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• **Supplementary File** •

Observer-based l_2-l_∞ control for singularly perturbed semi-Markov jump systems with an improved weighted TOD protocol

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Appendix A Notations and Specific Forms of Some Matrices

The notations employed in this work are standard. $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_1\times n_2}$ signify the Euclidean space of n_1-dimensional and the set of $n_1\times n_2$ real matrices, respectively; $\mathbb{Z}$ and $\mathbb{Z}_{[a,b]}$ denote the set of non-negative integers and the set of $\{c\in\mathbb{Z}|a\leqslant c\leqslant b\}$; $\mathcal{E}\{\cdot\}|_\chi$ is the conditional expectation operator conditioned on χ; $sym\{P\}$ means $P+P^T$; $tr(P)$ represents the trace of the square matrix P, and the minimum/maximum eigenvalue of matrix P is represented by $\lambda_{\max}\{P\}/\lambda_{\min}\{P\}$; $\vartheta(i)$ is Kronecker sign function satisfying $\vartheta(i)=1$ when $i=0$, and $\vartheta(i)=0$ otherwise; $\otimes$ signifies Kronecker product. In the following, some special forms of matrices utilized in the article are presented.

$$
\begin{aligned}
\tilde{A}_{a\xi(k)} &\triangleq \begin{bmatrix} \hat{\Phi}(\xi(k)) & \hat{C}_{a\xi(k)} & 0 \\ 0 & \hat{A}_{a\xi(k)} & -B_aK_{a\xi(k)} \\ -H_{a\xi(k)}\hat{\Phi}(\xi(k)) & \hat{H}_{a\xi(k)} & A_a-H_{a\xi(k)}C_a \end{bmatrix}, \bar{I}_{12}\triangleq\begin{bmatrix} 0 & I_{n_1} & 0 \\ I_{n_2} & 0 & 0 \\ 0 & 0 & I_{n_2}\end{bmatrix}, \bar{I}_{21}\triangleq\begin{bmatrix} 0 & I_{n_2} & 0 \\ I_{n_1} & 0 & 0 \\ 0 & 0 & I_{n_2}\end{bmatrix}\\
E_{I43} &\triangleq \mathrm{diag}\{I_{Nn+n_1},\bar{I}_{21}\}, \hat{\Phi}(\xi(k))\triangleq I_{Nn}-\Phi(\xi(k)), \hat{A}_{a\xi(k)}\triangleq A_a+B_aK_{a\xi(k)}, \bar{A}_{a\xi(k)}=E_{I34}\tilde{A}_{a\xi(k)}E_{I43}\\
\tilde{C}_a &\triangleq \begin{bmatrix} -I_{Nn} & C_a & 0\end{bmatrix}, \bar{D}_a=E_{I34}\tilde{D}_a, \bar{G}_a=\tilde{G}_aE_{I43}, \bar{E}_\epsilon=E_{I34}\tilde{E}_\epsilon E_{I43}, \tilde{E}_\epsilon\triangleq\mathrm{diag}\{I_{Nn},E_\epsilon,E_\epsilon\}, \tilde{G}_a\triangleq\begin{bmatrix} 0 & G_a & 0\end{bmatrix}\\
\bar{C}_a &= \tilde{C}_aE_{I43}, \tilde{D}_a\triangleq\begin{bmatrix} 0 & D_a^T & D_a^T\end{bmatrix}^T, \hat{C}_{a\xi(k)}\triangleq\Phi(\xi(k))\hat{\Delta}(k)C_a, \hat{H}_{a\xi(k)}\triangleq H_{a\xi(k)}[I-\Phi(\xi(k))\hat{\Delta}(k)]C_a, \hat{\Delta}(k)\triangleq I_{Nn}+\Delta(k).
\end{aligned}
$$

Remark 1. It can be observed that the expression of $\eta(k)$ has a special form $\tilde{E}_\epsilon\triangleq\mathrm{diag}\{I_{Nn},I_{n_1},\epsilon I_{n_2},I_{n_1},\epsilon I_{n_2}\}$ due to the augmented form of the system. Therefore, the contract matrices E_{I34} and E_{I43} satisfying $E_{I34}=E_{I43}^T$ and $E_{I34}E_{I43}=E_{I43}E_{I34}=I_{Nn+2n_x}$ are provided for matrix transformation, by which the matrix $\tilde{E}_\epsilon$ is transformed into $\bar{E}_\epsilon\triangleq\mathrm{diag}\{I_{Nn+2n_1},\epsilon I_{2n_2}\}$. Consequently, conventional analysis methods for singularly perturbed systems can be adopted here.

Appendix B Semi-Markov Chain and Quantization

Taking values in the set $\bar{M}\triangleq\{1,2,\ldots,M\}$, $\{\sigma(k)\}_{k\in\mathbb{Z}_{>0}}$ represents a semi-Markov chain. For $\forall m\in\mathbb{Z}_{>0}$, k_m means the time at the mth jump with the first jump being denoted as $k_0=0$. R_m signifies the mode index of the system at the mth jump. T_m denotes the sojourn-time of mode R_m, which satisfies $T_m=k_{m+1}-k_m$.

For $\forall l\in\bar{N}\triangleq\{1,\ldots,N\}$, the quantization levels is given as:

$$Q_l=\{\pm q_l^{(d)}|q_l^{(d)}=\rho_l^dq_l^{(0)},d=0,\pm1,\pm2,\ldots\}\cup\{0\}$$

where $\rho_l\in(0,1)$ denotes quantization density; $q_l^{(0)}$ stands for the input from quantizer. By denoting $\varphi_l\triangleq(1-\rho_l)/(1+\rho_l)$, the quantizer can be given as:

$$g_l(v)\triangleq\begin{cases} q_l^{(d)}, & v\in(q_l^{(d)}/(1+\varphi_l),q_l^{(d)}/(1-\varphi_l)]\\ 0, & v=0\\ -g_l(-v), & v<0.\end{cases}$$

Then, since quantization error is defined as $\tilde{y}(k)-y(k)\triangleq\Delta(k)y(k)$, one can get $\Delta(k)\leqslant\Upsilon\triangleq\mathrm{diag}\{\varphi_1,\ldots,\varphi_N\}\otimes I_n\leqslant I_{Nn}$.

Remark 2. Figure B1 is provided to show the structure of the observer-based networked control system with a constrained communication channel. During the process of controller design, the measurement output rather than state information is utilized for the construction. This is due mainly to the fact that the state of physical systems is often intractable to be measured by sensors directly. Therefore, to deal with this situation, an observer is designed to estimate system state and then a controller can be designed by virtue of the estimated state. It is clearly observed from the figure that the measurement output detected by the sensors is quantized before transmission. Subsequently, by comparing the variation values of each sensor node, the single

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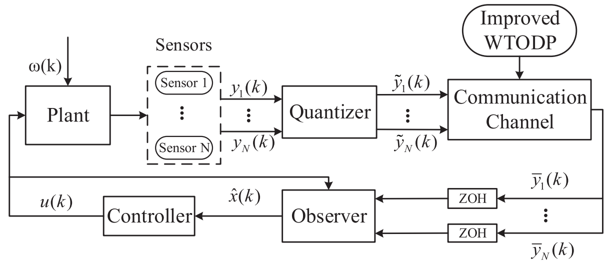

Figure B1 The structure of the observer-based networked control system.

node obtaining the transmission permission can be determined and then the data communication between sensors and observer is realized. By virtue of the obtained data, the state information of the system can be estimated by the observer, based on which the expected controller can be constructed. Worth mentioning, the process of determining data transmission is based on the assumption that time delays are neglected, and the feasibility of Figure B1 is based on an ideal computing and communication environment. Further introducing time delay into the transmission model is an important issue worthy study in future research.

Remark 3. To save limited communication resources, transmission protocols are widely applied. The features of the improved weighted try-once-discard protocol (WTODP) mechanism proposed in this paper mainly embodied in the following aspects. When the error of each node is not equal to zero simultaneously, only the single node with the maximum weighted error of data between the current instant and the previously transmitted instant gets the access to the communication network. Reversely, if the weighted error of all nodes is equal to zero, then there is no sensor node obtaining the transmit permission. Obviously, the data transmission in the traditional WTODP is carried out at each time instant even if the weighted error tends to zero and the data updating is not necessary. While in the improved WTODP, the data transmission is suspended when no sensor node needs to be updated. In contrast, the latter transmits less data, which is beneficial to further save communication resources. Meanwhile, the transmission frequency is lower than the former, which is conducive to reducing the wear of switching devices.

Appendix C Definitions and Lemmas

Definition 1. [1] For semi-Markov chain $\{\sigma(k)\}_{k\in\mathbb{Z}_{>0}}$, the semi-Markov kernel (SMK) $\bar{\Theta}(\tau) \triangleq \left[\bar{\theta}_{\alpha\beta}(\tau)\right]_{\alpha,\beta\in\bar{M}}$ can be defined as:

$$\bar{\theta}_{\alpha\beta}(\tau) \triangleq \Pr\{R_{m+1}=\beta, T_m=\tau|R_m=\alpha\} = \pi_{\alpha\beta} f_{\alpha\beta}(\tau), \forall\alpha,\beta\in\bar{M}$$

where $\pi_{\alpha\beta} \triangleq \Pr\{R_{m+1}=\beta|R_m=\alpha\}$ satisfying $1 \geqslant \pi_{\alpha\beta} \geqslant 0$ and $\pi_{\alpha\alpha}=0$ is the transition probability (TP); $f_{\alpha\beta}(\tau) \triangleq \Pr\{T_m=\tau|R_{m+1}=\beta, R_m=\alpha\}$ with $f_{\alpha\alpha}(\tau)=0$ is the probability density function of the sojourn-time (ST). Accordingly, the TP matrix can be given as $\Pi \triangleq [\pi_{\alpha\beta}]_{\alpha,\beta\in\bar{M}}$, which meets $\sum_{\beta=1}^{M}\pi_{\alpha\beta}=1$ for $\forall\alpha\in\bar{M}$.

Lemma 1. [1,2] For stochastic switched system $\bar{\eta}(k+1)=f_{\sigma(k)}(\bar{\eta}(k))$ with bounded sojourn-time and $\sigma(k)\in\bar{M}$, if there exists a set of functions $V_{\xi(k)}(\bar{\eta}(k),\sigma(k),\tau)\in\mathbb{R}_{\geqslant 0}$ with $k\in\mathbb{Z}_{[k_m,k_{m+1})}$, $\tau=k-k_m\in\left[1,\bar{d}_{\sigma(k)}\right]$ (time spent in current mode), and three class $\mathcal{K}_\infty$ functions $\psi_1(\cdot)$, $\psi_2(\cdot)$, $\psi_3(\cdot)$, such that for $\forall k\in\mathbb{Z}_{[k_m,k_{m+1})}$, given finite constants $l_{\sigma(k)}>0$ ($\sigma(k)\in\bar{M}$), and any initial conditions $\bar{\eta}(0)\in\mathbb{R}^{Nn+2n_x}$, $\sigma(0)\in\bar{M}$, there hold

$$\psi_1(\|\bar{\eta}(k)\|) \leqslant V_{\xi(k)}(\bar{\eta}(k),\sigma(k),\tau) \leqslant \psi_2(\|\bar{\eta}(k)\|) \tag{C1}$$

$$V_{\xi(k)}(\bar{\eta}(k),\sigma(k),\tau) \leqslant l_{\sigma(k_m)} V_{\xi(k_m)}(\bar{\eta}(k_m),\sigma(k_m),0) \tag{C2}$$

$$\mathcal{E}\{V_{\xi(k_{m+1})}(\bar{\eta}(k_{m+1}),\sigma(k_{m+1}),0)\}|_{\bar{\eta}(k_m),R_m} - V_{\xi(k_m)}(\bar{\eta}(k_m),\sigma(k_m),0) \leqslant -\psi_3(\|\bar{\eta}(k_m)\|) \tag{C3}$$

then, the system is δ-error mean-square stable (δ-EMSS).

Definition 2. [3] The closed-loop system is said to be δ-error mean-square stable (δ-EMSS) with a prescribed l_2-l_∞ performance level $\bar{\sigma}$, if closed-loop system is δ-EMSS, and under zero-initial conditions there exists a scalar $\bar{\sigma}>0$ such that for any nonzero $\omega(k)$, the following condition holds:

$$\sup_{0\leqslant l\leqslant\mathcal{N}}\mathcal{E}\{z^T(l)z(l)\} \leqslant \bar{\sigma}^2\sum_{l=0}^{\mathcal{N}}\omega^T(l)\omega(l). \tag{C4}$$

Lemma 2. [4,5] $\epsilon^2 J_1+\epsilon J_2+J_3<0$ holds for $\forall\epsilon\in(0,\bar{\epsilon}]$, $\bar{\epsilon}>0$, if *(a)* $J_1\geqslant 0$; *(b)* $J_3<0$ and *(c)* $\bar{\epsilon}^2J_1+\bar{\epsilon}J_2+J_3<0$ hold simultaneously.

Appendix D Proof of Theorem 1

Step 1: Under the condition of $\omega(k)\equiv 0$, we prove that the CLS (6) is δ-EMSS.

First of all, one can get from (7) that

$$V_{\xi(k)}(\bar{\eta}(k),\sigma(k),\tau) = \bar{\eta}^T(k)[P_{\sigma(k)}(\tau)+\bar{C}_{\sigma(k)}^T\check{Q}_{\xi(k)}\bar{C}_{\sigma(k)}]\bar{\eta}(k)$$

which means

$$\bar{\psi}_1\|\bar{\eta}(k)\|^2 \leqslant V_{\xi(k)}(\bar{\eta}(k),\sigma(k),\tau) \leqslant \bar{\psi}_2\|\bar{\eta}(k)\|^2 \tag{D1}$$

where $\bar{\psi}_1 \triangleq \min_{\forall a\in\bar{M},i\in\tilde{N},\tau\in[0,\bar{d}_a]}\lambda_{\min}\{\Gamma_{ai\tau}\}$, $\bar{\psi}_2 \triangleq \max_{\forall a\in\bar{M},i\in\tilde{N},\tau\in[0,\bar{d}_a]}\lambda_{\max}\{\Gamma_{ai\tau}\}$ with $\Gamma_{ai\tau}\triangleq P_a(\tau)+\bar{C}_a^T\check{Q}_i\bar{C}_a$. Therefore, the condition (C1) in Lemma 1 is satisfied.

On the other hand, the following inequality can be obtained from (8) easily under the condition of $\omega(k)\equiv 0$.

$$V_{\xi(k)}(\bar{\eta}(k),\sigma(k),\tau) \leqslant V_{\xi(k_m)}(\bar{\eta}(k_m),\sigma(k_m),0). \tag{D2}$$

Then, the condition (C2) can be derived with the given constants $l_{\sigma(k)} > 0$ ($\sigma(k) \in \bar{M}$) taken as a fixed value 1.

Moreover, by iterating over the inequality (8) within the interval $[k_m, k_{m+1})$, together with the utilizing of (9), it can be inferred that

$$\begin{aligned}&\mathcal{E}\left\{V_{\xi(k_{m+1})}(\bar{\eta}(k_{m+1}), \sigma(k_{m+1}), 0)\right\}|_{\bar{\eta}(k_m),\sigma(k_m)} \leqslant \mathcal{E}\left\{V_{\xi(k_{m+1})}(\bar{\eta}(k_{m+1}), \sigma(k_m), T_m)\right\}|_{\bar{\eta}(k_m),\sigma(k_m)}\\ \leqslant\ & \mathcal{E}\left\{V_{\xi(k_m)}(\bar{\eta}(k_m), \sigma(k_m), 0) + \sum\nolimits_{l=k_m}^{k_{m+1}-1} \omega^T(l)\omega(l)\right\}|_{\bar{\eta}(k_m),\sigma(k_m)}\\ \leqslant\ & V_{\xi(k_m)}(\bar{\eta}(k_m), \sigma(k_m), 0) + \sum\nolimits_{l=k_m}^{k_{m+1}-1} \omega^T(l)\omega(l). \qquad \text{(D3)}\end{aligned}$$

Synthesize (D1)-(D3), one can naturally concluded from Lemma 1 that the CLS (6) is δ-EMSS.

Step 2: Under zero-initial conditions, the $l_2 - l_\infty$ performance for system (6) is proved as follows.

Consider that $k \in [k_m, k_{m+1})$, one can infer from (8) that

$$V_{\xi(k_{m+1})}(\bar{\eta}(k_{m+1}), \sigma(k_{m+1}), T_m) \leqslant V_{\xi(k_m)}(\bar{\eta}(k_m), \sigma(k_m), 0) + \sum\nolimits_{l=k_m}^{k_{m+1}-1} \omega^T(l)\omega(l). \qquad \text{(D4)}$$

Together with the condition (9), one can obtain

$$\mathcal{E}\left\{V_{\xi(k_{m+1})}(\bar{\eta}(k_{m+1}), \sigma(k_{m+1}), 0)\right\}|_{\bar{\eta}(k_m),\sigma(k_m)} \leqslant V_{\xi(k_m)}(\bar{\eta}(k_m), \sigma(k_m), 0) + \sum\nolimits_{l=k_m}^{k_{m+1}-1} \omega^T(l)\omega(l).$$

Consider that $k \in [k_m, k_{m+1})$, the following can subsequently be derived

$$\mathcal{E}\left\{V_{\xi(k)}(\bar{\eta}(k), \sigma(k), \tau)\right\}|_{\bar{\eta}(k_0),\sigma(k_0)} \leqslant V_{\xi(k_0)}(\bar{\eta}(k_0), \sigma(k_0), 0) + \sum\nolimits_{l=k_0}^{k-1} \omega^T(l)\omega(l). \qquad \text{(D5)}$$

which combining with the zero-initial conditions means

$$\mathcal{E}\left\{V_{\xi(k)}(\bar{\eta}(k), \sigma(k), \tau)\right\}|_{\bar{\eta}(k_0),\sigma(k_0)} \leqslant \sum\nolimits_{l=k_0}^{k-1} \omega^T(l)\omega(l). \qquad \text{(D6)}$$

Furthermore, by virtue of the inequality (10), it can be elicited from (D6) and (6) that

$$\mathcal{E}\left\{z^T(k)z(k)\right\}|_{\bar{\eta}(k_0),\sigma(k_0)} \leqslant \gamma^2 \sum\nolimits_{l=k_0}^{k-1} \omega^T(l)\omega(l).$$

Appendix E Proof of Theorem 2

Obviously, it can be inferred from Lemma 2 and inequalities (12), (13) that for $\forall \epsilon \in (0, \bar{\epsilon}]$, we have

$$-(\bar{E}_\epsilon \mathcal{P}_{a\xi(k)}(\tau)\bar{E}_\epsilon)^{-1} \leqslant \bar{E}_\epsilon \mathcal{P}_{a\xi(k)}(\tau)\bar{E}_\epsilon - 2I < -\Omega_{\xi(k)a}(\tau) \qquad \text{(E1)}$$

where

$$\mathcal{P}_{a\xi(k)}(\tau) \triangleq P_a(\tau+1) + \sum\nolimits_{j=1}^{N} \lambda_{\bar{\xi}j} \bar{C}_a^T \check{Q}_j \bar{C}_a$$

satisfying $\sum_{j=1}^{N} \lambda_{\bar{\xi}j} = 1$.

Furthermore, by virtue of the Lemma 2.4 in [6] and Schur complement, together with the utilizing of (E1), the following inequality can be obtained under the condition of (11)

$$\Lambda_{\xi(k)a}(\tau) \triangleq \begin{bmatrix} \bar{\phi}^1_{\xi(k)a}(\tau) & \bar{A}^T_{a\xi(k)}\bar{E}_\epsilon \mathcal{P}_{a\xi(k)}(\tau)\bar{E}_\epsilon \bar{D}_a \\ * & \bar{D}_a^T \bar{E}_\epsilon \mathcal{P}_{a\xi(k)}(\tau)\bar{E}_\epsilon \bar{D}_a - I \end{bmatrix} < 0 \qquad \text{(E2)}$$

where

$$\bar{\phi}^1_{\xi(k)a}(\tau) \triangleq \phi^1_{\xi(k)a}(\tau) - \varrho_a E_{I34}^T \mathcal{C}_a^T \Upsilon^T \Upsilon_a \mathcal{C}_a E_{I43} + \bar{A}^T_{a\xi(k)}\bar{E}_\epsilon \mathcal{P}_{a\xi(k)}(\tau)\bar{E}_\epsilon \bar{A}_{a\xi(k)}.$$

Consider that $k \in \mathbb{Z}_{[k_m, k_{m+1})}$. Then, based on the above analysis, it can be elicited from (7) and constraints (i) and (ii) in Remark 1 of letter that for $\forall \sigma(k) \triangleq a \in \bar{M}$, $\tau \in [0, \bar{d}_a)$, the following condition holds

$$\begin{aligned}&V_{\xi(k+1)}(\bar{\eta}(k+1), a, \tau+1) - V_{\xi(k)}(\bar{\eta}(k), a, \tau) - \omega^T(k)\omega(k)\\ \leqslant\ & \bar{\eta}^T(k+1)P_a(\tau+1)\bar{\eta}(k+1) - \omega^T(k)\omega(k) + \check{y}^T(k+1)\sum\nolimits_{j=1}^{N} \lambda_{\bar{\xi}j}\check{Q}_j \check{y}(k+1)\\ &-\bar{\eta}^T(k)P_a(\tau)\bar{\eta}(k) - \check{y}^T(k)\check{Q}_{\xi(k)}\check{y}(k) + \sum\nolimits_{j=1}^{N} \vartheta(\xi(k)-0)\check{y}^T(k)\bar{l}_j \hat{Q}\Phi(j)\check{y}(k)\\ =\ & \hat{\eta}^T(k)\Lambda_{\xi(k)a}(\tau)\hat{\eta}(k) < 0\end{aligned}$$

with $\hat{\eta}(k) \triangleq \begin{bmatrix} \bar{\eta}^T(k) & \omega^T(k) \end{bmatrix}^T$, which means (8) is satisfied.

On the other hand, it can be obtained from (6), (7) and the concept of semi-Markov jump that

$$\begin{aligned}&\mathcal{E}\left\{V_{\xi(k_{m+1})}(\bar{\eta}(k_{m+1}), \sigma(k_{m+1}), 0)\right\}|_{\bar{\eta}(k_m),\sigma(k_m)} - \mathcal{E}\left\{V_{\xi(k_{m+1})}(\bar{\eta}(k_{m+1}), \sigma(k_m), T_m)\right\}|_{\bar{\eta}(k_m),\sigma(k_m)}\\ =\ & \mathcal{E}\left\{\check{y}(k_{m+1})^T \check{Q}_{\xi(k_{m+1})}\check{y}(k_{m+1}) + \bar{\eta}^T(k_{m+1})P_{\sigma(k_m)}(0)\bar{\eta}(k_{m+1})\right\}|_{\bar{\eta}(k_m),\sigma(k_m)}\end{aligned}$$

$$
\begin{aligned}
&-\mathcal{E}\left\{\breve{y}(k_{m+1})^T\check{Q}_{\xi(k_{m+1})}\breve{y}(k_{m+1})+\bar{\eta}^T(k_{m+1})P_{\sigma(k_m)}(T_m)\bar{\eta}(k_{m+1})\right\}|_{\bar{\eta}(k_m),\sigma(k_m)}\\
=&\sum\nolimits_{d=1}^{\bar{d}_a}\sum\nolimits_{b\in\bar{M}}\frac{\bar{\theta}_{ab}(d)}{\varsigma_a}\bar{\eta}^T(k_{m+1})P_b(0)\bar{\eta}(k_{m+1})-\sum\nolimits_{d=1}^{\bar{d}_a}\sum\nolimits_{b\in\bar{M}}\frac{\bar{\theta}_{ab}(d)}{\varsigma_a}\bar{\eta}^T(k_{m+1})P_a(d)\bar{\eta}(k_{m+1})\\
=&\sum\nolimits_{d=1}^{\bar{d}_a}\sum\nolimits_{b\in\bar{M}}\frac{\bar{\theta}_{ab}(d)}{\varsigma_a}\bar{\eta}^T(k_{m+1})(P_b(0)-P_a(d))\bar{\eta}(k_{m+1})\\
=&\sum\nolimits_{d=1}^{\bar{d}_a}\bar{\eta}^T(k_m+d)\hat{P}_a(d)\bar{\eta}(k_m+d).
\end{aligned}
$$

Since (14) means $\hat{P}_a(d)<0$, it can be deduced that

$$
\mathcal{E}\left\{V_{\xi(k_{m+1})}(\bar{\eta}(k_{m+1}),\sigma(k_{m+1}),0)\right\}|_{\bar{\eta}(k_m),\sigma(k_m)}-\mathcal{E}\left\{V_{\xi(k_{m+1})}(\bar{\eta}(k_{m+1}),\sigma(k_m),T_m)\right\}|_{\bar{\eta}(k_m),\sigma(k_m)}\leqslant 0.
$$

Therefore, (9) holds under the constraint of (14).

Moreover, when $k\in\mathbb{Z}_{[k_m,k_{m+1})}$, for $\forall\tau\in\left[0,\bar{d}_a\right)$, $a\in\bar{M}$, it can be obtained from (15) that

$$
z^T(k)z(k)-\gamma^2V_{\xi(k)}(\bar{\eta}(k),a,\tau)<0.
$$

Consequently, the establishment of (10) is guaranteed.

Remark 4. The computational complexity of the conditions in theorems mainly lies in the following three aspects.

(a) Since the quantization, improved WTODP and observer-based controller are taken into account in the analysis of singularly perturbed semi-Markov jump systems simultaneously, the disposing process encounters great obstacles. Firstly, it can be observed from the expression of systems model (1) that system parameters exhibit switching feature, and the semi-Markov jump signal $\sigma(k)$ is introduced to govern the switching. Moreover, the two-time-scale phenomenon of the system dynamics are considered, thus a singularly perturbation parameter ϵ is introduced into the investigated model. The existence of $\sigma(k)$ and ϵ brings heavy computing burden to the process of design suitable controller and observer gains.

(b) Secondly, the transmission of the sensor data is processed by the quantizer and the improved WTODP. Then, the constraints about the error between the original measurement output $y(k)$ and the output of the quantizer $\tilde{y}(k)$ should be analyzed explicitly based on the mechanism of the logarithmic quantizer. In addition, with regard to the improved WTODP, the sophisticated term $\breve{y}(k)^T\check{Q}_{\xi(k)}\breve{y}(k)$ which involves the weighted error between the current measurement output and the previously transmitted data is brought into $V_{\xi(k)}(\bar{\eta}(k),\sigma(k),\tau)$. Moreover, two protocol-dependent conditions are skillfully constructed. Although the feasibility and less conservatism of the obtained criteria are guaranteed, these processes make the conditions obtained more sophisticated.

(c) Thirdly, the relationships among these conditions are intricate which brings great difficulty to the solving of these conditions. In addition to the complex process of calculating the desired observer and controller gain matrices, the implementation of the control scheme under the affection of semi-Markov jump parameter, quantization and improved WTODP is also intractable. Especially, with the increase of the number of subsystems or sensors, the amount of computation increases in proportion.

How to further reduce the computational complexity of related issues is crucial in control scheme design.

Remark 5. The derived conditions in this paper are sufficient, which is mainly caused by the construction of Lyapunov functions and the utilization of some special inequalities. Notably, the conservatism of these conditions may bring great difficulties to the seeking of satisfactory controller and observer gain matrices. Therefore, exploring proper strategy such as using stricter inequalities or constructing more flexible Lyapunov functions to reduce the conservatism of the obtained criteria is an interesting issue worthy pursuing further.

Remark 6. It should be pointed out that the two conditions (i) and (ii) given in Remark 1 of the letter are in fact designed from the following two perspectives: (I) For condition (i), the comparisons of the weighted error of the node obtaining transmission permission and the combinational weighted error of all nodes are given at time instant $k+1$. It is in fact utilized to deal with the protocol related term $\breve{y}(k+1)^T\check{Q}_{\xi(k+1)}\breve{y}(k+1)$ which appears in the difference of Lyapunov function $V_{\xi(k)}(\bar{\eta}(k),\sigma(k),\tau)$. And it has the similar principle of the application of inequalities derived from the triggering laws in the study of event-triggered control. (II) For condition (ii) whether or not the weighted error value being zero is discussed at time instant k. This consideration is driven by the expectation to make full use of the available transmission information associated with the improved WTODP, which is expected to further reduce the conservatism of the obtained results. Furthermore, it can be noted from (i) that the node obtaining transmission at time instant $k+1$ is $\xi(k+1)$ with $\xi(k+1)\in\tilde{N}$. If $\xi(k+1)=0$, it means that no node obtains communication permission at time instant $k+1$ and $\Phi(\xi(k+1))=0$. Then, according to the principle (3) of the improved WTODP, we can obtain that $\varepsilon_y(k+1)=0$, i.e., $\breve{y}(k+1)=0$. Then, for any $\lambda_{\bar{\xi}j}>0$ satisfying $\sum_{j=1}^N\lambda_{\bar{\xi}j}=1$, $\bar{\xi}\in\bar{N}$, it can be naturally obtained that $\breve{y}(k+1)^T\check{Q}_{\xi(k+1)}\breve{y}(k+1)\leqslant\breve{y}(k+1)^T\sum_{j=1}^N\lambda_{\bar{\xi}j}\check{Q}_j\breve{y}(k+1)$.If $\xi(k+1)=q$, $q\in\bar{N}$, we can obtain $\Phi(\xi(k))=\mathrm{diag}\{0,\ldots,0,I,0,\ldots,0\}$ with the qth diagonal element been I. Since $\check{Q}_{\xi(k)}\triangleq\hat{Q}-\hat{Q}\Phi(\xi(k))$, it means $\check{Q}_{\xi(k+1)}=\mathrm{diag}\{Q_1,\ldots,Q_{q-1},0,Q_{q+1},\ldots Q_N\}$. Then, we can get

$$
\begin{aligned}
\breve{y}(k+1)^T\check{Q}_{\xi(k+1)}\breve{y}(k+1)\;&=\;\sum\nolimits_{i=1,i\neq q}^N\breve{y}_i^T(k+1)Q_i\breve{y}_i(k+1)\\
&=\;\sum\nolimits_{i=1}^N\breve{y}_i^T(k+1)Q_i\breve{y}_i(k+1)-\breve{y}_q^T(k+1)Q_q\breve{y}_q(k+1).
\end{aligned}
$$

Furthermore, $\breve{y}(k+1)^T\sum_{j=1}^N\lambda_{\bar{\xi}j}\check{Q}_j\breve{y}(k+1)=\sum_{i=1}^N\breve{y}_i^T(k+1)Q_i\breve{y}_i(k+1)-\sum_{i=1}^N\breve{y}_i^T(k+1)\lambda_{\bar{\xi}i}Q_i\breve{y}_i(k+1)$. According to the principle (3) of the improved WTODP, the weighted error of node q is no less than the weighted error of other node. Thus, $\breve{y}_q^T(k+1)Q_q\breve{y}_q(k+1)\geqslant\breve{y}_i^T(k+1)\lambda_{\bar{\xi}i}Q_i\breve{y}_i(k+1)$ for any $i\in\bar{N}$, which implies

$$
\breve{y}(k+1)^T\check{Q}_{\xi(k+1)}\breve{y}(k+1)\leqslant\breve{y}(k+1)^T\sum\nolimits_{j=1}^N\lambda_{\bar{\xi}j}\check{Q}_j\breve{y}(k+1).
$$

This is the deduce process of condition (i). As for condition (ii), it can be noted that if node obtains transmission permission at time instant k, then $\xi(k)=0$ and $\vartheta(\xi(k)-0)=1$. Based on the principle (3), we can get $\breve{y}(k)=0$. Therefore, $\sum_{j=1}^N\vartheta(\xi(k)-0)\breve{y}^T(k)\bar{l}_j\hat{Q}\Phi(j)\breve{y}(k)=0$ holds for $\forall\bar{l}_j$ $(j\in\bar{N})$. If node q obtains transmission permission at time instant k, then $\xi(k)\neq 0$ and $\vartheta(\xi(k)-0)=0$. Therefore, for any $\bar{l}_j, j\in\bar{N}$, $\sum_{j=1}^N\vartheta(\xi(k)-0)\breve{y}^T(k)\bar{l}_j\hat{Q}\Phi(j)\breve{y}(k)=0$.

Appendix F Simulation

The parameters of the networked system with three jumping modes and three sensor nodes is described as follows:

$$
\begin{aligned}
A_1 &= \begin{bmatrix} 1.02 & 0.11 & 1.01 \\ 0.12 & 0.85 & 0.99 \\ 0.12 & 0.17 & 0.98 \end{bmatrix}, A_2 = \begin{bmatrix} 0.95 & 0.02 & 0.42 \\ 0.09 & 0.97 & 1.06 \\ 0.23 & 0.15 & 0.95 \end{bmatrix}, A_3 = \begin{bmatrix} 0.98 & 0.08 & 1.03 \\ 0.15 & 0.84 & 1.12 \\ 0.25 & 0.24 & 1.03 \end{bmatrix} \\
D_1 &= \begin{bmatrix} 0.40 \\ 0.16 \\ 0.32 \end{bmatrix}, D_2 = \begin{bmatrix} 0.35 \\ 0.14 \\ 0.28 \end{bmatrix}, D_3 = \begin{bmatrix} 0.45 \\ 0.18 \\ 0.36 \end{bmatrix}, C_a = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.5 & 0.8 & 0.1 \\ 0.3 & 0.1 & 0.7 \end{bmatrix} \\
B_1 &= \mathrm{diag}\{1.2, 1.4, -0.8\}, B_2 = \mathrm{diag}\{0.8, 0.9, -1.3\}, B_3 = \mathrm{diag}\{1.1, 1.2, -0.7\} \\
G_a &= \begin{bmatrix} 0.2 & 0.4 & 0.5 \end{bmatrix}, D_3 = \begin{bmatrix} 0.45 & 0.18 & 0.36 \end{bmatrix}^T, \gamma = 0.8, a = \{1, 2, 3\}.
\end{aligned}
$$

For semi-Markov chain, the probability density functions of the ST and the TPs associated to SMC are presented as

$$
\begin{aligned}
\pi_{12} &= 0.8, \pi_{13} = 0.2, \pi_{21} = 0.7, \pi_{23} = 0.3, \pi_{31} = 0.6, \pi_{32} = 0.4, \pi_{11} = \pi_{22} = \pi_{33} = 0 \\
f_{11}(d) &= f_{22}(d) = f_{33}(d) = 0 f_{12}(d) = 0.6^d \cdot 0.4^{(10-d)} \cdot 10!/((10-d)! \cdot d!) f_{13}(d) = 0.4^d \cdot 0.6^{(10-d)} \cdot 10!/((10-d)! \cdot d!) \\
f_{21}(d) &= 0.9^{(d-1)^2} - 0.9^{d^2}, f_{32}(d) = 0.3^{(d-1)^{0.8}} - 0.3^{d^{0.8}} f_{23}(d) = 0.5^8 \cdot 8!/((10-d)! \cdot d!), f_{31}(d) = 0.4 \cdot 0.6^{d-1}.
\end{aligned}
$$

The corresponding upper bounds of ST for different modes are given as $\bar{d}_1 = 10$, $\bar{d}_2 = 8$, $\bar{d}_3 = 6$. The singularly perturbed matrix is selected to be $E_\epsilon = \mathrm{diag}\{1, 1, \epsilon\}$ with $\epsilon \in (0, 0.08]$.

Remark 7. Since the relationship among the parameters related to the semi-Markov chain is complicated, the following steps for obtaining a set suitable jumping sequences are provided with the hope to clearly show the relationship and utilization of these parameters.

(I) Give proper parameters and forms about the probability density functions of sojourn time $f_{\alpha\beta}(\tau)$ and transition probabilities $\pi_{\alpha\beta}$ for $\forall \alpha, \beta \in \bar{M}$; select an initial system mode $\sigma(1) \in \bar{M}$, the upper bounds of sojourn time $\bar{d}_\alpha, \alpha \in \bar{M}$, and the total length of time $\bar{L}$. Within the given time, let us continue with the following steps.

(II) As the total number of system mode is M and the upper bounds of sojourn time are $\bar{d}_a, \alpha \in \bar{M}$. Then, generate $\sum_{\alpha=1}^{M} \bar{d}_a$ probabilities based on $f_{\alpha\beta}(\tau)$ and $\pi_{\alpha\beta}$, $\forall \alpha, \beta \in \bar{M}$. This step is in fact utilized to determine the probability that the system stays in mode α for a duration of τ, $\alpha \in \bar{M}$, $\tau \in \bar{d}_a$.

(III) According to the probabilities obtained above, calculate M time lengths $\bar{l}_{\bar{m}}, \bar{m} \in \bar{M}$ which signify the actual lengths of duration for each mode.

(IV) Randomly generate a number $Pron$ evenly distributed on the interval $[0, 1]$. Then, compare the number $Pron$ with the elements in transition probability matrix to determine the next system mode. Meanwhile, based on the lengths $\bar{l}_{\bar{m}}, \bar{m} \in \bar{M}$ obtained in step (III), the duration of the next system mode is also determined.

(V) Preserve the modes obtained in step (IV) as well as their corresponding sojourn time.

(VI) Looping execution the above steps (II)-(V) until the desired sequence length is met. Then, s set of satisfactory semi-Markov jump sequences are obtained.

The quantization density of the adopted quantizer is regarded as $\rho_1 = \rho_2 = \rho_3 = 0.9$. As for the improved WTODP, the weight matrix is taken as $\hat{Q} = \mathrm{diag}\{0.6, 0.4, 0.5\}$. Furthermore, the external disturbance is selected to be $\omega(k) = \sin(2k)/(k^2)$.

Algorithm F1 Obtain transmission sequence under the improved weighted try-once-discard protocol.

Input: Weight matrix $\hat{Q}$; Total simulation time $Leth$; Node number N; jumping signal $\sigma(k)$; quantization parameters ρ_l, $q_l^{(0)}$;
Output: Transition signal $\xi(k)$.

Initialization with given $\xi(1) = 0$, $\bar{y}(0) = 0$, $x(0)$ and $\hat{x}(0)$;
for $k = 1 : Leth$ **do**
  Update the measurement output $y(k)$ and $\bar{y}(k-1)$ according to (1), (2), (4) and (5);
  for $i = 1 : N$ **do**
    Calculate the error between the current measurement output and the previously transmitted data of node i, i.e., $e_{y_i}(k) = \|y_i(k) - \bar{y}_i(k-1)\|^2_{\hat{Q}_i}$;
  end for
  Let $e_y\{k\} = [e_{y_1}(k), \ldots, e_{y_N}(k)]$;
  Determine the maximum error of these N nodes, i.e., calculate $Max(k) = \max(e_y\{k\})$;
  if $Max(k) == 0$ **then**
    $\xi(k) = 0$;
  else
    for $j = 1 : N$ **do**
      if $e_y\{k\}(1, j) == Max(k)$ **then**
        $\xi(k) = j$;
        break
      end if
    end for
  end if
end for

To facilitate the subsequent analysis, the generation process of the transmission sequence under the improved WTODP is provided in Algorithm F1. First of all, we verify the effectiveness of the proposed control scheme by comparing open-loop and

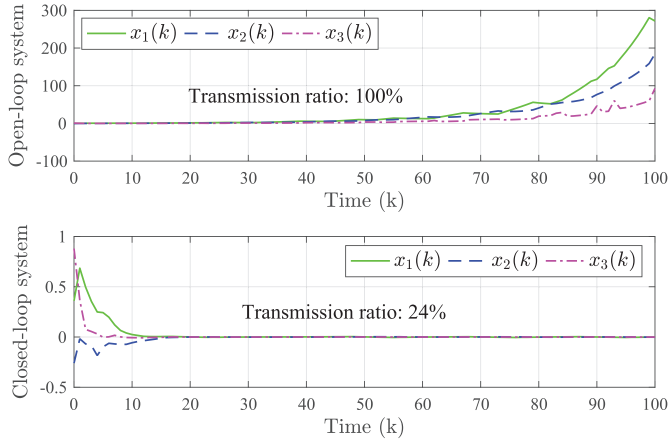

Figure F1 State responses of the open-loop system and the CLS.

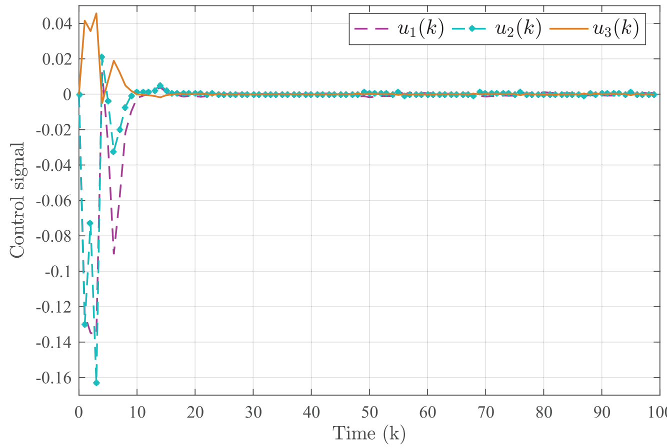

Figure F2 Control input signal $u(k)$.

closed-loop figures. Consider that the initial state of system and observer being $x(k) = \begin{bmatrix} 0.36 & -0.32 & 0.88 \end{bmatrix}^T$ and $\hat{x}(k) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$, respectively. Then, according to the proposed observer-based control scheme, together with the employ of Theorems 1 and 2 and Algorithm F1, simulation results can be seen in Figures F1-F5. The state responses of open-loop system and CLS are plotted in Figure F1 and controlled input is depicted in Figure F2. Indicated by these figures, one can conclude that the unstable system (state response divergence) tends to be stable (state response convergence) under the action of the devised observer-based controller, which manifests the validity of the control scheme.

The jump sequence of system modes and the transmission sequence of the channel are presented in Figure F3. The measurement output $y(k)$ of the plant and the data $\bar{y}(k)$ from the transmission channel obtained by the observer are depicted in Figure F4. As observed from Figures F3-F4, when the weighted data error of each node is zero, no node obtains the transmission permission. At this time, the data obtained by the observer is consistent with the previous moment under the action of the zero-order holder (ZOH). Otherwise, when the error of each node is not all zero, only the node with the largest data variation in the weighted case can transmit data.

Figure F5 is presented to show control results and transmission circumstances of the model under traditional WTODP. It can be noted from Figure F3 and Figure F5 that compared with utilizing traditional WTODP [7], which needs to transmit data all the time (transmission ratio 100%), the improved WTODP only transmits data at partial instant (transmission ratio 24% in this paper). Clearly, the proposed transmission protocol in this paper greatly reduces the amount of data transmission at each moment and the transmission ratio. Therefore, the proposed protocol is not only conducive to saving limited communication resources, but also can reduce the wear and tear of switching devices to a certain extent, which distinctly shows the superiority of the transport protocol.

In addition, the $l_2 - l_\infty$ performance index is calculated under zero initial conditions as follows:

$$\sqrt{\frac{\sup_{0\leqslant k\leqslant 100}\mathcal{E}\{z^T(k)z(k)\}}{\sum_{k=0}^{100}\mathcal{E}\{\omega^T(k)\omega(k)\}}} = 0.1338 < \gamma = 0.8$$

which further confirms that the designed controller performs well and the prescribed performance level is satisfied.

In what follows, the relationship between the quantization level ρ_l and the upper bound of singularly perturbation parameter $\bar{\epsilon}$ is investigated. For convenience, we denote $\rho_1 = \rho_2 = \rho_3 = \hat{\rho}$ and the maximum value of $\bar{\epsilon}$ is denoted as $\bar{\epsilon}_{\max}$. By adjusting the value of $\hat{\rho}$, the corresponding value of $\bar{\epsilon}_{\max}$ can be calculated with other parameters are the same as above. Then, Table I can be obtained. One can notice from it that the value of $\bar{\epsilon}_{\max}$ grows larger with the increase of $\hat{\rho}$. It implies that the decrease of the range of quantization error may be beneficial to the extension of the upper bound of perturbation parameter, which consistent with theoretical analysis. Worth mentioning, when the singularly perturbed parameter ϵ takes value within the calculated range

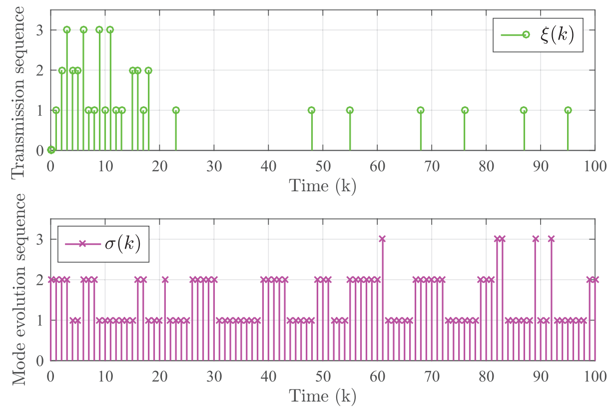

Figure F3 The transmission sequence $\xi(k)$ and the jump sequence $\sigma(k)$.

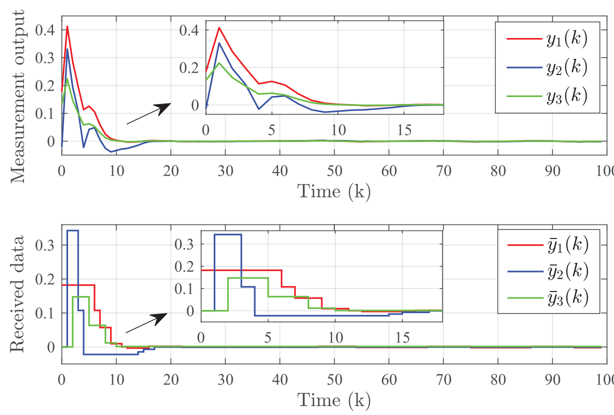

Figure F4 The measurement output $y(k)$ and the observer received data $\bar{y}(k)$.

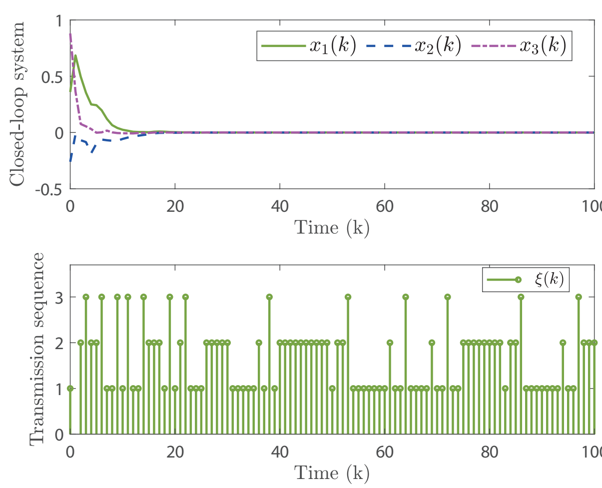

Figure F5 The state responses and transmission sequence obtained by using traditional try-once-discard protocol.

$(0, \bar{\epsilon}_{\max}]$, with the developed methods, a set of controller and observer gain matrices can be calculated to ensure the overall stability of the systems. However, when the value of parameter ϵ exceeds the calculated range, conditions in Theorem 2 will be unfeasible. The designed strategy may fail to explore suitable controller/observer gains. However, since the conditions obtained are sufficient, it does not mean the nonexistence of the desired controller/observer gains. Therefore, the case that the value of ϵ is out of the calculated range $(0, \bar{\epsilon}_{\max}]$ does not necessarily induce the instability of the closed-loop system. Seeking proper method to extend the upper bound of perturbation parameter has thus became an urgent problem to be solved.

Remark 8. Generally speaking, both event-triggered and weighted TOD transmission protocols determine the current transmission status according to the error between the current data and the data transmitted at the previous time. The difference lies in that event-triggered mechanism focuses on reducing the transmission frequency and not every time instant there will be data transmission. While WTODP focuses on reducing the amount of data transmission at each time instant, i.e, only one node can obtain the permission of data transmission. It can be noted that there is no need for data transmission after the system tends

Table F1 The maximum allowable perturbation upper bound $\bar{\epsilon}_{\max}$ under different quantization level $\hat{\rho}$

$\hat{\rho}$	0.9	0.8	0.7	0.6	0.5	0.4
$\bar{\epsilon}_{\max}$	0.4361	0.4138	0.3631	0.2805	0.1745	0.0279

to be stable and only the data preserved in ZOH need to be utilized. However, with regard to the traditional WTODP, even if the error mentioned above tends to zero as the closed-loop system approaching stable, there will still be a node can transmit data at each moment. Obviously, the transmission of these data is unnecessary, and it will result in the waste limited communication resources to a certain extent. Therefore, combined with the characteristics of event-triggered mechanism and WTODP, an improved WTODP which can not only reduce transmission volume but also transmission frequency is designed in our work. Specifically, at current time instant, only when the data is different from the previously transmitted data can one node obtain transmission permission. Moreover, the node obtaining transmission permission is the one with the largest weighted error. In summary, the improved WTODP can further lessen the transmission burden and can effectively save limited communication resources.

References

1 L. Zhang, Y. Leng, and P. Colaneri, "Stability and stabilization of discrete-time semi-Markov jump linear systems via semi-Markov kernel approach," *IEEE Trans. Automat. Control,* vol. 61, no. 2, pp. 503–508, Feb. 2016.

2 Z. Ning, L. Zhang, and P. Colaneri, "Semi-Markov jump linear systems with incomplete sojourn and transition information: Analysis and synthesis," *IEEE Trans. Automat. Control,* vol. 65, no. 1, pp. 159–174, Jan. 2020.

3 Y. Xu, R. Lu, J. Tao, H. Peng, and K. Xie, "Nonfragile l_2–l_∞ state estimation for discrete-time neural networks with jumping saturations," *Neurocomputing,* vol. 207, no. 26, pp. 15–21, Sept. 2016.

4 F. Li, S. Xu, and B. Zhang, "Resilient asynchronous H_∞ control for discrete-time Markov jump singularly perturbed systems based on hidden Markov model," *IEEE Trans. Syst. Man Cybern. Syst.,* vol. 50, no. 8, pp. 2860–2869, Aug. 2020.

5 J. Dong and G. H. Yang, "Robust H_∞ control for standard discrete-time singularly perturbed systems," *IET Control Theory Appl.,* vol. 1, no. 4, pp. 1141–1148, Jul. 2007.

6 L. Xie, "Output feedback H_∞ control of systems with parameter uncertainty," *Int. J. Control,* vol. 63, no. 4, pp. 741–750, Jun. 1996.

7 L. Zou, Z. Wang, Q. L. Han, and D. Zhou, "Ultimate boundedness control for networked systems with try-once-discard protocol and uniform quantization effects," *IEEE Trans Automat Control,* vol. 62, no. 12, pp. 6582–6588, Dec. 2017.