 0$, $k = 1,\ldots,l$. By means of L'Hôpital's rule we can get $\lim_{t\to\infty}|S_{k,n_k}(t)| \leqslant \lim_{t\to\infty}\frac{|d_1\gamma_1(t)|\kappa(t)}{\mathrm{Re}(\lambda_k)}$. We also have $|d_1\gamma_1(t)|\kappa(t) = |d_1\gamma_1(0) + c_5\sum_{i,j=1}^n\int_0^t g_1(s)\mathrm{d}\omega_{ji}(s)|\kappa(t)$, where $c_5 = d_1\sum_{k=1}^n \bar{v}_k\sum_{i,j=1}^n a_{ij}\sigma_{ji}$. Let $V(t) = c_5\sum_{i,j=1}^n\int_0^t g_1(s)\mathrm{d}\omega_{ji}(s)$. Then we have $\lim_{t\to\infty}\langle V\rangle(t) = c_3\int_0^t g_1^2(s)\mathrm{d}s$. Then from (C6) we can obtain $\lim_{t\to\infty}|S_{k,n_k}(t)| = 0$. From [22], (C4), (C6), and (C7), by the knowledge of martingale theory and some other estimation methods, we can obtain $\lim_{t\to\infty}\|Z_{k,n_k}(t)\| = 0$ and $\lim_{t\to\infty}\|Z'_{k,n_k}(t)\| = 0$. Then using the skills which are similar to the above analysis, we can obtain $\lim_{t\to\infty}\|\zeta_k(t)\| = 0$ for all $k = 1,\ldots,l$. Then we have the conclusion that the agents in both groups achieve ASWC.

**Remark 6.** From the analysis above, we can see that it is more difficult to analyze consensus problems of the second group. It is easy to see from (C6) that the time-varying intensity coefficient $\kappa(t)$ must be attenuated and its attenuation rate is related to $g_1(t)$, that is, the information received from another group vanishes at infinite time. We can understand this naturally in the following ways. Since different groups in MASs need to achieve different consensus and each agent in MASs has its own autonomous behavior, the group affected by agents in other groups can achieve consensus only when the influence is gradually attenuated.

**Remark 7.** In Theorem 2, condition (C6) reveals the attenuation rate of the influence between the two groups for achieving group consensus. In fact, condition (C6) can be proved to be necessary for the weak group consensus if $d_1 \neq 0$. Suppose that the weak group consensus holds and then $\mathbb{E}\|\zeta_{k,j}(t)\|^2 = 0, k = 1,\dots,l, j = 1,\dots,n_k$ for any $\lim_{t\to\infty}\|\zeta(0)\|^2 \neq 0$. Hence, from (12), (13), and $S_{k,n_k}(t)$ we can obtain that when $d_1 \neq 0$, $\lim_{t\to\infty}\kappa(t) = 0$, that is, the necessity of (C6) is obtained.

Above, we obtain some conditions on $g_1(t)$, $g_2(t)$, and $\kappa(t)$ for agents in both groups to achieve stochastic weak consensus. Note that the agents in different groups may be assigned different tasks, and different groups are then required to achieve different consensus behaviors, that is, agents in one group achieve stochastic weak consensus and agents in another group achieve stochastic strong consensus. We call this behavior a hybrid group consensus, which is discussed in Subsections 3.2 and 3.3.

### 3.2 Hybrid group consensus: weak+strong

Now, we first examine the hybrid group consensus, weak+strong consensus, that is, agents in two groups achieve stochastic weak consensus and stochastic strong consensus, respectively.

For the first group, we know that the agents can achieve stochastic weak consensus when conditions (C1)–(C3) hold. Then, to examine the stochastic strong consensus of the second group, we need the following conditions:

(C8) $\int_0^\infty g_2^2(t)\mathrm{d}t < \infty$;

(C9) $\int_0^\infty g_2(t)\kappa(t)\mathrm{d}t < \infty$.

**Remark 8.** Condition (C8) is called robustness condition. With this condition, the proposed consensus protocol can be robust against additive noises effectively [20], which together with (C1) guarantees stochastic strong consensus of MASs with additive noises. When each agent in group $\Omega_2$ only exchanges information with its neighbors in its own group, Zong et al. [22] told us that conditions (C4) and (C8) are sufficient to obtain the global stochastic strong consensus result. But in this paper, the agents in group $\Omega_2$ can receive the state information from group $\Omega_1$. In this case, existing some conditions are not sufficient to analyze the stochastic strong consensus of group $\Omega_2$, so we propose conditions (C9) to examine the consensus.

**Theorem 3.** For the MAS (1), suppose that Assumption 1 holds and $g_2(t)\kappa(t)$ is monotonic. Then, under protocol (2), when the agents in the first group can achieve stochastic weak consensus, the agents in the second group can achieve MSSC if conditions (C4)–(C6), (C8) and (C9) hold, and only if condition (C8) holds. Moreover, the agents in the second group can achieve ASSC if conditions (C4)–(C9) hold, and only if condition (C8) holds.

*Proof.* We first prove the two "if" parts. From the above analysis, we can get that the agents can achieve stochastic weak consensus for all $i \in \mathcal{V}_1$. Then we consider the second group. Let $\xi'(t) = Q_2^{-1}X_2(t) = [\xi'_1(t), \xi'_2(t), \dots, \xi'_m(t)]^{\mathrm{T}}$, $\bar{\xi}'(t) = [\xi'_2(t),\dots,\xi'_m(t)]^{\mathrm{T}}$, $\xi'_i(t) \in \mathbb{R}, i = 1,\dots,m$. Then we can get $\mathrm{d}\xi'_1(t) = -g_2(t)\kappa(t)\beta(t)\mathrm{d}t + \bar{\bar{\upsilon}}^{\mathrm{T}} g_2(t)\sum_{i,j=n+1}^{n+m} a_{ij}\sigma_{ji}\vartheta_{m,i-n}\mathrm{d}\omega_{ji}(t) + \bar{\bar{\upsilon}}^{\mathrm{T}} g_2(t)\kappa(t)\sum_{i=n+1}^{n+m}\sum_{j=1}^{n} a_{ij}\sigma_{ji}\vartheta_{m,i-n}\mathrm{d}\omega_{ji}(t)$, where $\beta(t) = \sum_{q=1}^{n} d'_q\gamma_q(t)$ is the first element of $Q_2^{-1}\mathcal{L}_{21}Q_1\gamma(t)$ and $[d'_1,\dots,d'_n]$ is the first row of $Q_2^{-1}\mathcal{L}_{21}Q_1$. Then we have

$$\begin{aligned}\xi'_1(t) =& \xi'_1(0) - \int_0^t g_2(s)\kappa(s)\beta(s)\mathrm{d}s \\ &+ \bar{\bar{\upsilon}}^{\mathrm{T}} \sum_{i,j=n+1}^{n+m} a_{ij}\sigma_{ji}\vartheta_{m,i-n}\int_0^t g_2(s)\mathrm{d}\omega_{ji}(s) \\ &+ \bar{\bar{\upsilon}}^{\mathrm{T}} \sum_{i=n+1}^{n+m}\sum_{j=1}^{n} a_{ij}\sigma_{ji}\vartheta_{m,i-n}\int_0^t g_2(s)\kappa(s)\mathrm{d}\omega_{ji}(s), \end{aligned} \tag{16}$$

where $\xi'_1(0) = \bar{\bar{\upsilon}}^{\mathrm{T}}X_2(0)$, $\bar{\bar{\upsilon}}^{\mathrm{T}}\sum_{i,j=n+1}^{n+m} a_{ij}\sigma_{ji}\vartheta_{m,i-n}\int_0^\infty g_2(s)\mathrm{d}\omega_{ji}(s)$, and $\bar{\bar{\upsilon}}^{\mathrm{T}}\sum_{i=n+1}^{n+m}\sum_{j=1}^{n} a_{ij}\sigma_{ji}\vartheta_{m,i-n} \times \int_0^\infty g_2(s)\kappa(s)\mathrm{d}\omega_{ji}(s)$ are well defined [20]. Then we have $\lim_{t\to\infty}\int_0^t g_2(s)\kappa(s)\beta(s)\mathrm{d}s = \lim_{t\to\infty} b_1 \times \int_0^t g_2(s)\kappa(s)\mathrm{d}s + \lim_{t\to\infty} b_2\sum_{i,j=1}^{n}\int_0^t g_2(s)\kappa(s)g_1(s)\mathrm{d}\omega_{ji}(s) + \lim_{t\to\infty}\sum_{q=2}^{n} d'_q\int_0^t \gamma_q(s)g_2(s)\kappa(s)\mathrm{d}s$, where

$b_1 = d_1'\gamma_1(0)$ and $b_2 = d_1'\bar{\upsilon}^{\rm T}\sum_{i,j=1}^{n} a_{ij}\sigma_{ji}\vartheta_{n,i}$. Noting that $\lim_{t\to\infty}\sum_{q=2}^{n} d_q'\gamma_q(t) = 0$, then from condition (C9) and Abel test we have $\lim_{t\to\infty}\sum_{q=2}^{n} d_q'\int_0^t \gamma_q(s)g_2(s)\kappa(s){\rm d}s < \infty$. From condition (C8) and the analysis above, we know $\sum_{i,j=1}^{n}\int_0^{\infty} g_2(s)\kappa(s)g_1(s){\rm d}\omega_{ji}(s)$ is also well defined, and then we have $\lim_{t\to\infty}\int_0^t g_2(s)\kappa(s)\beta(s){\rm d}s < \infty$. We also have

$$\begin{aligned}\mathbb{E}\|\xi_1'(t)|^2\| \leqslant& 4\|\xi_1'^2(0)\| + 4\mathbb{E}|\int_0^t g_2(s)\kappa(s)\beta(s){\rm d}s|^2\\ &+ a_1\int_0^t g_2^2(s){\rm d}s + a_2\int_0^t g_2^2(s)\kappa^2(s){\rm d}s,\end{aligned} \tag{17}$$

where $a_1$, $a_2$ are all constants, $a_1 = \sum_{k=1}^{m}\bar{\bar{\upsilon}}_k^2\sum_{i,j=n+1}^{n+m} a_{ij}^2\sigma_{ji}^2 > 0$, and $a_2 = \sum_{k=1}^{m}\bar{\bar{\upsilon}}_k^2\sum_{i=n+1}^{n+m}\sum_{j=1}^{n} a_{ij}^2 \sigma_{ji}^2 > 0$. Similar to the proof of $\int_0^{\infty} g_2(s)\kappa(s)\beta(s){\rm d}s < \infty$, we have $\lim_{t\to\infty}\mathbb{E}|\int_0^t g_2(s)\kappa(s)\beta(s){\rm d}s|^2 \leqslant \lim_{t\to\infty}\mathbb{E}|b_1\int_0^t g_2(s)\kappa(s){\rm d}s|^2 + \lim_{t\to\infty}\mathbb{E}|b_2\sum_{i,j=1}^{n}\int_0^t g_2(s)\kappa(s)g_1(s){\rm d}s|^2 < \infty$. Then we can get

$$\lim_{t\to\infty}\mathbb{E}\|\xi_1'(t) - \xi_1'^{\infty}\|^2 = 0, \quad \lim_{t\to\infty}\xi_1'(t) = \xi_1'^{\infty}, \tag{18}$$

where $\xi_1'^{\infty} = \xi_1'(0) - \int_0^{\infty} g_2(s)\kappa(s)\beta(s){\rm d}s + \bar{\bar{\upsilon}}^{\rm T}\sum_{i,j=n+1}^{n+m} a_{ij}\sigma_{ji}\vartheta_{m,i-n}\int_0^{\infty} g_2(s){\rm d}\omega_{ji}(s) + \bar{\bar{\upsilon}}^{\rm T}\sum_{i=n+1}^{n+m}\sum_{j=1}^{n} a_{ij} \times\sigma_{ji}\vartheta_{m,i-n}\int_0^{\infty} g_2(s)\kappa(s){\rm d}\omega_{ji}(s) < \infty$. We also have ${\rm d}\bar{\xi}'(t) = -g_2(t)\widetilde{\mathcal{L}}_{22}\xi'(t){\rm d}t - g_2(t)\kappa(t)P'\gamma(t){\rm d}t + {\rm d}W(t) + {\rm d}W'(t)$, where $P' = \overline{Q}_2\mathcal{L}_{21}Q_1$, $W(t) = \sum_{i,j=n+1}^{n+m} a_{ij}\sigma_{ji}\overline{Q}_2\vartheta_{m,i-n}\int_0^t g_2(s){\rm d}\omega_{ji}(s)$ and $W'(t) = \sum_{i=n+1}^{n+m}\sum_{j=1}^{n} a_{ij}\sigma_{ji}\overline{Q}_2\vartheta_{m,i-n}\int_0^t g_2(s)\kappa(s){\rm d}\omega_{ji}(s)$. By the similar skills used in the above analysis, we can easily get $\lim_{t\to\infty}\mathbb{E}\|\bar{\xi}'(t)\|^2 = 0$ and $\lim_{t\to\infty}\|\bar{\xi}'(t)\| = 0$. This together with (18) gives the conclusion that the first part of Theorem 3 holds and the final convergence state is $\lim_{t\to\infty} X_2(t) = \xi_1'^{\infty}\mathbf{1}_m$ when agents reach almost sure strong consensus for all $i \in \mathcal{V}_2$. And the MAS (1) achieves hybrid group consensus: weak+strong.

Then we prove the two "only if" parts. From (18), $\lim_{t\to\infty}\mathbb{E}|\xi_1'(t)|^2 < \infty$, $a_1 > 0$ and $a_2 > 0$, we can get the necessity of (C8).

Above, we have examined the case of hybrid group consensus: weak+strong. Then we investigate another case of hybrid group consensus: strong+weak, that is, agents in the two groups reach stochastic strong consensus and stochastic weak consensus, respectively.

### 3.3 Hybrid group consensus: strong+weak

For agents in the first group, we know that in addition to conditions (C1)–(C3), the following condition is also needed to explore stochastic strong consensus behaviors:

(C10) $\int_0^{\infty} g_1^2(t){\rm d}t < \infty$.

Then, for the hybrid group consensus behavior: strong+weak, we give the following theorem.

**Theorem 4.** For the MAS (1), suppose that Assumption 1 holds. Then, under protocol (2), the agents in the first group can achieve stochastic strong consensus if conditions (C1) and (C10) hold. Moreover, the agents in the second group can achieve MSWC if conditions (C4)–(C6) hold and the agents in the second group can achieve ASWC if conditions (C4)–(C7) hold.

*Proof.* By similar skills used in the analysis above, we have $\gamma_1(t) = \gamma_1(0) + \bar{\upsilon}^{\rm T}\sum_{i,j=1}^{n} a_{ij}\sigma_{ji}\vartheta_{n,i}\int_0^t g_1(s){\rm d}\omega_{ji}(s)$, and from (C10) we know that $\bar{\upsilon}^{\rm T}\sum_{i,j=1}^{n} a_{ij}\sigma_{ji}\vartheta_{n,i}\int_0^t g_1(s){\rm d}\omega_{ji}(s)$ is also well defined. Similar to the proof of Theorems 1 and 2, we can obtain that the MAS (1) achieves hybrid group consensus: strong+weak.

Above, we have investigated types (a), (b), and (c) of group consensus. Similarly, in Subsection 3.4, we study the type (d): stochastic strong group consensus.

### 3.4 Stochastic strong group consensus

Now, we study the stochastic strong group consensus: agents in both groups achieve stochastic strong consensus. From above analysis and conditions, we can get the following theorem directly.

**Theorem 5.** For the MAS (1), suppose that Assumption 1 holds and $g_2(t)\kappa(t)$ is monotonic. Then, under protocol (2), when the agents in the first group can achieve stochastic strong consensus, the agents

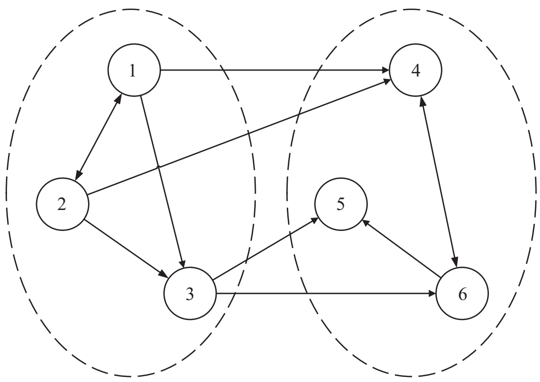


**Figure 1** The communication graph of a system with six agents.

in the second group can achieve MSSC if conditions (C4)–(C6), (C8) and (C9) hold, and only if condition (C8) holds. Moreover, the agents in the second group can achieve ASSC if conditions (C4)–(C9) hold, and only if condition (C8) holds.

*Proof.* The proof is similar to the above analysis in Subsection 3.3, and is omitted here.

**Remark 9.** From the above analysis we can find that the designs of $g_1(t)$ and $g_2(t)$ depend on the attenuation rate of $\kappa(t)$. These conditions tell us that when the unaffected group in the MAS can achieve stochastic consensus, in order to make the affected group achieve stochastic consensus, we can impose the joint condition on $g_1(t)$, $g_2(t)$, and $\kappa(t)$, where the attenuation rate of $\kappa(t)$ might be helpful for us to achieve the control of group consensus.

**Remark 10.** If the balance assumption in [13,14] holds, conditions (C1)–(C5), (C7), (C8) and (C10) are sufficient to analyze the problem studied in this paper. In fact, the balance assumption means that there does not exist substantial communication between the two groups; i.e., the close loop systems for the two groups are independent. Hence, in this case, the conditions of group consensus are the same as those of global consensus. This case excludes many types of topologies, and limits largely its application [18]. In this paper, we replace the assumption with Assumption 1, which may provide additional flexibility in applications.

## 4 Simulation

In this section, some simulation examples for the group consensus problems discussed in this paper are given to illustrate the effectiveness of the proposed group control protocol and conditions.

We give a system with six agents and the number of groups is 2, whose communication graph is given in Figure 1, where agents 1–3 belong to group $\Omega_1$, and agents 4–6 belong to group $\Omega_2$. We can get that $\mathcal{G}_1$ and $\mathcal{G}_2$ contain a spanning tree, respectively. The initial states are $X_1(0) = [4, -2, 5]^{\mathrm{T}}$ and $X_2(0) = [5, -8, -6]^{\mathrm{T}}$. Now, we assume that $\sigma_{ji} = 1, i, j = 1, 2, 3, 4, 5, 6$.

To verify that the given MAS can achieve stochastic weak group consensus under protocol (2), here, we consider time-varying control gain functions $g_1(t) = g_2(t) = (1+t)^{-0.4}$ and $\kappa(t) = (1+t)^{-0.8}$. Then it is easy to get that conditions (C1), (C3), and (C4)–(C7) hold. From Theorem 2 we have the conclusion that the given MAS can achieve weak group consensus in almost sure. The state trajectories of agents in the MAS are shown intuitively in Figure 2, which shows that all agents in the same group get together. Noting that condition (C2) also holds, then from Theorem 1 we know that agents in both groups can achieve MSWC. For such group consensus behavior, according to the definition of MSWC, we consider $\mathbb{E}|x_i(t) - x_1(t)|^2{}_{i=2,3}$ and $\mathbb{E}|x_i(t) - x_4(t)|^2{}_{i=5,6}$. We generate $10^3$ sample paths. Then, we have Figure 3, which shows that the given MAS achieves weak group consensus in mean square.

Considering $g_1(t) = (1+t)^{-0.5}$, $g_2(t) = (1+t)^{-1}$, and $\kappa(t) = (1+t)^{-0.5}$, the conditions (C1), (C3), and (C4)–(C9) hold. Theorem 3 tells us that the given MAS can achieve hybrid group consensus: weak+strong, which is shown in Figure 4 intuitively.

Now, we consider $g_1(t) = g_2(t) = (1+t)^{-1}$, $\kappa(t) = (1+t)^{-0.5}$. We can get that conditions (C1), (C10), and (C4)–(C9) hold. Other conditions are the same as above. From Theorem 5 we know that agents in both groups achieve ASSC. This type of stochastic strong group consensus behavior is shown intuitively in Figure 5.

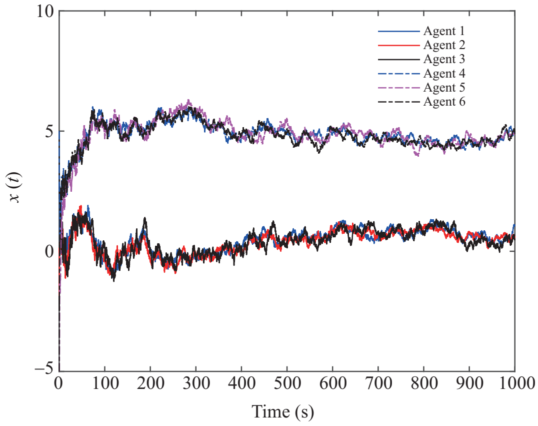


**Figure 2** (Color online) Almost-sure weak group consensus.

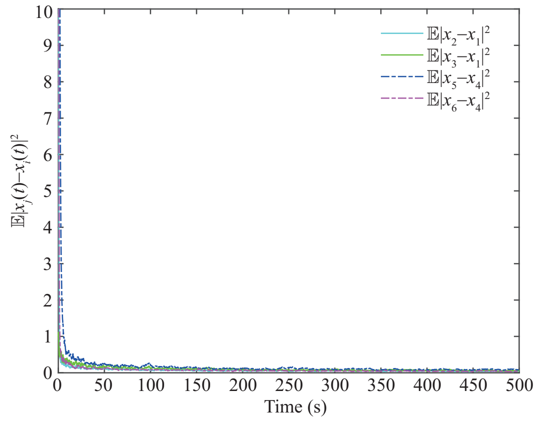


**Figure 3** (Color online) Mean square weak group consensus.

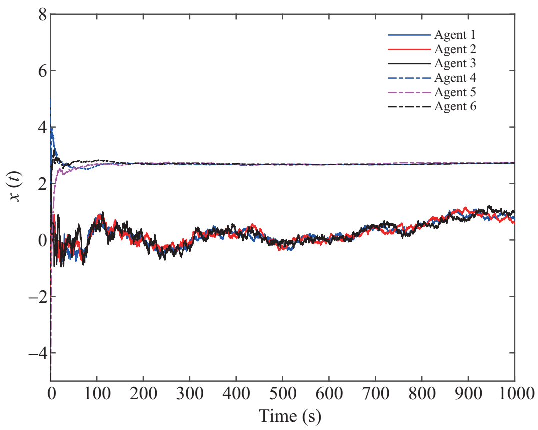


**Figure 4** (Color online) Hybrid group consensus: weak+strong.

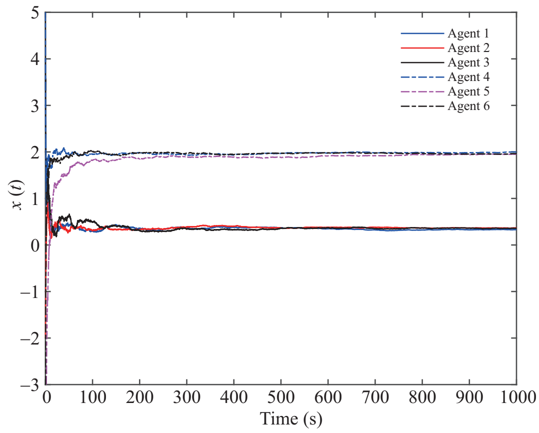


**Figure 5** (Color online) Almost-sure strong group consensus.

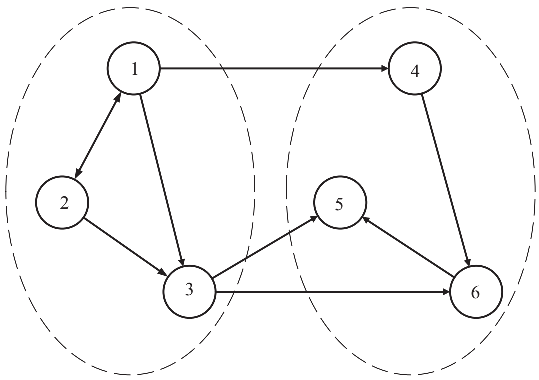


**Figure 6** The communication graph of a system with six agents.

Then, we give another system, whose communication graph is given in Figure 6, where agents 1–3 belong to group $\Omega_1$, and agents 4–6 belong to group $\Omega_2$. We can get that $\mathcal{G}_1$ and $\mathcal{G}_2$ contain a spanning tree, respectively. Other conditions are the same as above. We choose $g_1(t) = (1+t)^{-0.5}$, $g_2(t) = (1+t)^{-1}$, and $\kappa(t) = (1+t)^{-0.5}$. Then we have the conclusion that the given MAS can achieve hybrid group consensus: weak+strong, which is shown in Figure 7.

Now, we give a system with seven agents, where agents 1–3 belong to group $\Omega_1$, and agents 4–7 belong to group $\Omega_2$. $\mathcal{A} = [a_{ij}]_{7\times 7}$ with $a_{12} = a_{21} = a_{31} = a_{32} = a_{42} = a_{53} = a_{61} = a_{64} = a_{65} = a_{67} = a_{75} = 1$ and other values being zero. We can get that $\mathcal{G}_1$ contains a spanning tree and $\mathcal{G}_2$ does not contain

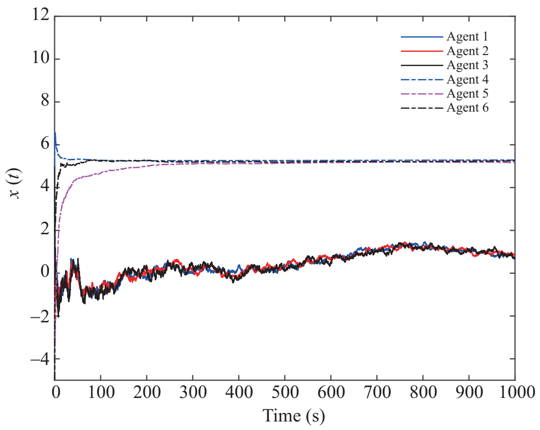


**Figure 7** (Color online) Hybrid group consensus: weak+strong.

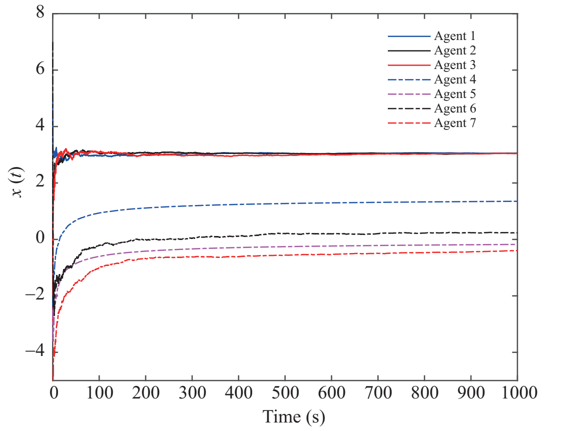


**Figure 8** (Color online) The counter-example: the given MAS cannot achieve group consensus.

a spanning tree but the whole topology graph contains a spanning tree. We give the initial states $X_1(0) = [5, -1, -3]^{\mathrm{T}}$, $X_2(0) = [-2, -2.5, 7, -6]^{\mathrm{T}}$, and assume that $\sigma_{ji} = 1, i, j = 1, 2, 3, 4, 5, 6, 7$. We consider $g_1(t) = g_2(t) = (1+t)^{-1}$ and $\kappa(t) = (1+t)^{-0.5}$. Then we obtain Figure 8, which shows that the given MAS cannot achieve group consensus. That is, the assumption that $\mathcal{G}_1$ and $\mathcal{G}_2$ contain a spanning tree respectively is necessary for the group consensus.

## 5 Conclusion

This paper addressed group consensus problems of MASs in additive noise environments. This paper supposed that the communication between the two groups was unidirectional, and agents in group $\Omega_2$ can receive the information from agents in group $\Omega_1$. Effects on the second group from the first group could be regarded as external robustness. However, the model cannot be considered a conventional dynamic system with an external disturbance since the effect is modeled as stochastic systems driven by additive noises, and the corresponding stability analysis has not been well established. By the semi-decoupled skill and some estimation methods, this paper concluded that the MASs can achieve pure group consensus and hybrid group consensus. Some coupling relationships between $\kappa(t)$, $g_1(t)$, and $g_2(t)$ were found. It is proved that the time-varying intensity coefficient $\kappa(t)$ must be attenuated, and its attenuation rate may be beneficial in designing appropriate time-varying control gain functions in the control protocol for achieving group consensus. Based on these findings, this paper developed sufficient conditions and some necessary conditions about the control gain functions for achieving different group consensus behaviors.

It is noted that it will be more complicated to study group consensus problems if the communication between the two groups is bidirectional. It will also be more challenging to analyze the four types of group consensus behaviors of MASs with both time-delays and noises. These issues still require further research.

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