SCIENCE CHINA
Information Sciences

• RESEARCH PAPER •

September 2022, Vol. 65 192201:1–192201:15
https://doi.org/10.1007/s11432-021-3448-1

Discrete-time delay systems: part 2. Sub-fully actuated case

Guangren DUAN[1,2]

[1]*Center for Control Science and Technology, Southern University of Science and Technology, Shenzhen* 518055*, China;*
[2]*Center for Control Theory and Guidance Technology, Harbin Institute of Technology, Harbin* 150001*, China*

Received 9 September 2021/Revised 19 November 2021/Accepted 11 February 2022/Published online 29 August 2022

Abstract In continuation to the first part of the paper, this second part further investigates the models and control of discrete-time sub-fully actuated systems with time-varying delays. Firstly, a new representation for general linear and nonlinear dynamical discrete-time fully actuated systems (FASs) with time-varying state delays and constant input delays is proposed, and the concept of sub-FAS is defined. The set of feasible points and the set of singular points for a sub-FAS are introduced. Secondly, like the global FAS case, controllers for a discrete-time sub-FAS can also be easily designed, which results in constant linear closed-loop systems with arbitrarily assignable eigenstructures, but unlike the global FAS case, a constraint must be added, which is expressed by the set of feasible points of the system and guarantees the realizability of the designed controllers. Finally, a general definition for controllable dynamical systems with time delays is given.

Keywords time-delay systems, discrete-time systems, sub-fully actuation, control designs, feasibility

Citation Duan G R. Discrete-time delay systems: part 2. Sub-fully actuated case. Sci China Inf Sci, 2022, 65(9): 192201, https://doi.org/10.1007/s11432-021-3448-1

1 Introduction

In the first part of this study [1], a very brief overview on discrete-time delay systems was given. Basically, it mentioned that time-delay systems are often encountered in practical engineering [2], and have remained to be a difficult problem for decades. On one hand, as far as state-space representations are used, nonlinearities have always been a vital problem. Nonlinear control problems can only be solved with state-space approaches under very strict assumptions, either on the structure of the system or on the magnitude of the nonlinear terms in the system. On the other hand, time-varying delays also add real difficulties, and make the general control problem truly a harder one. The problem is quite challenging due to complicated dynamical properties [3], and consequently, there is still an ample amount of technical problems open for scientific research [4]. As a matter of fact, today even the control of a discrete-time linear system with time-varying delays still remains to be an unsolvable problem in general. This is because in such a case, the commonly used state augmentation technique is usually difficult to apply, or even fails [5, 6]. Furthermore, most existing results based on various types of methods can be only obtained in a local sense (see [7–12]).

The first part of this study [1] also pointed out that there exists a typical route in control systems design using state-space approaches, that is, first to establish the stability results for the type of systems, and then to perform the control design based on the obtained stability conditions. Such a design route may suffer from two problems, one is that stability analysis of a nonlinear system is generally difficult, especially that of a nonlinear delay system, the other is that the design results are limited by the conservatism of the stability analysis results.

Is there an effective nonlinear control method which is applicable to nonlinear time-delay systems and is not dependent on the stability analysis of the open-loop systems? For a positive answer to this question, Ref. [1] generalized the proposed fully actuated system (FAS) approaches recently proposed for

Email: duangr@sustech.edu.cn; g.r.duan@hit.edu.cn; grduan@ieee.org

continuous-time dynamical systems (see [13–24]), and discrete-time ones [25], to discrete-time systems with delays.

Firstly, it was shown in [1] that a general FAS model representation exists for discrete-time systems with time-varying state delays and a constant input delay, which covers the discrete-time FAS model proposed in [25] as a special case. It is shown by two specific classes of systems, namely, the linear discrete-time systems with constant time delays and the nonlinear discrete-time strict-feedback systems with input delays, that certain discrete-time delay systems in state-space form can indeed be converted into a discrete-time FAS with time delays.

Secondly, it was proven in [1] that the control strategy based on the full-actuation feature of a FAS works as well on the type of proposed discrete-time delay FASs. As expected, in the case that the FASs have input delays, a prediction mechanism is needed and can be easily constructed based on the open-loop system. As an advantage of the FAS approach, the closed-loop system turns out to be a constant linear one with an arbitrarily assignable eigenstructure.

The first part of this study [1] concentrated on the case of discrete-time global FASs with time-varying delays, while in this paper, we further investigate the case of discrete-time sub-FASs with time-varying delays. It starts with an introduction of a new representation of a discrete-time FAS with multiple time-varying delays. For such a system, there may be quite a few integer-valued time-varying functions involved, which represent the time-varying delays in the system. With the new representation, the number of time-varying integer-valued functions is dramatically reduced, often to 1. Such a representation gives great convenience in dealing with sub-FASs, in which the full-actuation condition does not hold globally. As in the continuous-time system case [22], this gives arise to the problems of singularity and feasibility of the systems. Eventually, the concepts of set of feasible points and set of singular points associated with a sub-FAS are defined.

For control of a discrete-time sub-FAS with time-varying state delays only, a controller in state feedback form can also be conveniently designed, which again results in a constant linear closed-loop systems with an arbitrary system order and an arbitrarily assignable eigenstructure, but within the “feasible region” determined by the set of feasible points of the system. While for control of a discrete-time FAS with both time-varying state delays and a constant input delay, as expected, a prediction procedure is needed to give the states of certain steps ahead in order to make the controller realizable, and the prediction is easily carried out based on the open-loop system. Different from the one for a discrete-time sub-FAS with time-varying state delays only, now the designed controller eventually turns out to be a dynamical one.

Finally, as in the continuous-time system case [22], controllability of a general discrete-time dynamical system with time delays is defined, depending on whether it can be equivalently converted into a global FAS or a sub-FAS.

In the sequential sections, I_n denotes the identity matrix of order n, $\emptyset$ denotes the null set, $\mathbb{N}$ is the set of natural numbers, and $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ denote the spaces of n dimensional vectors and $m\times n$ dimensional matrices, respectively. Furthermore, $\det(A)$ and A^{-1} denote the determinant and the inverse of a matrix A, respectively.

For $x_i\in\mathbb{R}^m$, $i=1,2,\ldots,n$, let

$$x_{i\sim j}(k)=\begin{bmatrix} x_i(k)\\ x_{i+1}(k)\\ \vdots\\ x_j(k)\end{bmatrix},\quad i\leqslant j\leqslant n.$$

For $A_i\in\mathbb{R}^{m\times m}$, $i=0,1,\ldots,n$ as in [18], the following symbols are used:

$$A_{0\sim n}=\begin{bmatrix} A_0 & A_1 & \cdots & A_n\end{bmatrix},$$

$$\Psi(A_{0\sim n})=\begin{bmatrix} A_0 & A_1 & \cdots & A_n\\ I & & & \\ & \ddots & & \\ & & I & 0\end{bmatrix}.$$

For $x(k) \in \mathbb{R}^m$, it is well-known that the one-step backward operator is usually denoted by q^{-1}, which operates in the following way:

$$q^{-1}x(k) = x(k-1), \quad q^{-i}x(k) = x(k-i).$$

For convenience, in this paper define the above operation by the following notation:

$$x^{\lceil i \rceil}(k) = x(k-i).$$

Based on this notation, for $x \in \mathbb{R}^m$, $n_i \in \mathbb{N}$, $i = 1, 2, \ldots, n$, the following symbols are introduced and used in the paper:

$$x^{\lceil n_{1\sim p} \rceil}(k) = \begin{bmatrix} x^{\lceil n_1 \rceil}(k) \\ x^{\lceil n_2 \rceil}(k) \\ \vdots \\ x^{\lceil n_p \rceil}(k) \end{bmatrix}, \quad p \geqslant 1,$$

$$x^{\lceil n_1 \sim n_2 \rceil}(k) = \begin{bmatrix} x^{\lceil n_1 \rceil}(k) \\ x^{\lceil n_1+1 \rceil}(k) \\ \vdots \\ x^{\lceil n_2 \rceil}(k) \end{bmatrix}, \quad n_1 \leqslant n_2,$$

hence

$$x^{\lceil 0 \sim n \rceil}(k) = \begin{bmatrix} x(k) \\ x(k-1) \\ \vdots \\ x(k-n) \end{bmatrix}.$$

Further for $x_i \in \mathbb{R}^m$, $n_0 \in \mathbb{N}$, $n_0 < n_i$, $i = 1, 2, \ldots, n$, the following symbols are used:

$$x_{i\sim j}^{\lceil n_1 \sim n_2 \rceil}(k) = \begin{bmatrix} x_i^{\lceil n_1 \sim n_2 \rceil}(k) \\ x_{i+1}^{\lceil n_1 \sim n_2 \rceil}(k) \\ \vdots \\ x_j^{\lceil n_1 \sim n_2 \rceil}(k) \end{bmatrix}, \quad i \leqslant j, \ n_1 \leqslant n_2,$$

$$x_p^{\lceil n_p \rceil}(k)\,|_{p=i\sim j} = \begin{bmatrix} x_i^{\lceil n_i \rceil}(k) \\ x_{i+1}^{\lceil n_{i+1} \rceil}(k) \\ \vdots \\ x_j^{\lceil n_j \rceil}(k) \end{bmatrix}, \quad i \leqslant j,$$

$$x_p^{\lceil n_0 \sim n_p \rceil}(k)\,|_{p=i\sim j} = \begin{bmatrix} x_i^{\lceil n_0 \sim n_i \rceil}(k) \\ x_{i+1}^{\lceil n_0 \sim n_{i+1} \rceil}(k) \\ \vdots \\ x_j^{\lceil n_0 \sim n_j \rceil}(k) \end{bmatrix}, \quad i \leqslant j.$$

This paper is organized into 7 sections. Section 2 gives a new representation of a linear discrete-time FAS with time-varying delays. In Section 3, the new representation of a nonlinear discrete-time FAS with time-varying delays is introduced, and sub-FASs, as well as the set of singular points and the set of feasible points are defined. Sections 4 and 5 present the designs of controllers for the proposed discrete-time sub-FASs with state delays only, and with both state and input delays, respectively. Section 6 raises some further issues related to the FAS approaches for discrete-time delay systems, followed by a brief concluding remark in Section 7.

2 Linear time-delay FASs

It was pointed in [1] that any discrete-time delay FAS with interconnections can be written in the form of a discrete-time delay FAS without interconnections, just like a decentralized large-scale system can always be written into a centralized system form with a large dimension. Therefore, in this paper only the discrete-time delay system without interconnections proposed in [1] is considered.

2.1 FAS with time-varying delays

The following discrete-time FAS with time-varying state delays and a constant input delay was proposed in [1]:

$$x\left(k+1\right)=L\left(x^{\lceil\tau_{1\sim\zeta}(k)\rceil}\left(k\right),k\right)+Bu\left(k-h\right), \tag{1}$$

where x and $u\in\mathbb{R}^r$ are the state and input vectors, respectively, $B\in\mathbb{R}^{r\times r}$ is a nonsingular matrix,

$$x^{\lceil\tau_{1\sim\zeta}(k)\rceil}\left(k\right)=\begin{bmatrix}x\left(k-\tau_1\left(k\right)\right)\\x\left(k-\tau_2\left(k\right)\right)\\\vdots\\x\left(k-\tau_\zeta\left(k\right)\right)\end{bmatrix},$$

with

$$\tau_{1\sim\zeta}\left(k\right)=\begin{bmatrix}\tau_1\left(k\right)\\\tau_2\left(k\right)\\\vdots\\\tau_\zeta\left(k\right)\end{bmatrix}, \tag{2}$$

and $\tau_i\left(k\right), i=1,2,\ldots,\zeta$, being a group of nonnegative bounded integer-valued functions. Furthermore, $L\left(\cdot\right)$ is a linear vector function with respect to $x^{\lceil\tau_{1\sim\zeta}(k)\rceil}\left(k\right)$.

Clearly, a specific type of time-delay FASs in the form of (1) is the following:

$$x\left(k+1\right)=\sum_{i=1}^{\zeta}A_i x^{\lceil\tau_i(k)\rceil}\left(k\right)+Bu\left(k-h\right), \tag{3}$$

which can also be written, by our notations, as

$$x\left(k+1\right)=A_{1\sim\zeta}x^{\lceil\tau_{1\sim\zeta}(k)\rceil}\left(k\right)+Bu\left(k-h\right). \tag{4}$$

2.2 The equivalent form

To illustrate the idea and process, let us first examine the following specific linear r-dimensional vector function:

$$L\left(k\right)\triangleq L\left(x^{\lceil\tau_{1\sim2}(k)\rceil}\left(k\right)\right)=L_1x\left(k-\tau_1\left(k\right)\right)+L_2x\left(k-\tau_2\left(k\right)\right), \tag{5}$$

where L_1 and L_2 are two constant coefficient matrices, and

$$\tau_1\left(k\right)=\begin{cases}2, \text{ if mod}\left[\frac{k}{3}\right]=0,\\1, \text{ if mod}\left[\frac{k}{3}\right]=1,\\0, \text{ if mod}\left[\frac{k}{3}\right]=2,\end{cases}\quad \tau_2\left(k\right)=\begin{cases}3, \text{ if } k \text{ is odd},\\1, \text{ otherwise},\end{cases} \tag{6}$$

where mod $[\cdot]$ represents the remainder operation.

Clearly, by (5) we have

$$\begin{aligned}L\left(0\right)&=L_1x\left(0-2\right)+L_2x\left(0-1\right),\\L\left(1\right)&=L_1x\left(1-1\right)+L_2x\left(1-3\right),\\L\left(2\right)&=L_1x\left(2-0\right)+L_2x\left(2-1\right),\end{aligned}$$

$$\begin{aligned} L(3) &= L_1 x(3-2) + L_2 x(3-3), \\ L(4) &= L_1 x(4-1) + L_2 x(4-1), \\ &\vdots \end{aligned}$$

From this set of equations, we can get the following new representation for the function $L(k) \triangleq L(x^{\lceil \tau_{1\sim 2}(k) \rceil}(k))$ as

$$L(k) \triangleq L\left(x^{\lceil 0\sim\rho(k) \rceil}(k)\right) = \sum_{i=0}^{\rho(k)} A_i(k)\, x(k-i)$$

with $\rho(k)$ and $A_i(k), i = 0, 1, \ldots, \rho(k)$, being given as follows:

k	$A_0(k)$	$A_1(k)$	$A_2(k)$	$A_3(k)$	$\cdots$	$\rho(k)$
0	0	L_2	L_1	–	$\cdots$	2
1	0	L_1	0	L_2	$\cdots$	3
2	L_1	L_2	–	–	$\cdots$	1
3	0	0	L_1	L_2	$\cdots$	3
4	0	$L_1 + L_2$	–	–	$\cdots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Such a fact implies the following general result.

Lemma 1. Let $L(x^{\lceil \tau_{1\sim\zeta}(k) \rceil}(k))$ be a given linear function, with $\tau_i(k), i = 1, 2, \ldots, \zeta$, being a group of nonnegative bounded integer-valued functions. Let

$$\rho(k) = \max\{\tau_i(k), i = 1, 2, \ldots, \zeta\}.$$

Then there exists a function $L'(x^{\lceil 0\sim\rho(k) \rceil}(k), k)$, which is linear with respect to the first argument, such that

$$L\left(x^{\lceil \tau_{1\sim\zeta}(k) \rceil}(k)\right) = L'\left(x^{\lceil 0\sim\rho(k) \rceil}(k), k\right).$$

With the help of the above lemma, the linear FAS (1) with time-varying delays has the following new form:

$$x(k+1) = L'\left(x^{\lceil 0\sim\rho(k) \rceil}(k), k\right) + Bu(k-h). \tag{7}$$

Similarly, the system (3) can be written as

$$x(k+1) = \sum_{i=0}^{\rho(k)} A_i(k)\, x^{\lceil i \rceil}(k) + Bu(k-h), \tag{8}$$

which can also be written, by our notations, as

$$x(k+1) = A_{0\sim\rho(k)}(k)\, x^{\lceil 0\sim\rho(k) \rceil}(k) + Bu(k-h). \tag{9}$$

Clearly, this is a linear time-delay system with both time-varying delays and time-varying coefficients.

In the special case that $\tau_i(k) = \tau_i, i = 1, 2, \ldots, \zeta$, are all constant, the system (3) becomes the following system with constant delays:

$$x(k+1) = \sum_{i=1}^{\zeta} A_i x^{\lceil \tau_i \rceil}(k) + Bu(k-h). \tag{10}$$

In view of the above Lemma 1, we can define

$$\rho = \max\{\tau_i, i = 1, 2, \ldots, \zeta\},$$

and the above time-delay system (10) can be equivalently expressed as

$$x(k+1) = \sum_{i=0}^{\rho} \bar{A}_i x^{\lceil i \rceil}(k) + Bu(k-h), \tag{11}$$

with $\bar{A}_i, i = 1, 2, \ldots, \rho$, being a group of proper matrix coefficients.

2.3 Further generalizations

Obviously, the discrete-time system (8) can be further generalized into the following form:

$$x(k+1)=\sum_{i=0}^{\rho(k)} A_i(k)\, x^{\lceil i\rceil}(k)+\sum_{i=0}^{m} B_i u^{\lceil i\rceil}(k-h), \tag{12}$$

where $B_i\in\mathbb{R}^{r\times r}, i=1,2,\ldots,m$ are a group of constant coefficient matrices. It can be easily reasoned that, when $B=B_0$ is a square nonsingular matrix, the system (12) can be converted equivalently into the following FAS:

$$x(k+1)=A_{0\sim\rho(k)}(k)\, x^{\lceil 0\sim\rho(k)\rceil}(k)+Bu'(k-h), \tag{13}$$

where

$$u'(k)=B^{-1}\sum_{i=0}^{m} B_i u^{\lceil i\rceil}(k)=u(k)+\sum_{i=1}^{m} B^{-1}B_i u^{\lceil i\rceil}(k). \tag{14}$$

Clearly, the original control vector $u(k)$ can be uniquely solved through (14), and it gives a reasonable control sequence if the above system (14) is stable with respect to $u(k)$.

More generally, let us consider the following system represented by the step backward operator:

$$x(k+1)=\sum_{i=0}^{\rho(k)} A_i(k)\, x^{\lceil i\rceil}(k)+q^{-h}B\left(q^{-1}\right)u(k), \tag{15}$$

where $h\geqslant 0$, and

$$B\left(q^{-1}\right)=\sum_{i=0}^{m} B_i q^{-i}.$$

Such a system is a time-delay FAS if $B(q^{-1})$ is a unimodular matrix, that is,

$$\det B\left(q^{-1}\right)\neq 0,\quad \forall q^{-1}\in\mathbb{C}. \tag{16}$$

Pre-multiplying both sides of (15) by $B^{-1}\left(q^{-1}\right)$, we have

$$B^{-1}\left(q^{-1}\right)\left[x(k+1)-\sum_{i=0}^{\rho(k)} A_i(k)\, x^{\lceil i\rceil}(k)\right]=u(k-h), \tag{17}$$

which can be easily shown to be a discrete-time FAS with time delays.

3 Nonlinear time-delay sub-FASs

3.1 Nonlinear FASs

In [1], both affine and non-affine nonlinear discrete-time FASs with time-varying delays are defined. The affine one appears as

$$x(k+1)=f\left(x^{\lceil \tau_{1\sim\varsigma}(k)\rceil}(k),k\right)+B\left(x^{\lceil \tau_{1\sim\varsigma}(k)\rceil}(k),k\right)u(k-h), \tag{18}$$

where $\tau_{1\sim\varsigma}(k)$ is the time-delay vector defined in (2), $f(\cdot,k)\in\mathbb{R}^r$ and $B(\cdot,k)\in\mathbb{R}^{r\times r}$ are nonlinear functions. The system is called globally fully actuated if the matrix $B(\cdot,k)$ satisfies the following full-actuation condition:

$$\det B\left(x^{\lceil \tau_{1\sim\varsigma}(k)\rceil}(k),k\right)\neq 0,\quad \forall x^{\lceil \tau_{1\sim\varsigma}(k)\rceil}(k)\ \text{and}\ k\geqslant h. \tag{19}$$

Parallel to the above nonlinear affine discrete-time delay FAS (18), the non-affine discrete-time delay FAS is as follows:

$$x(k+1)=f\left(x^{\lceil \tau_{1\sim\varsigma}(k)\rceil}(k),k\right)+g\left(x^{\lceil \tau_{1\sim\varsigma}(k)\rceil}(k),k,u(k-h)\right), \tag{20}$$

where $g(\cdot,k,u)\in\mathbb{R}^r$ is a nonlinear function satisfying the full-actuation assumption: $g(x^{\lceil\tau_1\sim\varsigma(k)\rceil}(k),k,u)=\tilde{u}$ forms a differential homeomorphism from u to $\tilde{u}$ for all $x^{\lceil\tau_1\sim\varsigma(k)\rceil}(k)$ and $k\geqslant h$.

Under the above full-actuation conditions, we can introduce the new control vector:

$$\tilde{u}(k-h)=B\left(x^{\lceil\tau_1\sim\varsigma(k)\rceil}(k),k\right)u(k-h), \tag{21}$$

or

$$\tilde{u}(k-h)=g\left(x^{\lceil\tau_1\sim\varsigma(k)\rceil}(k),k,u(k-h)\right), \tag{22}$$

and the affine system (18), or the non-affine system (20), can be both written into the following standard form:

$$x(k+1)=f\left(x^{\lceil\tau_1\sim\varsigma(k)\rceil}(k),k\right)+\tilde{u}(k-h). \tag{23}$$

3.2 Equivalent forms

Like the linear system case, let us again start with a specific nonlinear function, but of one-dimension:

$$f(k)\triangleq f\left(x^{\lceil\tau_1\sim 2(k)\rceil}(k)\right)=\sin x(k-\tau_1(k))+x^2(k-\tau_2(k)), \tag{24}$$

where $\tau_1(k)$ and $\tau_2(k)$ are again as given in (6).

Clearly, following from (24) we have

$$\begin{aligned}
f(0)&=\sin x(0-2)+x^2(0-1),\\
f(1)&=\sin x(1-1)+x^2(1-3),\\
f(2)&=\sin x(2-0)+x^2(2-1),\\
f(3)&=\sin x(3-2)+x^2(3-3),\\
f(4)&=\sin x(4-1)+x^2(4-1),\\
&\vdots
\end{aligned}$$

From this set of equations, we can get the following new representation for the function $f(k)\triangleq f(x^{\lceil\tau_1\sim 2(k)\rceil}(k))$ as

$$f(k)\triangleq\phi\left(x^{\lceil 0\sim\rho(k)\rceil}(k),k\right), \tag{25}$$

with $\phi(x^{\lceil 0\sim\rho(k)\rceil}(k),k)$ being some nonlinear function, and $\rho(k)$ being a positive integer-valued function defined by

k	0	1	2	3	4	$\cdots$
$\rho(k)$	2	3	1	3	1	$\cdots$

For a general function $\phi(x^{\lceil\tau_1\sim\varsigma(k)\rceil}(k))$, with

$$x=\begin{bmatrix}x_1 & x_2 & \cdots & x_r\end{bmatrix}^{\mathrm{T}}\in\mathbb{R}^r,$$

and $\tau_i(k), i=1,2,\ldots,\zeta$, being integer-valued functions, we can firstly rewrite it into the following finer form:

$$\phi\left(x^{\lceil\tau_1\sim\varsigma(k)\rceil}(k)\right)=\phi\left(x_p^{\lceil\tau_{p,1}\sim\zeta_p(k)\rceil}(k)\Big|_{p=1\sim r}\right), \tag{26}$$

where $\tau_{p,i}(k), i=1,2,\ldots,\zeta_p$, are the ζ_p number of time delays in $x_p(k)$. Then, following the above illustration, we can further convert it into a form that is parallel to (25).

Lemma 2. Let $h(\cdot)$ be a given nonlinear function in the form of (26). Define

$$\rho_p(k)=\max\{\tau_{p,i}(k), i=1,2,\ldots,\zeta_p\}.$$

Then there exists a nonlinear function $\tilde{\phi}(x_p^{\lceil 0\sim\rho_p(k)\rceil}(k)|_{p=1\sim r},k)$ such that

$$\phi\left(x_p^{\lceil\tau_{p,1}\sim\zeta_p(k)\rceil}(k)\Big|_{p=1\sim r}\right)=\tilde{\phi}\left(x_p^{\lceil 0\sim\rho_p(k)\rceil}(k)\Big|_{p=1\sim r},k\right).$$

With the help of the above lemma, the affine FAS (18) with time-varying delays has the following new form:

$$x(k+1) = f(\psi(k), k) + B\left(x_p^{\lceil 0\sim\gamma_p(k)\rceil}(k)\Big|_{p=1\sim r}, k\right) u(k-h), \tag{27}$$

where $\psi(k)$ may take the following three forms:

$$\psi(k) = x^{\lceil \tau_1\sim\varsigma(k)\rceil}(k), \tag{28}$$

$$\psi(k) = x_p^{\left\lceil \tau_{p,1}\sim\zeta_p(k)\right\rceil}(k)\Big|_{p=1\sim r}, \tag{29}$$

or

$$\psi(k) = x_p^{\lceil 0\sim\rho_p(k)\rceil}(k)\Big|_{p=1\sim r}. \tag{30}$$

Because $x_p^{\lceil 0\sim\gamma_p(k)\rceil}(k)\,|_{p=1\sim r} \in \mathbb{R}^{\kappa(k)}$, with

$$\kappa(k) = r + \sum_{p=1}^{r} \gamma_p(k), \tag{31}$$

corresponding to (19), the full-actuation assumption becomes Assumption 1.

Assumption 1. $\det B(X,k) \neq 0$ or $\infty, \forall X \in \mathbb{R}^{\kappa(k)}$ and $k \geqslant h$.

Parallelly, the non-affine discrete-time delay FAS (20) has the following equivalent form:

$$x(k+1) = f(\psi(k), k) + g\left(x_p^{\lceil 0\sim\gamma_p(k)\rceil}(k)\Big|_{p=1\sim r}, k, u(k-h)\right), \tag{32}$$

where $\psi(k)$ takes one of the forms in (28)–(30), and represents the delay terms in the $f(\cdot)$ function, while the function $g(\cdot)$ satisfies the following assumption.

Assumption 2. $g(X,k,u) = \tilde{u}$ forms a differential homeomorphism from u to $\tilde{u}$ for all $X \in \mathbb{R}^{\kappa(k)}$ and $k \geqslant h$.

We point out that the above two assumptions are not at all too conservative, but are relatively general. To understand this point, it is suggested to refer to [1]. Furthermore, it is noted that a sufficient condition for the above Assumption 2 is that $\frac{\partial}{\partial u} g(X,k,u)$ is nonsingular for all $X \in \mathbb{R}^{\kappa(k)}$ and $k \geqslant h$.

As a further generalization of the discrete-time time-delay system (27), the following discrete-time system with time-varying multiple input delays is proposed:

$$x(k+1) = f_e\left(x^{\lceil 0\sim\rho(k)\rceil}(k), u^{\lceil 1\sim\xi(k)\rceil}(k-h), k\right) + B_e\left(x^{\lceil 0\sim\rho(k)\rceil}(k), u^{\lceil 1\sim\xi(k)\rceil}(k-h), k\right) u(k-h), \tag{33}$$

where $\xi(k) \geqslant 1$ is a positive integer-valued function, representing the number of delays in the control vector, $f_e(\cdot) \in \mathbb{R}^r$ and $B_e(\cdot) \in \mathbb{R}^{r\times r}$ are proper nonlinear vector and matrix functions, respectively, while the other variables are as stated before.

Parallel to the extended affine system (33), the following extended non-affine one can also be defined:

$$x(k+1) = f_e\left(x^{\lceil 1\sim\rho(k)\rceil}(k), u^{\lceil 1\sim\xi(k)\rceil}(k-h), k\right) + g_e\left(x^{\lceil 1\sim\rho(k)\rceil}(k), u^{\lceil 1\sim\xi(k)\rceil}(k-h), k, u(k-h)\right), \tag{34}$$

where $\tilde{u} = g_e(X, U, k, u)$ forms a differential homeomorphism from u to $\tilde{u}$ for all $X \in \mathbb{R}^{r\rho(k)}, U \in \mathbb{R}^{r\xi(k)}$ and $k \geqslant h$.

3.3 Sub-fully actuated systems

Let us again consider the time-delay affine system (27) and the non-affine system (32). We here look into the case that Assumptions 1 and 2 are not valid, and this leads to the concept of sub-fully actuated systems.

Recall that $x_p^{\lceil 0\sim\gamma_p(k)\rceil}(k)\,|_{p=1\sim r} \in \mathbb{R}^{\kappa(k)}$. Similar to the treatment in [22–24], we can introduce the following definition.

Definition 1. A vector $X \in \mathbb{R}^{\kappa(k)}$ is called a feasible point of system (27) at time k if it satisfies

$$\det B\left(X, k\right) \neq 0 \text{ or } \infty. \tag{35}$$

Furthermore, the following set:

$$\mathbb{F}_k = \left\{ X \right| \det B\left(X, k\right) \neq 0 \text{ or } \infty \}$$

is called the set of feasible points at time k of system (27).

For convenience, we call

$$\mathbb{S}_k = \mathbb{R}^{\kappa(k)} \backslash \mathbb{F}_k \tag{36}$$

the set of singular points at time k, and any element in $\mathbb{S}_k$ is called a singular point of the system (27) at time k.

Further, corresponding to Definition 2.1 in [22] for continuous-time systems, or Definition 4.2 in [25] for discrete-time systems with constant delays, the following definition for discrete-time sub-fully actuated time-delay systems can also be proposed.

Definition 2. Let $\mathbb{F}_k$ be the set of feasible points of system (27) at time k. Then the system (27) is called (globally) fully actuated at time k if $\mathbb{F}_k = \mathbb{R}^{\kappa(k)}$, and is called sub-fully actuated at time k if $\mathbb{F}_k \subset \mathbb{R}^{\kappa(k)}$ is not empty. Furthermore, the system (27) is called (globally) fully actuated if $\mathbb{F}_k = \mathbb{R}^{\kappa(k)}$ holds for all $k \geqslant h$, and is called sub-fully actuated if $\mathbb{F}_k$ is not empty for all $k \geqslant h$.

According to the above definition, it is obvious that the system (27) is (globally) fully actuated when Assumption 1 is met.

Parallel to Definition 1, the following definition for the non-affine system (32) can also be proposed (also, refer to Definition 4.3 in [25]).

Definition 3. Let $\mathbb{F}_k \subset \mathbb{R}^{\kappa(k)}$ be the largest set such that the following mapping:

$$\tilde{u} = g\left(X, k, u\right) \tag{37}$$

forms a differential homeomorphism from u to $\tilde{u}$ for all $X \in \mathbb{F}_k$. Then the set $\mathbb{F}_k$ is called the set of feasible points of system (32) at time k, and any $X \in \mathbb{F}_k$ is called a feasible point of system (32) at time k. Furthermore, the system (32) is called (globally) fully actuated if $\mathbb{F}_k = \mathbb{R}^{\kappa(k)}$ holds for all $k \geqslant h$, and is called sub-fully actuated if $\mathbb{F}_k \subset \mathbb{R}^{\kappa(k)}$ is not empty for all $k \geqslant h$.

With the set of feasible points at time k, namely, $\mathbb{F}_k$, well-defined above for (32), the concepts of singular points at time k can also be defined as in (36), and the definition of sub-full-actuation of the non-affine system (32) can be also immediately given simply by replacing the system (27) in Definition 3 by system (32).

Specially, when system (27) reduces to

$$x\left(k+1\right) = f\left(\psi\left(k\right), k\right) + B\left(x_p^{\lceil 0\sim\gamma_p(k)\rceil}\left(k\right)\Big|_{p=1\sim r}\right) u\left(k-h\right), \tag{38}$$

where the $B\left(\cdot\right)$ function is not dependent explicitly on the time k, and

$$\kappa\left(k\right) = r + \sum_{p=1}^{r} \gamma_p\left(k\right) = \kappa \tag{39}$$

is invariant with time k, we then have

$$\mathbb{F} = \left\{ X \right| \det B\left(X\right) \neq 0 \text{ or } \infty \},$$

and

$$\mathbb{S} = \mathbb{R}^{\kappa} \backslash \mathbb{F}, \tag{40}$$

which are not dependent on k. In this case, $\mathbb{F}$ and $\mathbb{S}$ are simply called the set of feasible points and the set of singular points of the system (38), respectively. Furthermore, now the system (38) is said to be (globally) fully actuated if $\mathbb{S} = \emptyset$, and is said to be sub-fully actuated if $\mathbb{F} \neq \emptyset$.

Finally, we point out that the set of feasible points and sub-full-actuation for the extended system (33) can also be similarly defined.

4 Controller designs: case of state delays only

For simplicity, let us first treat the FAS with state delays only.

Consider the following discrete-time sub-FAS with state delays only:

$$x(k+1) = f(\psi(k), k) + B\left(x_p^{\lceil 0\sim\gamma_p(k)\rceil}(k)\Big|_{p=1\sim r}, k \right) u(k), \tag{41}$$

where $\psi(k)$ is given by any of the three equations (28), (29) or (30).

Due to the full-actuation feature, the following important fact can be easily verified, which reveals a great advantage about the control of the sub-FAS (41).

Theorem 1. Let $\mathbb{F}_k$ be the set of feasible points of system (41) at time k. Further, let $\varpi \geqslant 1$ be an integer, and $A_{0\sim\varpi-1} \in \mathbb{R}^{r\times\varpi r}$ be an arbitrarily given matrix. Then the following controller:

$$\begin{cases} u(k) = B^{-1}\left(x_p^{\lceil 0\sim\gamma_p(k)\rceil}(k)\Big|_{p=1\sim r}, k \right)\left[-f(\psi(k), k) + u^*(k) \right], \\ u^*(k) = A_{0\sim\varpi-1}x^{\lceil 0\sim\varpi-1\rceil}(k) + v(k), \end{cases} \tag{42}$$

for system (41), with $v(k)$ being an external input, produces the following constant linear closed-loop system:

$$x(k+1) = A_{0\sim\varpi-1}x^{\lceil 0\sim\varpi-1\rceil}(k) + v(k) \tag{43}$$

subject to the following constraint:

$$x^{\lceil 1-\rho(k)\rceil}(k) \in \mathbb{F}_k,\ \ k \geqslant h. \tag{44}$$

As pointed out in [1], the integer ϖ may be chosen within the range of

$$1 \leqslant \varpi \in \max\{\rho(k), k \geqslant h\}.$$

In practical applications, the integer ϖ may be chosen according to the requirement on the closed-loop system. For example, if there is a requirement on both $x(k)$ and $x(k-1)$, then ϖ should be chosen greater than 2.

For the non-affine time-delay FAS with state delays only:

$$x(k+1) = f(\psi(k), k) + g\left(x_p^{\lceil 0\sim\gamma_p(k)\rceil}(k)\Big|_{p=1\sim r}, k, u(k) \right), \tag{45}$$

a result similar to Theorem 1 still holds. The controller can be obtained using the differential homeomorphism property of the mapping $\tilde{u} = g(X, k, u)$, as

$$\begin{cases} u(k) = g^{-1}\left(x_p^{\lceil 0\sim\gamma_p(k)\rceil}(k)\Big|_{p=1\sim r}, k, \tilde{u}(k) \right), \\ \tilde{u}(k) = -f(\psi(k), k) + u^*(k), \\ u^*(k) = A_{0\sim\varpi-1}x^{\lceil 0\sim\varpi-1\rceil}(k) + v(k), \end{cases} \tag{46}$$

and the same closed-loop system as in (43) is obtained.

When the linear closed-loop system (43) is written in state-space form, it appears as follows:

$$x^{\lceil 0\sim\varpi-1\rceil}(k+1) = \Psi(A_{0\sim\varpi-1})\,x^{\lceil 0\sim\varpi-1\rceil}(k) + \Gamma_c v(k), \tag{47}$$

where

$$\Gamma_c = \begin{bmatrix} I_r \\ 0_{r(\varpi-1)\times r} \end{bmatrix}.$$

Therefore, the linear system (43), or equivalently, (47), is stable if and only if the matrix $\Psi(A_{0\sim\varpi-1})$ is Schur. Such a fact gives a design method for the feedback gain matrix $A_{0\sim\varpi-1}$. Noting that

$$\Psi(A_{0\sim\varpi-1}) = \Psi(0_{0\sim\varpi-1}) + \Gamma_c A_{0\sim\varpi-1},$$

and $[\Psi(0_{0\sim\varpi-1}), \Gamma_{\rm c}]$ is controllable, the solution to the feedback gain in the proposed controllers really reduces to a problem of eigenstructure assignment, hence can be easily solved. Particularly, a complete parameterization of the designs can be given. Specifically, as done in the continuous-time case, complete parametric expressions for the feedback gain $A_{0\sim\varpi-1}$ in the controllers as well as the eigenstructure of the closed-loop system can be established, which provides all the design degrees of freedom to be used for further improving the system performance (see [13, 22–24] for the continuous-time case).

Regarding the feasibility requirement, that is, the constraint (44), we remark that it is essential for control of sub-FASs. It vanishes only for global FASs. Noticing that an eigenstructure assignment problem has the same formulation in the algebra level, for treatment of this constraint, there are at least two methods:

(1) Method based on Lyapunov matrix equation. It is well-known that, for constant stable linear systems, there exists a converged exponential trajectory envelope which is represented by the solution to the related Lyapunov matrix equation associated with the matrix $\Psi(A_{0\sim\varpi-1})$. Therefore, an efficient way to solve the feedback gain $A_{0\sim\varpi-1}$ can be given by adjusting the system trajectory envelope to get a larger range of the system initial values, within which the required constraint is met (see Lemma 5.2 in [23] for the continuous-time case).

(2) Method based on eigenstructure assignment. Theoretically, the trajectory of a linear system is totally determined by its eigenstructure. Through eigenstructure assignment techniques, we can provide a parametric solution to the feedback gain $A_{0\sim\varpi-1}$ and also express the closed-loop system trajectory in terms of the closed-loop eigenvalues and eigenvectors. Therefore, by properly adjusting the closed-loop eigenvalues and eigenvectors, again a less conservative range of the system initial values can be obtained such that the constraint is satisfied (see Lemma 6.1 in [24]).

Example 1. Consider a system in the form of

$$x(k+1) = f(\psi(k), k) + x_1^2(k-\gamma(k))\, u(k),$$

where $x, u \in \mathbb{R}^r$, x_1 is the first element in x, $\psi(k)$ represents the delay term in the function f, and $\gamma(k)$ is a positive integer-valued function.

By Theorem 1, the controller is designed, when $\varpi = 2$, as

$$u(k) = \frac{1}{x_1^2(k-\gamma(k))}\left[-f(\psi(k), k) + A_0 x(k) + A_1 x(k-1) + v\right],$$

where v is a nonzero constant. The corresponding closed-loop system is given by

$$\begin{cases} x(k+1) = A_0 x(k) + A_1 x(k-1) + v, \\ x_1(k-\gamma(k)) \neq 0,\ k \geqslant 0, \end{cases} \tag{48}$$

whose stability is determined by that of

$$\Psi(A_{0\sim 1}) = \begin{bmatrix} A_0 & A_1 \\ I_r & 0 \end{bmatrix}.$$

With the following initial values:

$$x(-1),\ x(0) \text{ and } \psi(0),$$

the controller sequence and the state trajectory can be produced.

It is important to point out that, when the linear system in (48) is stable, the corresponding constraint can be satisfied by properly restricting the system initial value ranges.

5 Controller designs: case of state and input delays

Due to the existence of the input delay, the nonlinear discrete-time delay FAS (18) is free of control at the beginning, and runs the first h steps by itself in the following way.

When the system initial values $\psi(0)$, $x_p^{\lceil 0\sim\gamma_p(0)\rceil}(0)|_{p=1\sim r}$, and $u(-h)$ are available, it follows from the system equation (18) that

$$x(1)=f(\psi(0),0)+B\left(x_p^{\lceil 0\sim\gamma_p(0)\rceil}(0)\Big|_{p=1\sim r},0\right)u(-h). \tag{49}$$

By now, $\psi(1)$ and $x_p^{\lceil 0\sim\gamma_p(1)\rceil}(1)|_{p=1\sim r}$ are available. Further, when $u(-h+1)$ is available (often set to zero), we can obtain

$$x(2)=f(\psi(1),1)+B\left(x_p^{\lceil 0\sim\gamma_p(1)\rceil}(1)\Big|_{p=1\sim r},1\right)u(-h+1). \tag{50}$$

Carrying on with this process, when $u(-1)$ is given, we finally have

$$x(h)=f(\psi(h-1),h-1)+B\left(x_p^{\lceil 0\sim\gamma_p(h-1)\rceil}(h-1)\Big|_{p=1\sim r},h-1\right)u(-1). \tag{51}$$

Therefore, by now,

$$\psi(k)\ \text{ and }\ x_p^{\lceil 0\sim\gamma_p(k)\rceil}(k)\Big|_{p=1\sim r},\quad 0\leqslant k\leqslant h \tag{52}$$

are all available. From here on we can add the effect of control.

Similar to the control of the nonlinear discrete-time FAS (41) with states delay only, for control of the nonlinear discrete-time FAS (18) with time delays, we naturally design the following controller utilizing the full-actuation feature of the system:

$$\begin{cases} u(k-h)=B^{-1}\left(x_p^{\lceil 0\sim\gamma_p(k)\rceil}(k)\Big|_{p=1\sim r},k\right)\left[-f(\psi(k),k)+u^*(k-h)\right],\\ u^*(k-h)=A_{0\sim\varpi-1}x^{\lceil 0\sim\varpi-1\rceil}(k)+v(k),\ \ k\geqslant h. \end{cases} \tag{53}$$

With the starting values in (52), we can get

$$\begin{cases} u(0)=B^{-1}\left(x_p^{\lceil 0\sim\gamma_p(h)\rceil}(h)\Big|_{p=1\sim r},h\right)\left[-f(\psi(h),h)+u^*(0)\right],\\ u^*(0)=A_{0\sim\varpi-1}x^{\lceil 0\sim\varpi-1\rceil}(h)+v(h). \end{cases} \tag{54}$$

To further get $u(1)$, the value of $x(h+1)$ may be needed, and can be obtained via the open-loop system (18) as

$$x(h+1)=f(\psi(h),h)+B\left(x_p^{\lceil 0\sim\gamma_p(h)\rceil}(h)\Big|_{p=1\sim r},h\right)u(0). \tag{55}$$

Hence $x_p^{\lceil 0\sim\gamma_p(h+1)\rceil}(h+1)|_{p=1\sim r}$ is now available, and we can then further get $u(1)$ using (53) again.

With such a process, we can realize the control of the system (27) with both state delays and an input delay.

Theorem 2. Let $\mathbb{F}_k$ be the set of feasible points of system (27) at time k. Further, let ϖ be an integer greater than 1, and $A_{0\sim\varpi-1}\in\mathbb{R}^{r\times\varpi r}$ be an arbitrarily given matrix. Then the following controller:

$$\begin{cases} u(k-h)=B^{-1}\left(x_p^{\lceil 0\sim\gamma_p(k)\rceil}(k)\Big|_{p=1\sim r},k\right)\left[-f(\psi(k),k)+u^*(k-h)\right],\\ u^*(k-h)=A_{0\sim\varpi-1}x^{\lceil 0\sim\varpi-1\rceil}(k)+v(k),\\ x(k)=f(\psi(k-1),k-1)+B\left(x_p^{\lceil 0\sim\gamma_p(k-1)\rceil}(k-1)\Big|_{p=1\sim r},k-1\right)u(k-h-1),\\ k\geqslant h, \end{cases} \tag{56}$$

for system (27) produces the same constant linear closed-loop system (43) provided that the constraint (44) is met, where $v(k)$ is an external input.

Please note that the initial value of the closed-loop system is $x^{\lceil 0\sim\varpi-1\rceil}(h)$, which is provided by the first h steps self-running of the open-loop system (27).

Example 2. Consider the following discrete-time sub-FAS:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} x_2(k-\rho_2(k)) \\ \sin x_1(k-\rho_1(k)) \end{bmatrix} + \begin{bmatrix} 1-x_2(k-\gamma(k)) & 0 \\ 3k\mathrm{e}^{-k} & 1 \end{bmatrix} u(k-2),$$

where $\rho_i(k), i=1,2$, and $\gamma(k)$ are non-negative integer-valued functions. Obviously, we have $r=h=2$, and

$$f(k) \triangleq f\left(x_1^{\lceil 0\sim\rho_1(k)\rceil}(k), x_2^{\lceil 0\sim\rho_2(k)\rceil}(k)\right) = \begin{bmatrix} x_2(k-\rho_2(k)) \\ \sin x_1(k-\rho_1(k)) \end{bmatrix},$$

$$B(k) \triangleq B\left(x_1^{\lceil 0\sim\gamma(k)\rceil}(k), k\right) = \begin{bmatrix} 1-x_2(k-\gamma(k)) & 0 \\ 3k\mathrm{e}^{-k} & 1 \end{bmatrix}.$$

Choosing

$$\varpi = 1, \quad A_0 = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$$

by Theorem 2, we can design for the system the following controller:

$$\begin{cases} u(k-2) = B^{-1}(k)\left[-f(k)+u^*(k-2)\right], \\ u^*(k-2) = \begin{bmatrix} a_1x_1(k) \\ a_2x_2(k) \end{bmatrix} + v(k), \\ x_{1\sim 2}(k) = f(k-1)+B(k-1)u(k-3), \\ k \geqslant 2, \end{cases} \tag{57}$$

where a_1 and a_2 are two real scalars with modular less than 1.

By two steps of self-running, the open-loop system provides the following initial values for the above controller:

$$x_1(1-\rho_1(1)), x_2(1-\rho_2(1)), x_2(1-\gamma(1)) \text{ and } u(-1).$$

Hence the controller works and a closed-loop system is obtained, which contains the following constant linear system:

$$x_1(k+1) = a_1x_1(k)+v_1(k), \ k \geqslant 2, \tag{58}$$

and the following constrained one:

$$\begin{cases} x_2(k+1) = a_2x_2(k)+v_2(k), \\ x_2(k-1) \neq 1, \ k \geqslant 2. \end{cases} \tag{59}$$

Clearly, in the conservative case that a_2 is chosen positive, the initial value $x_2(2)$ is negative, $v_2(k)$ is not greater than zero, and the constraint in (59) always holds true.

6 Further issues

6.1 Further extension

Let us consider the controller of the extended FAS (33). A straight forward extension can be given as follows:

$$\begin{cases} u(k-h) = B_e^{-1}(k)\left[-f_e(k)+u^*(k-h)\right], \\ u^*(k-h) = A_{0\sim\varpi-1}x^{\lceil 0\sim\varpi-1\rceil}(k)+v(k), \\ x(k) = f_e(k-1)+B_e(k-1)u(k-h-1), \\ k \geqslant h, \end{cases} \tag{60}$$

where

$$f_e(k) \triangleq f_e\left(x^{\lceil 0\sim\rho(k)\rceil}(k), u^{\lceil 1\sim\xi(k)\rceil}(k-h), k\right), \tag{61}$$

$$B_e(k) \triangleq B_e\left(x^{\lceil 0\sim\rho(k)\rceil}(k), u^{\lceil 1\sim\xi(k)\rceil}(k-h), k\right). \tag{62}$$

Again, a constant linear closed-loop system is produced:

$$x(k+1) = A_{0\sim\varpi-1}x^{\lceil 0\sim\varpi-1\rceil}(k) + v(k). \tag{63}$$

Please note that a constraint on the closed-loop system (63) also exists if the extended FAS (33) is only sub-fully actuated but not globally fully actuated.

Parallel to (60), the controller for the general non-affine FAS (34) can also be easily given.

6.2 Controllability

It is clearly seen in Sections 4 and 5, as also revealed in [22] for the continuous-time case, once a system is modeled in a global FAS form, the control of the system can be immediately realized. Furthermore, the closed-loop system can be made to be a constant linear one with an arbitrarily assignable eigen-polynomial or eigenstructure. As commented in [22], such a fact adequately explains the concept of complete controllability. In this subsection, we simply generalize the definition of controllability of continuous-time dynamical systems proposed in [22] into the case of discrete-time delay systems.

Introduce the following general nonlinear system:

$$F\left(\tilde{x}_p^{\lceil \tau_{p,1\sim\zeta_p}(k)\rceil}(k)\,|_{p=1\sim\eta}, \tilde{u}_p^{\lceil \sigma_{p,1\sim\xi_p}(k)\rceil}(k), k\right) = 0, \tag{64}$$

where $F(\cdot)$ is a vector function of appropriate dimension, and $\tilde{x}$ and $\tilde{u}$ are respectively the state and input vectors of r dimension, $\tau_{p,1\sim\zeta_p}(k)$ and $\sigma_{p,1\sim\xi_p}(k)$ are the delays in the state and input variables $\tilde{x}_p$ and $\tilde{u}_p, p = 1, 2, \ldots, r$, respectively. Regarding the controllability of the system (64), we propose the following definition.

Definition 4. The dynamical system (64) is called completely controllable if it can be equivalently converted into a globally FAS in the form of (27) or (32), and is called sub-controllable if it can be equivalently converted into a sub-FAS in the form of (27) or (32).

In the context of Kalman's controllability theory, controllability is only defined for systems in state-space representations. With the above definition, it is obvious that, all discrete-time dynamical control systems, not only those in state-space representations, possess the concepts of controllability. It needs to point out that the above definition obviously corresponds in some ways to Definition 3 in [14] and Definition 3.2 in [22] for the continuous-time systems. As a matter of fact, such a definition is a straight forward generalization of the linear system case. It is well-known that a linear system is controllable if and only if its closed-loop eigenvalues are arbitrarily assignable by state feedback, or equivalently, its closed-loop eigen-polynomial is arbitrarily assignable by state feedback. While as it is clearly seen in the above Theorems 1 and 2, and also those corresponding ones in [14, 22], by the new definition for controllability, a nonlinear system is completely controllable if it has a state feedback controller such that the closed-loop system is constant linear, and the closed-loop eigen-polynomial is arbitrarily assignable.

For the closely related concept of stabilizability for the discussed time-delay systems, one can also refer to [22] for the continuous-time case. For more comments on the definitions of controllability and stabilizability of dynamical systems, one can further refer to [22]. As indicated in [22], a system that is neither a FAS nor a sub-FAS may be equivalently decomposed into two subsystems, one is a FAS or a sub-FAS, the other is an autonomous one. The system is called stabilizable if the autonomous subsystem is stable in certain sense. The control of a stabilizable system is thus generally realized by controlling the fully or sub-fully actuated subsystem.

6.3 Designs for performance

The results proposed in the paper, together with those in the first part [1] are very fundamental and are also vitally important since they lay a solid basis for the FAS approaches to discrete-time delay systems. These basic results allow us to further address, as in the continuous-time FAS case, many design problems associated with discrete-time delay systems, such as robust control [18, 20], adaptive control [19, 20], disturbance rejection [21], optimal control [23], and signal tracking control [24].

7 Concluding remarks

In addition to the first part of this study, this second part further considers control of discrete-time sub-FASs with time-varying delays. By now the general FAS approaches for control of discrete-time delay systems are roughly sketched.

It is shown that a general discrete-time sub-FAS with time-varying state delays and a constant input delay can be expressed by a new representation. With the help of this new representation, the set of feasible points and the set of singular points of the system can be well defined to reflect the full-actuation condition of the sub-FASs. It is also shown that the controller of a discrete-time sub-FAS can also be carried out similar to the global FAS case, but with the state of the control system satisfying the feasibility condition.

Furthermore, the results can also be generalized into the time-varying input delay case.

Acknowledgements This work was partially supported by Major Program of National Natural Science Foundation of China (Grant Nos. 61690210, 61690212), National Natural Science Foundation of China (Grant No. 61333003), and Fundamental Science Center Program of National Natural Science Foundation of China (Grant No. 62188101). The author is grateful to his Ph.D. students Guangtai TIAN, Qin ZHAO, Xiubo WANG, Weizhen LIU, Kaixin CUI, Liyao HU, and Prof. Y. CUI, for helping him with reference selection and proofreading. His particular thanks go to his student Tianyi ZHAO for his help in working out the examples.

References

1 Duan G R. Discrete-time delay systems: part 1. Global fully actuated case. Sci China Inf Sci, 2022, 65: 182201
2 Xu S, Lam J. A survey of linear matrix inequality techniques in stability analysis of delay systems. Int J Syst Sci, 2008, 39: 1095–1113
3 Zhang X-M, Han Q-L, Seuret A, et al. Overview of recent advances in stability of linear systems with time-varying delays. IET Control Theor Appl, 2019, 13: 1–16
4 Mahmoud M S. Robust Control and Filtering for Time-Delay Systems. Boca Raton: CRC Press, 2018
5 Fridman E. Introduction to Time-Delay Systems: Analysis and Control. Berlin: Springer, 2014
6 Zhou B. Truncated Predictor Feedback for Time-Delay Systems. Berlin: Springer, 2014
7 Chen Y, Wang Z. Local stabilization for discrete-time systems with distributed state delay and fast-varying input delay under actuator saturations. IEEE Trans Automat Contr, 2020, 66: 1337–1344
8 González A. Robust stabilization of linear discrete-time systems with time-varying input delay. Automatica, 2013, 49: 2919–2922
9 González A, García P. Output-feedback anti-disturbance predictor-based control for discrete-time systems with time-varying input delays. Automatica, 2021, 129: 109627
10 He Y, Wu M, Liu G P, et al. Output feedback stabilization for a discrete-time system with a time-varying delay. IEEE Trans Automat Contr, 2008, 53: 2372–2377
11 Zhang H, Xie L, Duan G. H_∞ control of discrete-time systems with multiple input delays. IEEE Trans Automat Contr, 2007, 52: 271–283
12 Wang L, Xiao F. A new approach to consensus problems in discrete-time multiagent systems with time-delays. Sci China Ser F-Inf Sci, 2007, 50: 625–635
13 Duan G R. High-order system approaches: I. Full-actuation and parametric design (in Chinese). Acta Autom Sin, 2020, 46: 1333–1345
14 Duan G R. High-order system approaches: II. Controllability and fully-actuation (in Chinese). Acta Autom Sin, 2020, 46: 1571–1581
15 Duan G R. High-order system approaches: III. Observability and observer design (in Chinese). Acta Autom Sin, 2020, 46: 1885–1895
16 Duan G R. High-order fully actuated system approaches: part I. Models and basic procedure. Int J Syst Sci, 2021, 52: 422–435
17 Duan G R. High-order fully actuated system approaches: part II. Generalized strict-feedback systems. Int J Syst Sci, 2021, 52: 437–454
18 Duan G R. High-order fully actuated system approaches: part III. Robust control and high-order backstepping. Int J Syst Sci, 2021, 52: 952–971
19 Duan G R. High-order fully actuated system approaches: part IV. Adaptive control and high-order backstepping. Int J Syst Sci, 2021, 52: 972–989
20 Duan G R. High-order fully actuated system approaches: part V. Robust adaptive control. Int J Syst Sci, 2021, 52: 2129–2143
21 Duan G R. High-order fully-actuated system approaches: part VI. Disturbance attenuation and decoupling. Int J Syst Sci, 2021, 52: 2161–2181
22 Duan G R. High-order fully actuated system approaches: part VII. Controllability, stabilisability and parametric designs. Int J Syst Sci, 2021, 52: 3091–3114
23 Duan G R. High-order fully actuated system approaches: part VIII. Optimal control with application in spacecraft attitude stabilisation. Int J Syst Sci, 2022, 53: 54–73
24 Duan G R. High-order fully-actuated system approaches: part IX. Generalised PID control and model reference tracking. Int J Syst Sci, 2022, 53: 652–674
25 Duan G R. High-order fully actuated system approaches: part X. Basics of discrete-time systems. Int J Syst Sci, 2022, 53: 810–832