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Observer-based distributed consensus for nonlinear multi-agent systems with limited data rate

Jing ZHANG, Shuai LIU* & Xianfu ZHANG

School of Control Science and Engineering, Shandong University, Jinan 250061, China

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Abstract In this paper, we investigate the observer-based distributed consensus problem for nonlinear multi-agent systems with a limited data rate. We assume that each agent displays strict-feedback nonlinear dynamics. An observer is constructed to estimate the system states as they cannot be directly measured. Uniform quantizers and logarithmic quantizers are considered in communication channels among agents and observer-based distributed control protocols are proposed for each agent. For the uniform quantizer, the δ-asymptotic distributed consensus, with δ being the resolution of the quantizer, can be achieved under the proposed control protocol, i.e., the consensus errors tend to zero as δ approaches zero. For the logarithmic quantizer, the asymptotic consensus can be achieved. The simulation results reveal the conclusion.

Keywords nonlinear multi-agent systems, uniform quantizer, logarithmic quantizer, dynamic gain, δ-asymptotic distributed consensus

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1 Introduction

Multi-agent systems have become the focus of intensive research owing to their universality, such as smart grid and traffic control. Information is usually transmitted among agents through digital channels. In real systems, it is impossible for an agent to obtain the information of all agents due to communication restrictions on digital channels. Distributed consensus is to design a distributed control protocol, in which each agent can only access the information of its own and its neighbors, so that all agents can reach an agreement over their states. Therefore, it is very meaningful to study the distributed consensus problem. Recently, distributed consensus for nonlinear multi-agent systems has received significant attention. Refs. [1–3] addressed tracking control problems for nonlinear time-delay multi-agent systems. Leader-follower consensus control protocols were constructed for nonlinear multi-agent systems in [4–7]. Ref. [8] investigated both adaptive leader-follower and leaderless consensus problems for strict-feedback nonlinear multi-agent systems.

However, the above research results are considered under the assumption that the states of systems are directly accessible for feedback. Note that the states' information is often unavailable in practical systems. Therefore, it is crucial to design an observer to estimate the system states. A great deal of work has been conducted in this field. For example, a series of high-gain observers were constructed to estimate the system states for nonlinear systems in [9,10] and for the nonlinear multi-agent system in [11]. Over the last few decades, fuzzy systems and neural network systems were extensively used to deal with nonlinear functions. Since then, many researches have been done about fuzzy observers in [12,13] and neural network observers in [14,15]. A full-order unknown input observer and a reduced-order unknown input observer have been proposed to estimate the full states' error in [16]. Recently, a novel adaptive observer was designed in [17] without using the output information.

Communication plays an important role in multi-agent systems. Because of limited capability of digital communication channels, the transmitted data may suffer from many problems, such as data loss. To

* Corresponding author (email: lius0025@ntu.edu.sg)

this end, an increasing number of researchers are investigating how to reduce the communication burden. Quantization has been proposed to deal with the problem. Many important quantized results have been reported in linear systems [18–22] and nonlinear systems [23–33]. In [18], a static uniform quantizer was utilized for the consensus problem of multi-agent networks. Logarithmic quantizer-based continuous-time and sampled-data-based protocols were constructed in [19], which can achieve β-asymptotic average consensus. Ref. [20] addressed the average-consensus control for the discrete-time multi-agent system, in which each agent preprocesses the information by an encoder before sending it to the channel. Meanwhile, its neighbors process the received information with designed decoders. The control problem becomes more challenging when the systems are nonlinear. By introducing the uniform quantizer, logarithmic quantizer, and hysteretic quantizer, the authors studied stabilization problems for nonlinear systems in [23–25]. A series of output feedback controls were studied in [26–30] for nonlinear systems with quantizers. Ref. [31] discussed the leader-following consensus problem for second-order nonlinear multi-agent systems with uniform quantizers. The distributed consensus has been solved in [32] for a class of nonlinear multi-agent systems with hysteretic quantizers, where the nonlinear term was approximated by a neural network system. Ref. [33] considered the data rate consensus of nonlinear output feedback multi-agent systems, where the nonlinearity exists only in the last term of the system state. Owing to the influence of nonlinear functions in multi-agent systems, the leaderless consensus is very complex and is rarely studied.

Inspired by the aforementioned studies, this paper investigates the distributed consensus for nonlinear multi-agent systems with a limited data rate. The main contributions of this paper are as follows. (1) The uniform quantizer and the logarithmic quantizer are used in the communication channels between each pair of agents. (2) A quantized-based observer is constructed for each agent to estimate the unknown states. By jointly designing the observer and the quantizer, we propose an observer-based distributed consensus protocol using a dynamic gain method. Each agent can access the estimated states of its own and its neighbors. For communication with the uniform quantizer, we can get an upper bound of consensus errors in terms of quantization resolution δ by designing control protocol u_i and dynamic gain $l(t)$. When δ converges to zero, the consensus errors converge to zero, which means δ-asymptotic distributed consensus is achieved. In the logarithmic quantization setting, an exact asymptotic consensus can be achieved under the proposed control protocol.

The paper is organized as follows: Section 2 introduces some knowledge about graph theory, uniform quantizer, and logarithmic quantizer, which will be used later. The problem is formulated in Section 3. Distributed control protocol and convergence analysis with uniform quantizer and logarithmic quantizer are investigated in Section 4, whereas Section 5 presents simulation results. Ultimately, Section 6 presents the conclusion.

Notation. $\mathbb{R}$, $\mathbb{R}^n$, and $\mathbb{R}^{m\times n}$ indicate the sets of real numbers, n-dimensional real vectors, and $m\times n$ real matrices, respectively. Let $\mathbb{R}^+$, $\mathbb{Z}$, and $\mathbb{Z}^+$ be the sets of positive real numbers, integers, and positive integers, respectively. $\bar{\mathbb{R}}^+=\mathbb{R}^+\cup\{0\}$. $\lambda_{\max}(M)$ and $\lambda_{\min}(M)$ denote the maximum and minimum eigenvalues of the symmetric matrix M, respectively. A^{T} denotes the transpose of matrix A and $\|P\|$ is the Euclidean norm of $P\in\mathbb{R}^{m\times n}$. $\otimes$ represents the kronecker product. I represents the identity matrix with appropriate dimension. $\mathrm{diag}[a_1\cdots a_n]$ stands for a diagonal matrix, in which a_i is the i-th diagonal element.

2 Preliminary knowledge

In this section, we will introduce some knowledge of the graph theory, uniform quantizer, and logarithmic quantizer.

2.1 Graph theory

Suppose that there are N nodes. Let $\mathcal{G}=(\mathcal{V},\mathcal{E})$ be a directed graph, where $\mathcal{V}=\{1,2,\ldots,N\}$ represents the set of nodes and $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ represents the set of edges. $(j,k)\in\mathcal{E}$ if node k can receive information from j and we call that node j is a neighbor of node k. We use $\mathcal{N}_k=\{j\in\mathcal{V}|(j,k)\in\mathcal{E}\}$ to represent the set of neighbors of the k-th agent with $(k,k)\notin\mathcal{E}$. $a_{kj}>0$ if and only if $(j,k)\in\mathcal{E}$ and $a_{kj}=0$ otherwise. The Laplacian matrix $L=[l_{kj}]\in\mathbb{R}^{N\times N}$ is described as $l_{kj}=-a_{kj}, k\neq j$ and $l_{kk}=\sum_{i=1 i\neq k}^{N}a_{ki}$. A directed graph $\mathcal{G}$ is balanced if $\sum_{j=1}^{N}a_{kj}=\sum_{j=1}^{N}a_{jk}$ for all k.

2.2 Uniform quantizer

Uniform quantizer $q(\cdot):\mathbb{R}\to\Gamma$ has the following form:

$$q(x)=\begin{cases} k\delta, & \left(k-\frac{1}{2}\right)\delta\leqslant x<\left(k+\frac{1}{2}\right)\delta,\\ 0, & -\frac{1}{2}\delta<x<\frac{1}{2}\delta,\\ -q(-x), & x<0,\end{cases} \tag{1}$$

where $\Gamma=\{\pm k\delta:k\in\mathbb{Z}^+\}$ and $\delta\in\mathbb{R}^+$ represents the resolution of the quantizer. We can see from the expression of the quantizer that

$$|q(x)-x|\leqslant\frac{\delta}{2}. \tag{2}$$

2.3 logarithmic quantizer

Logarithmic quantizer $q(\cdot):\mathbb{R}\to\Gamma$ has the following form:

$$q(x)=\begin{cases} \omega(i), & \frac{\omega(i)}{1+\beta}<x\leqslant\frac{\omega(i)}{1-\beta},\\ 0, & x=0,\\ -q(-x), & x<0,\end{cases} \tag{3}$$

where $\Gamma=\{\pm\omega(i):\omega(i)=\rho^i\omega(0),i\in\mathbb{Z}\}\cup\{0\}$, $\rho\in(0,1)$, $\omega(0)\in\mathbb{R}^+$ and $\beta=\frac{1-\rho}{1+\rho}$. We can see from the expression of the quantizer that

$$|q(x)-x|\leqslant\beta|x|.$$

Then, we know that there exists a function $\triangle(x)\in[-\beta,\beta)$ such that the quantization error satisfies

$$q(x)-x=\triangle(x)x,\quad\forall x\in\mathbb{R}. \tag{4}$$

3 Problem statement

In this section, we consider nonlinear multi-agent systems in the following form:

$$\begin{cases} \dot{x}_{k,i}(t)=x_{k,i+1}(t)+f_i(\bar{x}_{k,i}(t)), \quad i=1,\ldots,n-1,\\ \dot{x}_{k,n}(t)=u_k(t)+f_n(\bar{x}_{k,n}(t)),\\ y_k(t)=x_{k,1}(t),\quad k\in\mathcal{V},\end{cases} \tag{5}$$

where $x_k(t)=[x_{k,1}(t)\ \cdots\ x_{k,n}(t)]^{\mathrm{T}}\in\mathbb{R}^n$, $u_k(t)\in\mathbb{R}$ and $y_k(t)\in\mathbb{R}$ represent the system state, the control input and control output of agent k, respectively; $\bar{x}_{k,i}(t)=[x_{k,1}(t)\ \cdots\ x_{k,i}(t)]^{\mathrm{T}}\in\mathbb{R}^i$; $f_i(\bar{x}_{k,i}(t))$, $i=1,2,\ldots,n$ are continuous nonlinear functions. Note that $x_{k,1}(t)$ can be directly obtained by measurement out $y_k(t)$ and all the rest of the state is not available for feedback.

The communication topology of multi-agent system (5) is assumed to be described by a graph $\mathcal{G}$, where each agent is corresponding to a node in $\mathcal{G}$. We assume that each communication channel, corresponding to an edge in $\mathcal{G}$, is with quantization. We say that the system (5) achieves distributed consensus if a distributed control protocol, which only uses the information of its own and its neighbours, is designed for each agent such that the states of all the agents satisfy: $\lim_{t\to\infty}\|x_k-x_j\|=0$, $k,j\in\mathcal{V}$.

Because of the impact of quantization, it is difficult to make the system reach a distributed consensus. Therefore, we introduce the following definition of δ-asymptotic distributed consensus.

Definition 1. For system (5), δ-asymptotic distributed consensus is achieved if and only if

$$\limsup_{t\to\infty}\|x_k-x_j\|<+\infty,$$

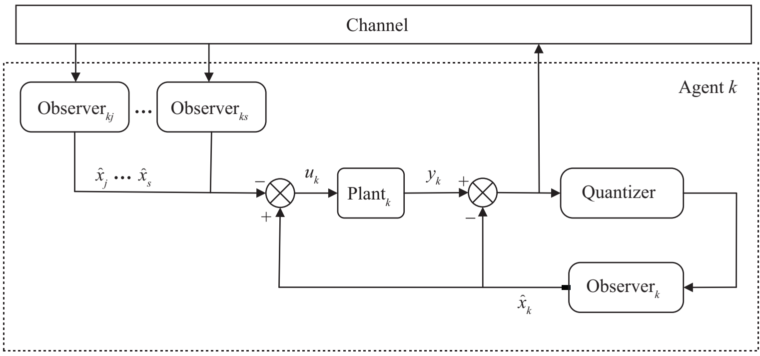

Figure 1 Architecture.

and

$$\lim_{\delta\to 0}\lim_{t\to\infty}\|x_k - x_j\| = 0, \quad k, j \in \mathcal{V},$$

where δ is the resolution of the uniform quantizer.

In the following, we shall design an observer-based distributed control protocol for each agent based on quantized information from neighbors such that the nonlinear multi-agent systems reach δ-asymptotic distributed consensus or consensus.

Lemmas 1 and 2, Assumptions 1 and 2 are introduced to accomplish this goal.

Lemma 1 ([34]). For $a \in \mathbb{R}$, $b \in \mathbb{R}$, and $\varepsilon \in \mathbb{R}^+$, the following inequality holds:

$$ab \leqslant \frac{\varepsilon a^p}{p} + \frac{b^q}{\varepsilon q},$$

where $\frac{1}{p} + \frac{1}{q} = 1$ with $p \in [1, \infty)$ and $q \in [1, \infty)$.

Lemma 2 ([35]). For $p \in [1, \infty)$ and $a_i \in \bar{\mathbb{R}}^+$, the following inequality holds:

$$(a_1 + \cdots + a_n)^p \leqslant n^{p-1}(a_1^p + \cdots + a_n^p).$$

Assumption 1. $f_i(\bar{x}_{k,i}(t))$ satisfies the following condition for $\theta \in \mathbb{R}^+$:

$$|f_i(\bar{x}_{k,i}(t)) - f_i(\bar{x}_{j,i}(t))| \leqslant \theta \sum_{l=1}^{i} |x_{k,l} - x_{j,l}|.$$

Assumption 2. $\mathcal{G}$ is a balanced directed graph and contains a spanning tree.

4 Consensus analysis and control protocol design

4.1 Observer design and control protocol design

Note that the full state information of each agent is not always available in many practical applications, which makes it difficult to design distributed control protocol in the absence of full neighbors' information. Therefore, in this subsection, we will design an observer with quantizer to estimate the system state by using the information of the outputs for each agent and propose the distributed control protocol. The architecture is shown in Figure 1.

An observer is designed in the following way:

$$\begin{cases} \dot{\hat{x}}_{k,i} = \hat{x}_{k,i+1} + f_i(\bar{\hat{x}}_{k,i}) + l^i(t)m_i q(y_k - \hat{x}_{k,1}), \\ \dot{\hat{x}}_{k,n} = u_k + f_n(\bar{\hat{x}}_{k,n}) + l^n(t)m_n q(y_k - \hat{x}_{k,1}), \end{cases} \tag{6}$$

where $\hat{x}_{k,i}$ is the estimation of $x_{k,i}, i = 1, \ldots, n$; $l(t) \in [1, \infty)$ is a dynamic gain to be designed later; $q(\cdot)$ is quantizer and $m_i, i = 1, 2, \ldots, n$ are coefficients of the Hurwitz polynomial $p_1(s) = s^n + m_1 s^{n-1} + \cdots + m_{n-1}s + m_n$. The observer$_k$, $k \in \mathcal{V}$ and the observer$_{kj}$, $k \in \mathcal{V}$, $j \in \mathcal{N}_k$ are expressed in (6).

The distributed consensus protocol is constructed as

$$u_k = -K\Lambda(t)\sum_{j\in\mathcal{N}_k} a_{kj}(\hat{x}_k - \hat{x}_j), \tag{7}$$

where a_{kj} and $\mathcal{N}_k$ are defined in graph theory, $\hat{x}_i = [\hat{x}_{i,1}\cdots\hat{x}_{i,n}]^{\rm T}$, $i\in\mathcal{V}$, $\Lambda(t) = {\rm diag}[l^n(t)\cdots l(t)]$, $K = [k_1\cdots k_n]$ can be found such that $A-\lambda_i BK$ is a Hurwitz matrix with $B = [0\ \cdots\ 0\ 1]^{\rm T}$, $A = [0_{(N-1)\times 1}, I_{(N-1)\times(N-1)}; 0_{1\times 1}, 0_{1\times(N-1)}]$ and for all eigenvalues λ_i, $i = 2,\ldots,N$ of L except $\lambda_1 = 0$.

4.2 Convergence analysis with uniform quantizer

In this subsection, we give the proof process and analyze the convergence of the system.

Define the following scaled estimation as well as estimation error for $\sigma\in\mathbb{R}^+$, $i = 1,2,\ldots,n$ and $k\in\mathcal{V}$:

$$\eta_{k,i} = \frac{\hat{x}_{k,i}}{l^{i-1+\sigma}(t)},\quad \tilde{x}_{k,i} = \frac{x_{k,i}-\hat{x}_{k,i}}{l^{i-1+\sigma}(t)}. \tag{8}$$

By proving that $\tilde{x}_{k,i}$ converges to zero and $l(t)$ is bounded, we can get that the state $\hat{x}_{k,i}$ of the observer converges to the state $x_{k,i}$ of system (5). By the linear transformation $z = (\Phi\otimes I_n)\eta$ with decomposing $z = [z_1^{\rm T}, z_2^{\rm T}]^{\rm T}$ and proving that $z_2 = [(\eta_1-\eta_2)^{\rm T}\cdots(\eta_1-\eta_N)^{\rm T}]^{\rm T}$ converges to zero, we can get that $\hat{x}_{k,i}-\hat{x}_{j,i}$ converges to zero, $\forall k,j\in\mathcal{V}$. That is to say, the states of the observer reach a consensus. The definitions of $\Phi,\eta_1,\ldots,\eta_N,\eta$ will be given later. Then we can obtain that the original multi-agent system achieves consensus. Based on the above idea, we shall analyze the consensus of system (5) under the observer (6) and control protocol (7) in details as follows.

According to (5), (6), and (8), we can obtain that

$$\dot{\tilde{x}}_{k,i} = l(t)\tilde{x}_{k,i+1} + \frac{f_i(\bar{x}_{k,i})-f_i(\bar{\hat{x}}_{k,i})}{l^{i-1+\sigma}(t)} - l(t)m_i\tilde{x}_{k,1} - \frac{m_i}{l^{\sigma-1}(t)}(q(y_k-\hat{x}_{k,1})-(y_k-\hat{x}_{k,1})) - \frac{\dot{l}(t)}{l(t)}(i-1+\sigma)\tilde{x}_{k,i},$$

and

$$\dot{\tilde{x}}_{k,n} = \frac{f_n(\bar{x}_{k,n})-f_n(\bar{\hat{x}}_{k,n})}{l^{n-1+\sigma}(t)} - l(t)m_n\tilde{x}_{k,1} - \frac{m_n}{l^{\sigma-1}(t)}(q(y_k-\hat{x}_{k,1})-(y_k-\hat{x}_{k,1})) - \frac{\dot{l}(t)}{l(t)}(n-1+\sigma)\tilde{x}_{k,n}.$$

Let $\tilde{x}_k = [\tilde{x}_{k,1}\ \cdots\ \tilde{x}_{k,n}]^{\rm T}$ for $k\in\mathcal{V}$. One has

$$\dot{\tilde{x}}_k = l(t)\bar{A}\tilde{x}_k + \bar{f}_k - \frac{1}{l^{\sigma-1}(t)}[m_1\ \cdots\ m_n]^{\rm T}(q(y_k-\hat{x}_{k,1})-(y_k-\hat{x}_{k,1})) - \frac{\dot{l}(t)}{l(t)}(D+\sigma I)\tilde{x}_k,$$

where $D = {\rm diag}[0\ 1\ \cdots\ n-1]$, $f_k = [\frac{f_1(\bar{x}_{k,1})-f_1(\bar{\hat{x}}_{k,1})}{l^{\sigma}(t)},\ldots,\frac{f_n(\bar{x}_{k,n})-f_n(\bar{\hat{x}}_{k,n})}{l^{n-1+\sigma}(t)}]^{\rm T}$ and $\bar{A} = [[-m_1\cdots-m_{n-1}]^{\rm T}, I_{(N-1)\times(N-1)}; -m_n, 0_{1\times(N-1)}]$.

Because $m_i, i = 1,2,\ldots,n$ are coefficients of a Hurwitz polynomial, we know that $\bar{A}$ is a Hurwitz matrix; then $\forall h\in\mathbb{R}^+$, there exists a positive definite matrix P such that

$$\bar{A}^{\rm T}P + P\bar{A} = -hI. \tag{9}$$

Because $D = {\rm diag}[0\ 1\cdots n-1]$, we know that the eigenvalues of D are $\nu_i = i-1$, $i = 1,\ldots,n$ and the eigenvectors of D are $z_i = [0\cdots1\cdots0]^{\rm T}, i = 1,\ldots,n$, where 1 is in the i-th row of z_i. Then $z_i^{\rm T}(DP+PD)z_i = z_i^{\rm T}DPz_i + z_i^{\rm T}PDz_i = 2\nu_i z_i^{\rm T}Pz_i \geqslant 0$. $\forall x\in\mathbb{R}^n$, x can be represented by z_i, that is $x = \sum_{i=1}^n \kappa_i z_i$, where $\kappa_i\in\mathbb{R}$, $i = 1,\ldots,n$ and not all zero; then $x^{\rm T}(DP+PD)x = \sum_{i=1}^n \kappa_i^2 z_i^{\rm T}(DP+PD)z_i \geqslant 0$. According to the above formula, we know that $DP+PD$ is a positive semi-definite matrix and

$$(D+\sigma I)P + P(D+\sigma I) \geqslant 2\sigma P.$$

Define $V_{1,k} = \tilde{x}_k^{\rm T}P\tilde{x}_k$. Taking derivative of $V_{1,k}$ with respect to t yields that

$$\dot{V}_{1,k} = \tilde{x}_k^{\rm T}\left(l(t)\bar{A}^{\rm T}P + l(t)P\bar{A} - \frac{\dot{l}(t)}{l(t)}(D+\sigma I)P - \frac{\dot{l}(t)}{l(t)}P(D+\sigma I)\right)\tilde{x}_k + 2\tilde{x}_k^{\rm T}P\bar{f}_k$$

$$
\begin{aligned}
&-\frac{2}{l^{\sigma-1}(t)}\tilde{x}_k^{\mathrm{T}}P[m_1\cdots m_n]^{\mathrm{T}}(q(y_k-\hat{x}_{k,1})-(y_k-\hat{x}_{k,1}))\\
&\leqslant\tilde{x}_k^{\mathrm{T}}\left(-l(t)hI-2\sigma\frac{\dot{l}(t)}{l(t)}P\right)\tilde{x}_k+2\tilde{x}_k^{\mathrm{T}}P\bar{f}_k-\frac{2}{l^{\sigma-1}(t)}\tilde{x}_k^{\mathrm{T}}P[m_1\cdots m_n]^{\mathrm{T}}(q(y_k-\hat{x}_{k,1})-(y_k-\hat{x}_{k,1})).
\end{aligned}
$$

By Assumption 1 and (8), we can calculate $\|\bar{f}_k\|\leqslant\sqrt{\frac{n(n+1)}{2}}\theta\|\tilde{x}_k\|$. Then, we have

$$
2\tilde{x}_k^{\mathrm{T}}P\bar{f}_k\leqslant2\|\tilde{x}_k\|\|P\|\|\bar{f}_k\|\leqslant\sqrt{2n(n+1)}\|P\|\theta\tilde{x}_k^{\mathrm{T}}\tilde{x}_k.
$$

Because the communication is under uniform quantizer, according to (2), we have $|q(y_k-\hat{x}_{k,1})-(y_k-\hat{x}_{k,1})|\leqslant\frac{\delta}{2}$. Moreover,

$$
\frac{2}{l^{\sigma-1}(t)}\tilde{x}_k^{\mathrm{T}}P[m_1\cdots m_n]^{\mathrm{T}}(q(y_k-\hat{x}_{k,1})-(y_k-\hat{x}_{k,1}))\leqslant r\|P\|^2m^2\tilde{x}_k^{\mathrm{T}}\tilde{x}_k+\frac{\delta^2}{4rl^{2\sigma-2}(t)},
$$

where $m=\sqrt{m_1^2+\cdots+m_n^2}$ and $r\in\mathbb{R}^+$. The derivative of $V_{1,k}$ is converted into

$$
\dot{V}_{1,k}\leqslant\tilde{x}_k^{\mathrm{T}}\left(-l(t)hI-2\sigma\frac{\dot{l}(t)}{l(t)}P+\sqrt{2n(n+1)}\|P\|\theta I+r\|P\|^2m^2I\right)\tilde{x}_k+\frac{\delta^2}{4rl^{2\sigma-2}(t)}. \tag{10}
$$

According to the define of $\eta_{k,i}$, we have from (6) that

$$
\dot{\eta}_{k,i}=l(t)\eta_{k,i+1}+\frac{f_i(\bar{\tilde{x}}_{k,i})}{l^{i-1+\sigma}(t)}+l(t)m_i\tilde{x}_{k,1}+\frac{m_i}{l^{\sigma-1}(t)}(q(y_k-\hat{x}_{k,1})-(y_k-\hat{x}_{k,1}))-\frac{\dot{l}(t)}{l(t)}(i-1+\sigma)\eta_{k,i},
$$

and

$$
\dot{\eta}_{k,n}=\frac{u_k}{l^{n-1+\sigma}(t)}+\frac{f_n(\bar{\tilde{x}}_{k,n})}{l^{n-1+\sigma}(t)}+l(t)m_n\tilde{x}_{k,1}+\frac{m_n}{l^{\sigma-1}(t)}(q(y_k-\hat{x}_{k,1})-(y_k-\hat{x}_{k,1}))-\frac{\dot{l}(t)}{l(t)}(n-1+\sigma)\eta_{k,n}.
$$

Let $\eta_k=[\eta_{k,1}\cdots\eta_{k,n}]^{\mathrm{T}}$ for $k\in\mathcal{V}$. One obtains that

$$
\begin{aligned}
\dot{\eta}_k\,=&\;l(t)A\eta_k+\tilde{f}_k+l(t)[m_1\cdots m_n]^{\mathrm{T}}\tilde{x}_{k,1}+\frac{1}{l^{\sigma-1}(t)}[m_1\cdots m_n]^{\mathrm{T}}(q(y_k-\hat{x}_{k,1})-(y_k-\hat{x}_{k,1}))\\
&-\frac{\dot{l}(t)}{l(t)}(D+\sigma I)\eta_k+[0\cdots0\ 1]^{\mathrm{T}}\frac{u_k}{l^{n-1+\sigma}(t)},
\end{aligned}\tag{11}
$$

where $\tilde{f}_k=[\frac{f_1(\bar{\tilde{x}}_{k,1})}{l^{\sigma}(t)}\cdots\frac{f_n(\bar{\tilde{x}}_{k,n})}{l^{n-1+\sigma}(t)}]^{\mathrm{T}}$. Letting $\eta=[\eta_1^{\mathrm{T}}\cdots\eta_N^{\mathrm{T}}]^{\mathrm{T}}$, combining (7) with (11), it is inferred that

$$
\begin{aligned}
\dot{\eta}\,=&\;l(t)((I_N\otimes A)-(L\otimes(BK)))\eta-\frac{\dot{l}(t)}{l(t)}(I_N\otimes(D+\sigma I))\eta+\tilde{f}+l(t)(I_N\otimes G)[\tilde{x}_1^{\mathrm{T}}\cdots\tilde{x}_N^{\mathrm{T}}]^{\mathrm{T}}\\
&+\frac{1}{l^{\sigma-1}(t)}(I_N\otimes[m_1\cdots m_n]^{\mathrm{T}})Q,
\end{aligned}\tag{12}
$$

where $\tilde{f}=[\tilde{f}_1^{\mathrm{T}}\cdots\tilde{f}_N^{\mathrm{T}}]^{\mathrm{T}}$, $Q=[(q(y_1-\hat{x}_{1,1})-(y_1-\hat{x}_{1,1}))\cdots(q(y_N-\hat{x}_{N,1})-(y_N-\hat{x}_{N,1}))]^{\mathrm{T}}$ and $G=[[m_1\cdots m_n]^{\mathrm{T}},0_{(N)\times(N-1)}]$.

Define the linear transformation $z=(\Phi\otimes I_n)\eta$, where $\Phi=[1_N\ \phi]$ with $\phi=[1_{1\times N};-I_{(N-1)\times(N-1)}]$. According to (12), we have

$$
\begin{aligned}
\dot{z}\,=&\;l(t)((I_N\otimes A)-(\Phi L\Phi^{-1}\otimes(BK)))z-\frac{\dot{l}(t)}{l(t)}(I_N\otimes(D+\sigma I))z+(\Phi\otimes I_n)\tilde{f}+l(t)(\Phi\otimes G)[\tilde{x}_1^{\mathrm{T}}\cdots\tilde{x}_N^{\mathrm{T}}]^{\mathrm{T}}\\
&+\frac{1}{l^{\sigma-1}(t)}(\Phi\otimes[m_1\ \cdots\ m_n]^{\mathrm{T}})Q,
\end{aligned}
$$

where $\Phi L\Phi^{-1}=\mathrm{diag}[0,\tilde{L}]$ and $\tilde{L}\in\mathbb{R}^{(N-1)\times(N-1)}$ is with all its eigenvalues having positive real parts. Decompose $z=[z_1^{\mathrm{T}},z_2^{\mathrm{T}}]^{\mathrm{T}}$ with $z_1\in\mathbb{R}^n$ and $z_2\in\mathbb{R}^{n(N-1)}$, and then

$$
\dot{z}_2\,=l(t)((I_{N-1}\otimes A)-(\tilde{L}\otimes(BK)))z_2-\frac{\dot{l}(t)}{l(t)}(I_{N-1}\otimes(D+\sigma I))z_2+(\phi^{\mathrm{T}}\otimes I_n)\tilde{f}
$$

$$+l(t)(\phi^{\mathrm{T}} \otimes G)[\tilde{x}_1^{\mathrm{T}} \cdots \tilde{x}_N^{\mathrm{T}}]^{\mathrm{T}} + \frac{1}{l^{\sigma-1}(t)}(\phi^{\mathrm{T}} \otimes [m_1 \cdots m_n]^{\mathrm{T}})Q.$$

According to $\Phi L \Phi^{-1} = \mathrm{diag}[0, \tilde{L}]$, we know that λ_i, $i = 2, \ldots, N$ are the eigenvalues of $\tilde{L}$. Because $A - \lambda_i BK$ is a Hurwitz matrix, $(I_{N-1} \otimes A) - (\tilde{L} \otimes (BK))$ is a Hurwitz matrix. There exists a positive definite matrix P_2 satisfying

$$((I_{N-1} \otimes A) - (\tilde{L} \otimes (BK)))^{\mathrm{T}} P_2 + P_2((I_{N-1} \otimes A) - (\tilde{L} \otimes (BK))) = -I. \tag{13}$$

Define $V_2 = z_2^{\mathrm{T}} P_2 z_2$. Taking time derivative of V_2 with respect to t yields that

$$\begin{aligned}
\dot{V}_2 =\ & z_2^{\mathrm{T}} \Bigg(l(t)((I_{N-1} \otimes A) - (\tilde{L} \otimes (BK)))^{\mathrm{T}} P_2 + l(t) P_2((I_{N-1} \otimes A) - (\tilde{L} \otimes (BK))) \\
& - \frac{\dot{l}(t)}{l(t)}(I_{N-1} \otimes (D + \sigma I))P_2 - \frac{\dot{l}(t)}{l(t)} P_2 (I_{N-1} \otimes (D + \sigma I)) \Bigg) z_2 + 2 z_2^{\mathrm{T}} P_2 (\phi^{\mathrm{T}} \otimes I_n) \tilde{f} \\
& + 2 z_2^{\mathrm{T}} P_2 l(t)(\phi^{\mathrm{T}} \otimes G)[\tilde{x}_1^{\mathrm{T}} \cdots \tilde{x}_N^{\mathrm{T}}]^{\mathrm{T}} + 2 z_2^{\mathrm{T}} P_2 \frac{1}{l^{\sigma-1}(t)}(\phi^{\mathrm{T}} \otimes [m_1 \cdots m_n]^{\mathrm{T}})Q \\
\leqslant\ & z_2^{\mathrm{T}} \left(-l(t) I - 2\sigma \frac{\dot{l}(t)}{l(t)} P_2 \right) z_2 + 2 z_2^{\mathrm{T}} P_2 (\phi^{\mathrm{T}} \otimes I_n) \tilde{f} + 2 z_2^{\mathrm{T}} P_2 l(t)(\phi^{\mathrm{T}} \otimes G)[\tilde{x}_1^{\mathrm{T}} \cdots \tilde{x}_N^{\mathrm{T}}]^{\mathrm{T}} \\
& + 2 z_2^{\mathrm{T}} P_2 \frac{1}{l^{\sigma-1}(t)}(\phi^{\mathrm{T}} \otimes [m_1 \cdots m_n]^{\mathrm{T}})Q.
\end{aligned} \tag{14}$$

By Assumption 1 and (8), one has $\|(\phi^{\mathrm{T}} \otimes I_n)\tilde{f}\| \leqslant \sqrt{\frac{n(n+1)}{2}}\theta \|z_2\|$. Then we have

$$2 z_2^{\mathrm{T}} P_2 (\phi^{\mathrm{T}} \otimes I_n)\tilde{f} \leqslant 2\|z_2\|\|P_2\|\|(\phi^{\mathrm{T}} \otimes I_n)\tilde{f}\| \leqslant \sqrt{2n(n+1)}\|P_2\|\theta z_2^{\mathrm{T}} z_2.$$

Based on Lemma 1, one have

$$2 z_2^{\mathrm{T}} P_2 l(t)(\phi^{\mathrm{T}} \otimes G)[\tilde{x}_1^{\mathrm{T}} \cdots \tilde{x}_N^{\mathrm{T}}]^{\mathrm{T}} \leqslant \frac{1}{2} l(t) z_2^{\mathrm{T}} z_2 + 2 l(t)\|P_2\|^2 m^2 N \sum_{k=1}^{N} \tilde{x}_k^{\mathrm{T}} \tilde{x}_k,$$

and

$$2 z_2^{\mathrm{T}} P_2 \frac{1}{l^{\sigma-1}(t)}(\phi^{\mathrm{T}} \otimes [m_1 \cdots m_n]^{\mathrm{T}})Q \leqslant r\|P_2\|^2 m^2 N z_2^{\mathrm{T}} z_2 + \frac{\delta^2 N}{4 r l^{2\sigma-2}(t)}.$$

Eq. (14) can be rewritten as

$$\begin{aligned}
\dot{V}_2 \leqslant\ & z_2^{\mathrm{T}} \left(-\frac{1}{2} l(t) I - 2\sigma \frac{\dot{l}(t)}{l(t)} P_2 + \sqrt{2n(n+1)}\|P_2\|\theta I + r\|P_2\|^2 m^2 N I \right) z_2 \\
& + 2 l(t)\|P_2\|^2 m^2 N \sum_{k=1}^{N} \tilde{x}_k^{\mathrm{T}} \tilde{x}_k + \frac{\delta^2 N}{4 r l^{2\sigma-2}(t)}.
\end{aligned} \tag{15}$$

Choose a Lyapunov candidate function as

$$V = \sum_{k=1}^{N} V_{1,k} + V_2.$$

From (10) and (15), it follows that

$$\begin{aligned}
\dot{V} \leqslant\ & \left(-l(t) h - 2\sigma \lambda_{\min}(P) \frac{\dot{l}(t)}{l(t)} + \sqrt{2n(n+1)}\|P\|\theta + r\|P\|^2 m^2 + 2 l(t)\|P_2\|^2 m^2 N \right) \sum_{k=1}^{N} \tilde{x}_k^{\mathrm{T}} \tilde{x}_k \\
& + \left(-\frac{1}{2} l(t) - 2\sigma \lambda_{\min}(P_2) \frac{\dot{l}(t)}{l(t)} + \sqrt{2n(n+1)}\|P_2\|\theta + r\|P_2\|^2 m^2 N \right) z_2^{\mathrm{T}} z_2 + \frac{\delta^2 N}{2 r l^{2\sigma-2}(t)}.
\end{aligned} \tag{16}$$

Based on the above analysis, we are in the position to show the result that the state of the system achieves consensus.

Theorem 1. Consider system (5) with uniform quantized communication described by (1). Under Assumptions 1 and 2 and control protocol (7), if the control gain $l(t)$ is designed as

$$\dot{l}(t) = \max\{G_1, G_2, 0\}, \quad l(0) = 1, \tag{17}$$

where

$$G_1 \triangleq \frac{l(t)}{2\sigma\lambda_{\min}(P)}(-l(t)h + \sqrt{2n(n+1)}\|P\|\theta + r\|P\|^2m^2 + 2l(t)\|P_2\|^2m^2N + \alpha_1), \tag{18}$$

and

$$G_2 \triangleq \frac{l(t)}{2\sigma\lambda_{\min}(P_2)}\left(-\frac{1}{2}l(t) + \sqrt{2n(n+1)}\|P_2\|\theta + r\|P_2\|^2m^2N + \alpha_2\right) \tag{19}$$

with $\alpha_1 \in \mathbb{R}^+$, $\alpha_2 \in \mathbb{R}^+$, and

$$h > 2\|P_2\|^2m^2N, \tag{20}$$

the δ-asymptotic distributed consensus can be achieved.

Proof. According to (20), we know that $-h + 2\|P_2\|^2m^2N < 0$. Combining (17) we know that $l(t)$ is a non-decreasing function. $\dot{l}(t) > 0$ when $l(t) < \max\{b_1, b_2\} \triangleq M$ with $b_1 = \frac{\sqrt{2n(n+1)}\|P\|\theta + r\|P\|^2m^2 + \alpha_1}{h - 2\|P_2\|^2m^2N}$ and $b_2 = 2(\sqrt{2n(n+1)}\|P_2\|\theta + r\|P_2\|^2m^2N + \alpha_2)$. $\dot{l}(t)$ becomes zero when $l(t) \geqslant M$, which implies that $l(t)$ is upper-bounded by M.

Substituting (17)–(19) into (16), it follows that

$$\dot{V}(t) \leqslant -\alpha_1\sum_{k=1}^{N}\tilde{x}_k^{\mathrm{T}}\tilde{x}_k - \alpha_2 z_2^{\mathrm{T}}z_2 + \frac{\delta^2N}{2rl^{2\sigma-2}(t)} \leqslant -c_1V + c_2,$$

where $c_1 = \min\{\frac{\alpha_1}{\lambda_{\max(P)}}, \frac{\alpha_2}{\lambda_{\max(P_2)}}\}$, $c_2 = \frac{\delta^2N}{2r}$ when $\sigma \geqslant 1$ or $c_2 = \frac{\delta^2NM^{2-2\sigma}}{2r}$ when $0 < \sigma < 1$, which implies that

$$\lim_{t\to\infty} V(t) \leqslant \lim_{t\to\infty}\left(\left(V(0) - \frac{c_2}{c_1}\right)e^{-c_1t} + \frac{c_2}{c_1}\right) = \frac{c_2}{c_1}.$$

Then

$$\lim_{\delta\to0}\lim_{t\to\infty} V(t) = 0.$$

That is $\lim_{\delta\to0}\lim_{t\to\infty}\|\tilde{x}\| = 0$ and $\lim_{\delta\to0}\lim_{t\to\infty}\|z_2\| = 0$. According to $\lim_{\delta\to0}\lim_{t\to\infty}\|\tilde{x}\| = 0$, we know that $\lim_{\delta\to0}\lim_{t\to\infty}\|x_k - \hat{x}_k\| = 0, \forall k \in \mathcal{V}$. According to $\lim_{\delta\to0}\lim_{t\to\infty}\|z_2\| = 0$, we know that $\lim_{\delta\to0}\lim_{t\to\infty}\|\hat{x}_1 - \hat{x}_k\| = 0$. Therefore, we can get that $\lim_{\delta\to0}\lim_{t\to\infty}\|x_k - x_j\| = 0$, $\forall k, j \in \mathcal{V}$. The proof of Theorem 1 is completed.

Remark 1. From the analysis, we know that both $\tilde{x}_{k,1}$ and $l(t)$ are bounded. In this case, the quantization level can be bounded by $R = \frac{\sup(x_{k,1} - \hat{x}_{k,1})}{\delta} = \sqrt{\frac{\max\{V(0), \frac{c_2}{c_1}\}}{\lambda_{\min(P_1)}}}\frac{l^\sigma(t)}{\delta}$ and the communication bandwidth can be set according to transmission rate $\lceil\log_2(2R)\rceil$-bits.

4.3 Converge analysis with logarithmic quantizer

In this subsection, we shall analyze the distributed consensus problem under communication with logarithmic quantization. Because the communication is under logarithmic quantizer, according to (4), we have

$$q(y_k - \hat{x}_{k,1}) = (1 + \triangle_k(x))(x_{k,1} - \hat{x}_{k,1}), \quad \triangle_k(x) \in [-\beta, \beta). \tag{21}$$

Using (5), (6), (8), and (21), we can obtain that

$$\dot{\tilde{x}}_{k,i} = l(t)\tilde{x}_{k,i+1} + \frac{f_i(\bar{x}_{k,i}) - f_i(\bar{\hat{x}}_{k,i})}{l^{i-1+\sigma}(t)} - l(t)m_i\tilde{x}_{k,1} - l(t)m_i\triangle_k(x)\tilde{x}_{k,1} - \frac{\dot{l}(t)}{l(t)}(i - 1 + \sigma)\tilde{x}_{k,i},$$

and

$$\dot{\tilde{x}}_{k,n} = \frac{f_n(\bar{x}_{k,n}) - f_n(\bar{\hat{x}}_{k,n})}{l^{n-1+\sigma}(t)} - l(t)m_n\tilde{x}_{k,1} - l(t)m_n\triangle_k(x)\tilde{x}_{k,1} - \frac{\dot{l}(t)}{l(t)}(n-1+\sigma)\tilde{x}_{k,n}.$$

Let $\tilde{x}_k = [\tilde{x}_{k,1} \cdots \tilde{x}_{k,n}]^{\mathrm{T}}$ for $k \in \mathcal{V}$. One has

$$\dot{\tilde{x}}_k = l(t)\bar{A}\tilde{x}_k + \bar{f}_k - l(t)[m_1 \cdots m_n]^{\mathrm{T}}\triangle_k(x)\tilde{x}_{k,1} - \frac{\dot{l}(t)}{l(t)}(D+\sigma I)\tilde{x}_k.$$

Define $V_{1,k} = \tilde{x}_k^{\mathrm{T}} P \tilde{x}_k$. Taking derivative of V_1 with respect to t yields that

$$\begin{aligned}\dot{V}_{1,k} &= \tilde{x}_k^{\mathrm{T}}\left(l(t)\bar{A}^{\mathrm{T}}P + l(t)P\bar{A} - \frac{\dot{l}(t)}{l(t)}(D+\sigma I)P - \frac{\dot{l}(t)}{l(t)}P(D+\sigma I)\right)\tilde{x}_k + 2\tilde{x}_k^{\mathrm{T}}P\bar{f}_k \\ &\quad -2\tilde{x}_k^{\mathrm{T}}Pl(t)[m_1 \cdots m_n]^{\mathrm{T}}\triangle_k(x)\tilde{x}_{k,1} \\ &\leqslant \tilde{x}_k^{\mathrm{T}}\left(-l(t)hI - 2\sigma\frac{\dot{l}(t)}{l(t)}P\right)\tilde{x}_k + 2\tilde{x}_k^{\mathrm{T}}P\bar{f}_k - 2l(t)\tilde{x}_k^{\mathrm{T}}P[m_1 \cdots m_n]^{\mathrm{T}}\triangle_k(x)\tilde{x}_{k,1}. \end{aligned} \tag{22}$$

Since $\triangle_k(x) \in [-\beta, \beta)$, one can have

$$2l(t)\tilde{x}_k^{\mathrm{T}}P[m_1 \cdots m_n]^{\mathrm{T}}\triangle_k(x)\tilde{x}_{k,1} \leqslant 2l(t)\|\tilde{x}_k\|\|\triangle_k(x)\|\|P\|\|[m_1 \cdots m_n]\|\|\tilde{x}_k\| \leqslant 2l(t)m\|P\|\beta\tilde{x}_k^{\mathrm{T}}\tilde{x}_k.$$

Eq. (22) is converted into

$$\dot{V}_{1,k} \leqslant \tilde{x}_k^{\mathrm{T}}\left(-l(t)hI - 2\sigma\frac{\dot{l}(t)}{l(t)}P + \sqrt{2n(n+1)}\|P\|\theta I + 2l(t)m\|P\|\beta I\right)\tilde{x}_k. \tag{23}$$

According to the define of $\eta_{k,i}$, (6) and (21), we have

$$\dot{\eta}_{k,i} = l(t)\eta_{k,i+1} + \frac{f_i(\bar{\hat{x}}_{k,i})}{l^{i-1+\sigma}(t)} + l(t)m_i(1+\triangle_k(x))\tilde{x}_{k,1} - \frac{\dot{l}(t)}{l(t)}(i-1+\sigma)\eta_{k,i},$$

and

$$\dot{\eta}_{k,n} = \frac{u_k}{l^{n-1+\sigma}(t)} + \frac{f_n(\bar{\hat{x}}_{k,n})}{l^{n-1+\sigma}(t)} + l(t)m_n(1+\triangle_k(x))\tilde{x}_{k,1} - \frac{\dot{l}(t)}{l(t)}(n-1+\sigma)\eta_{k,n}.$$

Let $\eta_k = [\eta_{k,1} \cdots \eta_{k,n}]^{\mathrm{T}}$ for $k \in \mathcal{V}$. One obtains that

$$\dot{\eta}_k = l(t)A\eta_k + \tilde{f}_k + l(t)[m_1 \cdots m_n]^{\mathrm{T}}(1+\triangle_k(x))\tilde{x}_{k,1} - \frac{\dot{l}(t)}{l(t)}(D+\sigma I)\eta_k + [0 \cdots 0\ 1]^{\mathrm{T}}\frac{u_k}{l^{n-1+\sigma}(t)}. \tag{24}$$

Letting $\eta = [\eta_1^{\mathrm{T}} \cdots \eta_N^{\mathrm{T}}]^{\mathrm{T}}$, combining (7) with (24), it is inferred that

$$\dot{\eta} = l(t)((I_N \otimes A) - (L \otimes (BK)))\eta - \frac{\dot{l}(t)}{l(t)}(I_N \otimes (D+\sigma I))\eta + \tilde{f} + l(t)(\bar{\triangle}(x) \otimes G)[\tilde{x}_1^{\mathrm{T}} \cdots \tilde{x}_N^{\mathrm{T}}]^{\mathrm{T}}, \tag{25}$$

where $\bar{\triangle}(x) = \mathrm{diag}[1+\triangle_1(x) \cdots 1+\triangle_N(x)]$. Define the linear transformation $z = (\Phi \otimes I_n)\eta$. According to (25), we have

$$\dot{z} = l(t)((I_N \otimes A) - (\Phi L \Phi^{-1} \otimes (BK)))z - \frac{\dot{l}(t)}{l(t)}(I_N \otimes (D+\sigma I))z + (\Phi \otimes I_n)\tilde{f} + l(t)((\Phi\bar{\triangle}(x)) \otimes G)[\tilde{x}_1^{\mathrm{T}} \cdots \tilde{x}_N^{\mathrm{T}}]^{\mathrm{T}},$$

where $\Phi L \Phi^{-1} = \mathrm{diag}[0, \tilde{L}]$ and $\tilde{L} \in \mathbb{R}^{(N-1)\times(N-1)}$ is with all its eigenvalues having positive real parts. Decompose $z = [z_1^{\mathrm{T}}, z_2^{\mathrm{T}}]^{\mathrm{T}}$ with $z_1 \in \mathbb{R}^n$ and $z_2 \in \mathbb{R}^{n(N-1)}$, and then

$$\dot{z}_2 = l(t)((I_{N-1} \otimes A) - (\tilde{L} \otimes (BK)))z_2 - \frac{\dot{l}(t)}{l(t)}(I_{N-1} \otimes (D+\sigma I))z_2 + (\phi^{\mathrm{T}} \otimes I_n)\tilde{f} + l(t)((\phi^{\mathrm{T}}\bar{\triangle}(x)) \otimes G)[\tilde{x}_1^{\mathrm{T}} \cdots \tilde{x}_N^{\mathrm{T}}]^{\mathrm{T}}.$$

Define $V_2 = z_2^{\mathrm{T}} P_2 z_2$. Taking time derivative of V_2 with respect to t yields that

$$\begin{aligned}\dot{V}_2 =& z_2^{\mathrm{T}}\Bigg(l(t)((I_{N-1}\otimes A)-(\tilde{L}\otimes(BK)))^{\mathrm{T}}P_2 + l(t)P_2((I_{N-1}\otimes A)-(\tilde{L}\otimes(BK)))\\ &-\frac{\dot{l}(t)}{l(t)}(I_{N-1}\otimes(D+\sigma I))P_2 - \frac{\dot{l}(t)}{l(t)}P_2(I_{N-1}\otimes(D+\sigma I))\Bigg)z_2\\ &+2z_2^{\mathrm{T}}P_2(\phi^{\mathrm{T}}\otimes I_n)\tilde{f} + 2z_2^{\mathrm{T}}P_2 l(t)((\phi^{\mathrm{T}}\bar{\triangle}(x))\otimes G)[\tilde{x}_1^{\mathrm{T}}\cdots\tilde{x}_N^{\mathrm{T}}]^{\mathrm{T}}\\ &\leqslant z_2^{\mathrm{T}}\left(-l(t)I - 2\sigma\frac{\dot{l}(t)}{l(t)}P_2\right)z_2 + 2z_2^{\mathrm{T}}P_2(\phi^{\mathrm{T}}\otimes I_n)\tilde{f} + 2z_2^{\mathrm{T}}P_2 l(t)((\phi^{\mathrm{T}}\bar{\triangle}(x))\otimes G)[\tilde{x}_1^{\mathrm{T}}\cdots\tilde{x}_N^{\mathrm{T}}]^{\mathrm{T}}. \end{aligned}\tag{26}$$

Based on Lemma 1, one have

$$2z_2^{\mathrm{T}}P_2 l(t)((\phi^{\mathrm{T}}\bar{\triangle}(x))\otimes G)[\tilde{x}_1^{\mathrm{T}}\cdots\tilde{x}_N^{\mathrm{T}}]^{\mathrm{T}} \leqslant \frac{1}{2}l(t)z_2^{\mathrm{T}}z_2 + 2l(t)\|P_2\|^2 m^2(1+\beta)^2 N\sum_{k=1}^{N}\tilde{x}_k^{\mathrm{T}}\tilde{x}_k.$$

Eq. (26) can be rewritten as

$$\dot{V}_2 \leqslant z_2^{\mathrm{T}}\left(-\frac{1}{2}l(t)I - 2\sigma\frac{\dot{l}(t)}{l(t)}P_2 + \sqrt{2n(n+1)}\|P_2\|\theta I\right)z_2 + 2l(t)\|P_2\|^2 m^2(1+\beta)^2 N\sum_{k=1}^{N}\tilde{x}_k^{\mathrm{T}}\tilde{x}_k. \tag{27}$$

Choose a Lyapunov candidate function as

$$V = \sum_{k=1}^{N} V_{1,k} + V_2.$$

From (23) and (27), it follows that

$$\begin{aligned}\dot{V} \leqslant& \left(-l(t)h - 2\sigma\lambda_{\min}(P)\frac{\dot{l}(t)}{l(t)} + \sqrt{2n(n+1)}\|P\|\theta + 2l(t)m\|P\|\beta + 2l(t)\|P_2\|^2 m^2(1+\beta)^2 N\right)\sum_{k=1}^{N}\tilde{x}_k^{\mathrm{T}}\tilde{x}_k\\ &+\left(-\frac{1}{2}l(t) - 2\sigma\lambda_{\min}(P_2)\frac{\dot{l}(t)}{l(t)} + \sqrt{2n(n+1)}\|P_2\|\theta\right)z_2^{\mathrm{T}}z_2.\end{aligned}\tag{28}$$

Based on the above analysis, we are in the position to show the result that the state of the system achieves consensus.

Theorem 2. Consider system (5) with logarithmic quantized communication described by (3). Under Assumptions 1 and 2 and control protocol (7), if the control gain $l(t)$ is designed as

$$\dot{l}(t) = \max\{G_1, G_2, 0\}, \quad l(0) = 1, \tag{29}$$

where

$$G_1 \triangleq \frac{l(t)}{2\sigma\lambda_{\min}(P)}(-l(t)h + \sqrt{2n(n+1)}\|P\|\theta + 2l(t)m\|P\|\beta + 2l(t)\|P_2\|^2 m^2(1+\beta)^2 N + \alpha_3), \tag{30}$$

and

$$G_2 \triangleq \frac{l(t)}{2\sigma\lambda_{\min}(P_2)}\left(-\frac{1}{2}l(t) + \sqrt{2n(n+1)}\|P_2\|\theta + \alpha_4\right) \tag{31}$$

with $\alpha_3 \in \mathbb{R}^+$, $\alpha_4 \in \mathbb{R}^+$, and $h > 8\|P_2\|^2 m^2 N$, for any given $\bar{\beta} \in (0, \beta_1)$, where

$$\begin{aligned}&\beta_1 = \max\beta\\ &\text{s.t. } -h + 2m\|P\|\beta + 2\|P_2\|^2 m^2(1+\beta)^2 N \leqslant 0,\end{aligned}$$

the distributed consensus can be achieved.

Proof. Firstly, we will prove the boundedness of $l(t)$. According to the definite of β_1, we know that $-h + 2m\|P\|\beta_1 + 2\|P_2\|^2 m^2(1+\beta_1)^2 N \leqslant 0$. When $\bar{\beta} \in (0, \beta_1)$,

$$-h + 2m\|P\|\bar{\beta} + 2\|P_2\|^2 m^2(1+\bar{\beta})^2 N < 0.$$

We can get from (29) that $l(t)$ is a nondecreasing function. $\dot{l}(t) > 0$ when $l(t) < \max\{b_3, b_4\}$ with $b_3 = \frac{\sqrt{2n(n+1)}\|P\|\theta + \alpha_3}{h - 2m\|P\|\beta - 2\|P_2\|^2 m^2(1+\beta)^2 N}$ and $b_4 = 2(\sqrt{2n(n+1)}\|P_2\|\theta + \alpha_4)$. $\dot{l}(t)$ becomes zero when $l(t) \geqslant \max\{b_3, b_4\}$, which implies that $l(t)$ is bounded.

Finally, we will prove that the state of the system achieves consensus. Substituting (29)–(31) into (28), it follows that

$$\dot{V}(t) \leqslant -\alpha_3 \sum_{k=1}^{N} \tilde{x}_k^{\mathrm{T}} \tilde{x}_k - \alpha_4 z_2^{\mathrm{T}} z_2 \leqslant -c_1 V,$$

where $c_1 = \min\{\frac{\alpha_3}{\lambda_{\max(P)}}, \frac{\alpha_4}{\lambda_{\max(P_2)}}\}$, which implies that

$$\lim_{t\to\infty} V(t) \leqslant \lim_{t\to\infty} V(0) e^{-c_1(t-t_0)} = 0.$$

That is $\lim_{t\to\infty} \|\tilde{x}\| = 0$ and $\lim_{t\to\infty} \|z_2\| = 0$. According to the definition of $\|\tilde{x}\|$ and $\|z_2\|$, we know that $\lim_{t\to\infty} \|x_k - \hat{x}_k\| = 0$, $\forall k \in \mathcal{V}$ and $\lim_{t\to\infty} \|\hat{x}_1 - \hat{x}_k\| = 0$. Therefore, we can get that $\lim_{t\to\infty} \|x_k - x_j\| = 0$, $\forall k, j \in \mathcal{V}$. The proof of Theorem 2 is completed.

Remark 2. Compared with [11], this paper studies the leaderless distributed consensus problem for nonlinear system rather than leader-follower consensus problem. For nonlinear systems, the leaderless distributed consensus is more complex than leader-follower consensus, because it is difficult to deal with the nonlinear term of the derivative of $\sum_{j\in\mathcal{N}_k}(x_k - x_j)$. In this paper, by using a linear transformation, we get that $\eta_1 - \eta_k$ converges to zero, that is to say, $\hat{x}_1 - \hat{x}_k$ converges to zero. Thus, the nonlinear leaderless consensus problem is solved.

Remark 3. The quantized input problems of the nonlinear systems are considered in [23–30]. Compared with [23–30], the quantizer is used in the communication channel between agents to reduce the communication burden between agents. Compared with [33], we study the quantized communication problem for more general nonlinear systems.

5 Simulation examples

In this section, we give two numerical examples to show the effectiveness of this method. Consider the nonlinear multi-agent system:

$$\begin{cases} \dot{x}_{k,1} = x_{k,2} - x_{k,1}, \\ \dot{x}_{k,2} = u_k + \sin(x_{k,2}), \\ y_k = x_{k,1}, \quad k = 1, 2, 3. \end{cases}$$

Design the following observer to estimate the system state

$$\begin{cases} \dot{\hat{x}}_{k,1} = \hat{x}_{k,2} - \hat{x}_{k,1} + 2l(t)q(y_k - \hat{x}_{k,1}), \\ \dot{\hat{x}}_{k,2} = u_k + \sin(\hat{x}_{k,2}) + l^2(t)q(y_k - \hat{x}_{k,1}), \quad k = 1, 2, 3. \end{cases}$$

The communication graph is shown in Figure 2.

According to the system and the observer, we know that $\theta = 1$. The Laplacian matrix is $L = [1, 0, -1; -1, 1, 0; 0, -1, 1]$.

Case 1. Consider uniform quantizer given in (1) with $\delta = 0.1$. The initial conditions of the agents and the observers are $x_{1,1}(0) = 1.1$, $x_{1,2}(0) = 1$, $x_{2,1}(0) = 2$, $x_{2,2}(0) = 2.5$, $x_{3,1}(0) = 3.9$, $x_{3,2}(0) = 3.2$, $\hat{x}_{1,1}(0) = 1.15$, $\hat{x}_{1,2}(0) = 1.1$, $\hat{x}_{2,1}(0) = 2.1$, $\hat{x}_{2,2}(0) = 2.6$, $\hat{x}_{3,1}(0) = 3.95$, $\hat{x}_{3,2}(0) = 3.5$, and $l(0) = 1$.

Through the design process in Subsection 4.2, the consensus protocol can be constructed as $u_1 = -K\Lambda(t)(\hat{x}_1 - \hat{x}_3)$, $u_2 = -K\Lambda(t)(\hat{x}_2 - \hat{x}_1)$, and $u_3 = -K\Lambda(t)(\hat{x}_3 - \hat{x}_2)$, where $K = [5, 2.5]$ and $\Lambda(t) = \mathrm{diag}[l^2(t), l(t)]$.

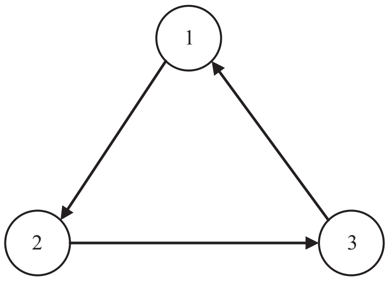

Figure 2 Communication graph $\mathcal{G}$.

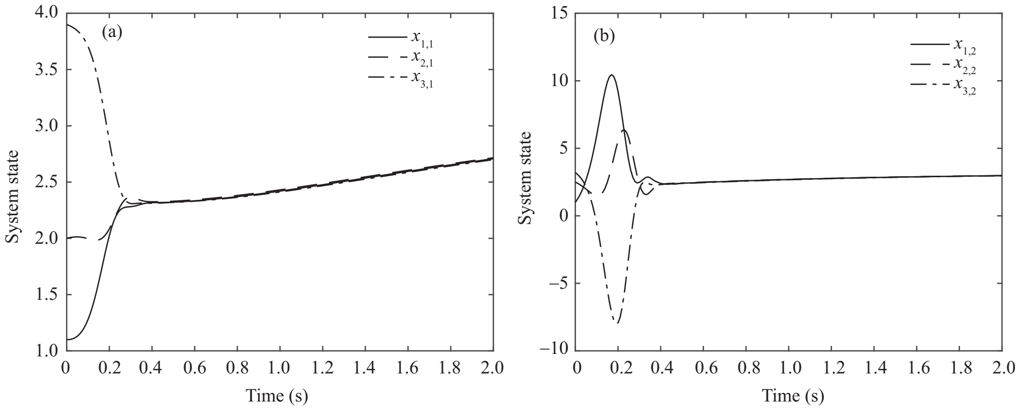

Figure 3 The trajectories of (a) $x_{1,1}$, $x_{2,1}$, $x_{3,1}$ of case 1, and (b) $x_{1,2}$, $x_{2,2}$, $x_{3,2}$ of case 1.

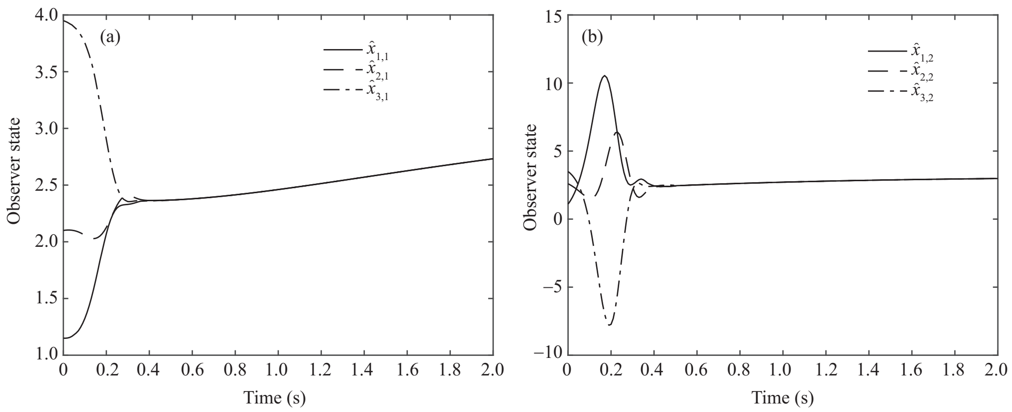

Figure 4 The trajectories of (a) $\hat{x}_{1,1}$, $\hat{x}_{2,1}$, $\hat{x}_{3,1}$ of case 1, and (b) $\hat{x}_{1,2}$, $\hat{x}_{2,2}$, $\hat{x}_{3.2}$ of case 1.

According to (9), (13), and Theorem 1 with $m_1 = 2$, $m_2 = 1$, $\sigma = 2$, and $r = 0.001$, we obtain that

$$\dot{l}(t) = \max\{G_1, G_2, 0\}, \quad l(0) = 1,$$

where $G_1 = -0.4320l^2(t) + 13.1228l(t)$ and $G_2 = -2.1739l^2(t) + 28.5597l(t)$.

The system states trajectories are shown in Figure 3 and the observer states trajectories are shown in Figure 4. Figure 5(a) expresses the trajectories of u_i, $i = 1, 2, 3$. Figure 5(b) illustrates the trajectory of $l(t)$. We can get from the simulation results that the system achieves δ-asymptotic distributed consensus.

Case 2. Logarithmic quantizer defined in (3) with $\beta = 0.06$ and $\omega(0) = 1$ is considered. The initial conditions of the agents and the observers are $x_{1,1}(0) = 1.4$, $x_{1,2}(0) = 1.8$, $x_{2,1}(0) = 2.6$, $x_{2,2}(0) = 2.5$, $x_{3,1}(0) = 3.1$, $x_{3,2}(0) = 3.2$, $\hat{x}_{1,1}(0) = 1.45$, $\hat{x}_{1,2}(0) = 2.5$, $\hat{x}_{2,1}(0) = 2.65$, $\hat{x}_{2,2}(0) = 3$, $\hat{x}_{3,1}(0) = 3.15$, $\hat{x}_{3,2}(0) = 3.5$, and $l(0) = 1$.

Through the design process in Subsection 4.3, the consensus protocol can be constructed as $u_1 = -K\Lambda(t)(\hat{x}_1 - \hat{x}_3)$, $u_2 = -K\Lambda(t)(\hat{x}_2 - \hat{x}_1)$, and $u_3 = -K\Lambda(t)(\hat{x}_3 - \hat{x}_2)$, where $K = [10, 5]$ and $\Lambda(t) = \mathrm{diag}[l^2(t), l(t)]$.

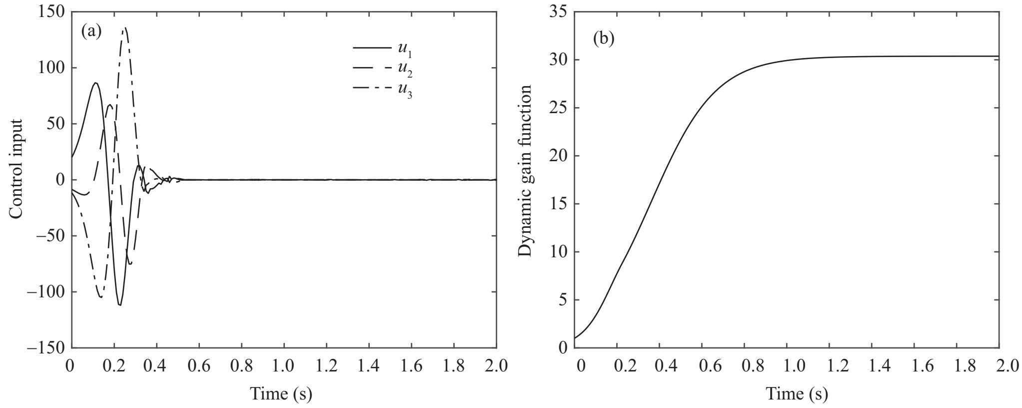

Figure 5 The trajectories of (a) u_1, u_2, u_3 of case 1, and (b) $l(t)$ of case 1.

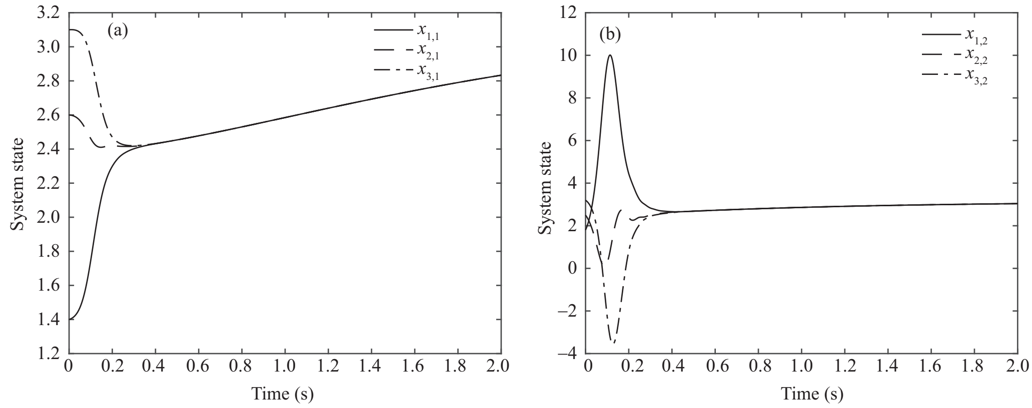

Figure 6 The trajectories of (a) $x_{1,1}$, $x_{2,1}$, $x_{3,1}$ of case 2, and (b) $x_{1,2}$, $x_{2,2}$, $x_{3,2}$ of case 2.

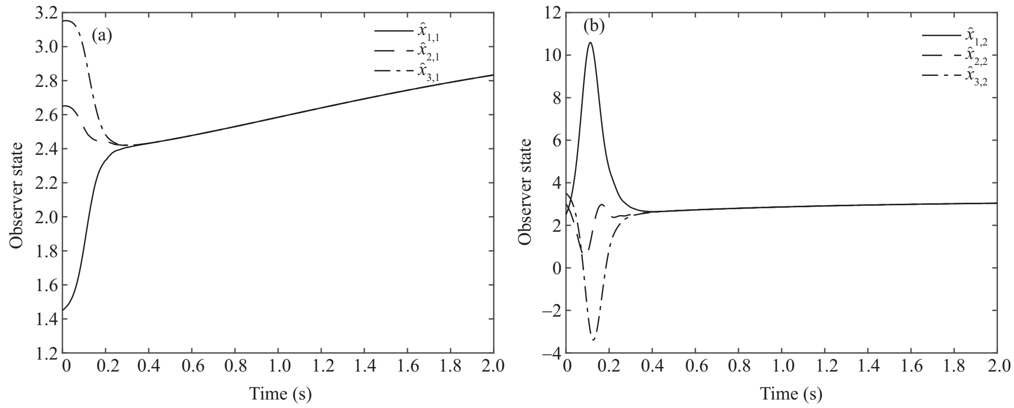

Figure 7 The trajectories of (a) $\hat{x}_{1,1}$, $\hat{x}_{2,1}$, $\hat{x}_{3,1}$ of case 2, and (b) $\hat{x}_{1,2}$, $\hat{x}_{2,2}$, $\hat{x}_{3,2}$ of case 2.

According to (9), (13), and Theorem 2 with $m_1 = 2$, $m_2 = 1$, and $\sigma = 2$, we can obtain that

$$\dot{l}(t) = \max\{G_1, G_2, 0\}, \quad l(0) = 1,$$

where $G_1 = -0.4466l^2(t) + 10.0984l(t)$ and $G_2 = -4.5372l^2(t) + 50.5199l(t)$.

The system states trajectories are shown in Figure 6 and the observer states trajectories are shown in Figure 7. Figure 8(a) expresses the trajectories of u_i, $i = 1, 2, 3$. Figure 8(b) illustrates the trajectory of $l(t)$. We can get from the simulation results that the system achieves asymptotic consensus.

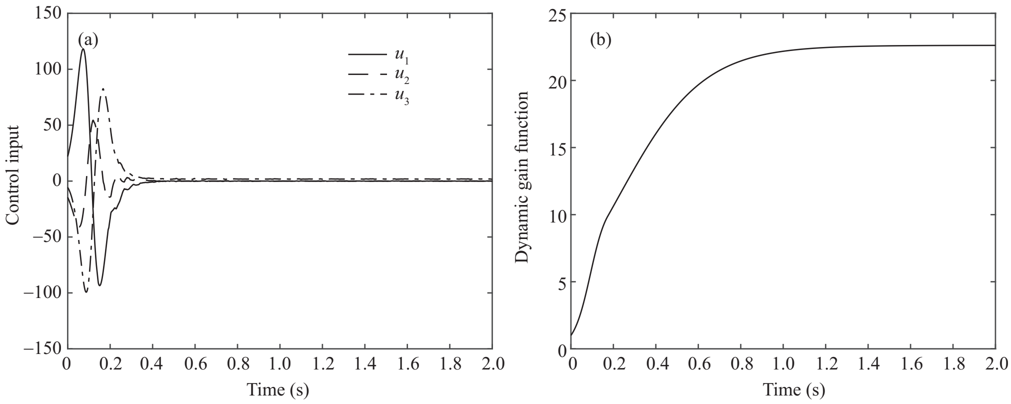

Figure 8 The trajectories of (a) u_1, u_2, u_3 of case 2, and (b) $l(t)$ of case 2.

6 Conclusion

In this paper, we propose distributed control protocols based on the observer, estimating the state of the original system, for nonlinear multi-agent systems via a dynamic gain method. The uniform quantizer and logarithmic quantizer are used to reduce the data transmission rate. Results show that the system consensus errors are uniformly bounded for the uniform quantizer. In particular, when δ converges to zero, the consensus errors converge to zero. The consensus errors of systems converge to zero for the logarithmic quantizer. The future interest is to continue to consider the quantizer consensus for nonlinear systems so that it can reduce the communication burden to a greater extent.

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