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Is Fully Distributed Adaptive Protocol Applicable to Graphs Containing Directed Spanning Tree?

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Appendix A The proof of Lemma 1

We prove it by constructing these two node subsets as follows. By the definition of spanning tree, we can find a node i_0 which has directed paths to any other nodes. Let $i_0 \in \mathcal{V}_1$. For each node $i_j \neq i_0$, $i_j \in \mathcal{V}_1$ if it has directed path to i_0, and $i_j \in \mathcal{V}_2$ otherwise. Clearly, such $\mathcal{V}_1$ and $\mathcal{V}_2$ satisfy i) and ii).

Appendix B The proof of Theorem 1

Without loss of generality, we assume that $\mathcal{V}_1 = \{1, \cdots, k\}$ and $\mathcal{V}_2 = \{k+1, \cdots, N\}$. Then, the Laplacian matrix can be divided into $\mathcal{L} = \begin{bmatrix} \mathcal{L}_{11} & 0 \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix}$, where $\mathcal{L}_{11} \in \mathbf{R}^{k\times k}$ represents the Laplacian matrix of the strongly connected subgraph among $\mathcal{V}_1$, and $\mathcal{L}_{22} \in \mathbf{R}^{(N-k)\times(N-k)}$. We have the following result to show the property of $\mathcal{L}_{22}$.

Lemma 1. $\mathcal{L}_{22}$ is a non-singular M-matrix.

Proof. Since the graph $\mathcal{G}$ contains a directed spanning tree, zero is a simple eigenvalue of $\mathcal{L}$. Note that the eigenvalues of $\mathcal{L}_{11}$ and $\mathcal{L}_{22}$ form all the eigenvalues of $\mathcal{L}$, and $\mathcal{L}_{11}$ contains a zero eigenvalue as it is the Laplacian matrix of the strongly connected subgraph among $\mathcal{V}_1$. Thus, all the eigenvalues of $\mathcal{L}_{22}$ are in the open right half plane. Consequently, $\mathcal{L}_{22}$ is a non-singular M-matrix by noticing that all the nondiagonal elements are nonpositive.

The following two lemmas reveal the properties of $\mathcal{L}_{11}$ and $\mathcal{L}_{22}$, respectively.

Lemma 2 ([1]). For a strongly connected graph $\mathcal{G}_1$ with Laplacian matrix $\mathcal{L}_{11}$, the matrix $\hat{\mathcal{L}}_{11} = R\mathcal{L}_{11} + \mathcal{L}_{11}^T R$ represents a weighted symmetric Laplacian matrix of an undirected connected graph, where $R = \mathrm{diag}(r_1, \cdots, r_k) > 0$ with $r = [r_1, \cdots, r_k]$ being the left zero unit eigenvector of $\mathcal{L}_{11}$. Moreover, $\min_{z^T x=0} x^T \hat{\mathcal{L}}_{11} x \geqslant \frac{\lambda_2(\hat{\mathcal{L}}_{11})}{k} x^T x$, where $\lambda_2(\hat{\mathcal{L}}_{11})$ is the smallest nonzero eigenvalue of $\hat{\mathcal{L}}_{11}$ and z is a vector with all positive elements.

Lemma 3 ([1]). For the non-singular M-matrix $\mathcal{L}_{22}$, there exists a diagonal positive definite matrix $G > 0$ such that $\mathcal{L}_0 = G\mathcal{L}_{22} + \mathcal{L}_{22}^T G > 0$.

The necessity is obvious. In the following, we shall show the consensus realization with $\mathcal{G}$ containing a directed spanning tree.

Let $\bar{x}_1 = [x_1^T, \cdots, x_k^T]^T$, $\bar{x}_2 = [x_{k+1}^T, \cdots, x_N^T]^T$, and $x = [\bar{x}_1^T, \bar{x}_2^T]^T$. Further let $\bar{\xi}_1 = [\xi_1^T, \cdots, \xi_k^T]^T$, $\bar{\xi}_2 = [\xi_{k+1}^T, \cdots, \xi_N^T]^T$, and $\xi = [\bar{\xi}_1^T, \bar{\xi}_2^T]^T$. Then, we have

$$\begin{aligned}\bar{\xi}_1 =& (\mathcal{L}_{11} \otimes I_n)\bar{x}_1, \\ \bar{\xi}_2 =& (\mathcal{L}_{22} \otimes I_n)\bar{x}_2 + (\mathcal{L}_{21} \otimes I_n)\bar{x}_1.\end{aligned} \tag{B1}$$

The closed-loop dynamics of $\bar{\xi}_1$ and $\bar{\xi}_2$ can be written as

$$\begin{aligned}\dot{\bar{\xi}}_1 =& [I_N \otimes A + \mathcal{L}_{11}(D_1 + \bar{\rho}_1) \otimes BK]\bar{\xi}_1, \\ \dot{\bar{\xi}}_2 =& [I_N \otimes A + \mathcal{L}_{22}(D_2 + \bar{\rho}_2) \otimes BK]\bar{\xi}_2 + [\mathcal{L}_{21}(D_1 + \bar{\rho}_1) \otimes BK]\bar{\xi}_1, \\ \dot{d}_i =& \xi_i^T PBB^T P\xi_i,\end{aligned} \tag{B2}$$

where $D_1 = \mathrm{diag}(d_1, \cdots, d_k)$, $D_2 = \mathrm{diag}(d_{k+1}, \cdots, d_N)$, $\rho_i = \xi_i^T P\xi_i, i = 1, \cdots, N$, $\bar{\rho}_1 = \mathrm{diag}(\rho_1, \cdots, \rho_k)$ and $\bar{\rho}_2 = \mathrm{diag}(\rho_{k+1}, \cdots, \rho_N)$.

Consider the following Lyapunov function candidate

$$V = \alpha V_1 + V_2 \tag{B3}$$

with

$$\begin{aligned}V_1 =& \sum_{i=1}^{k} \frac{r_i}{2}[(2d_i + \rho_i)\rho_i + (d_i - \beta_1)^2], \\ V_2 =& \sum_{i=k+1}^{N} \frac{g_i}{2}[(2d_i + \rho_i)\rho_i + (d_i - \beta_2)^2],\end{aligned} \tag{B4}$$

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where α, β_1 and β_2 are positive constants, r_i and g_i is determined by Lemma 2 and Lemma 3, respectively.

The time-derivative of V is given by

$$\dot{V} = \alpha\dot{V}_1 + \dot{V}_2, \tag{B5}$$

where

$$\begin{aligned}\dot{V}_1 =& \sum_{i=1}^{k} r_i[(d_i+\rho_i)\dot{\rho}_i + (d_i+\rho_i-\beta_1)\dot{d}_i] \\ =& 2\bar{\xi}_1^T[(D_1+\bar{\rho}_1)R\otimes P]\dot{\bar{\xi}}_1 + \sum_{i=1}^{k} r_i(d_i+\rho_i-\beta_1)\xi_i^T PBB^TP\xi_i \\ =& \bar{\xi}_1^T[(D_1+\bar{\rho}_1)R\otimes(PA+A^TP)]\bar{\xi}_1 - \bar{\xi}_1^T[(D_1+\bar{\rho}_1)\hat{\mathcal{L}}_{11}(D_1+\bar{\rho}_1)\otimes PBB^TP]\bar{\xi}_1 \\ &+ \bar{\xi}_1^T[(D_1+\bar{\rho}_1-\beta_1 I_k)R\otimes PBB^TP]\bar{\xi}_1,\end{aligned} \tag{B6}$$

and

$$\begin{aligned}\dot{V}_2 =& \sum_{i=k+1}^{N} g_i[(d_i+\rho_i)\dot{\rho}_i + (d_i+\rho_i-\beta_2)\dot{d}_i] \\ =& 2\bar{\xi}_2^T[(D_2+\bar{\rho}_2)G\otimes P]\dot{\bar{\xi}}_2 + \sum_{i=k+1}^{N} g_i(d_i+\rho_i-\beta_2)\xi_i^T PBB^TP\xi_i \\ =& \bar{\xi}_2^T[(D_2+\bar{\rho}_2)G\otimes(PA+A^TP)]\bar{\xi}_2 - \bar{\xi}_2^T[(D_2+\bar{\rho}_2)\mathcal{L}_0(D_2+\bar{\rho}_2)\otimes PBB^TP]\bar{\xi}_2 \\ &- 2\bar{\xi}_2^T[(D_2+\bar{\rho}_2)G\mathcal{L}_{21}(D_1+\bar{\rho}_1)\otimes PBB^TP]\bar{\xi}_1 + \bar{\xi}_2^T[(D_2+\bar{\rho}_2-\beta_2 I_{N-k})G\otimes PBB^TP]\bar{\xi}_2.\end{aligned} \tag{B7}$$

Define $\chi = [(D_1+\bar{\rho}_1)^{-1}\otimes I_n](r^T\otimes 1_n)$, which has all positive elements. We have

$$\begin{aligned}\chi^T[(D_1+\bar{\rho}_1)\otimes B^TP]\bar{\xi}_1 &= (r\otimes 1_n^T)(I_k\otimes B^TP)\bar{\xi}_1 \\ &= (r\mathcal{L}_{11}\otimes 1_n^T)(I_k\otimes B^TP)\bar{x}_1 = 0.\end{aligned}$$

In light of Lemma 2, we can obtain

$$\begin{aligned}&-\bar{\xi}_1^T[(D_1+\bar{\rho}_1)\hat{\mathcal{L}}_{11}(D_1+\bar{\rho}_1)\otimes PBB^TP]\bar{\xi}_1 \\ \leqslant& -\frac{\lambda_2(\hat{\mathcal{L}}_{11})}{k}\bar{\xi}_1^T[(D_1+\bar{\rho}_1)^2\otimes PBB^TP]\bar{\xi}_1.\end{aligned} \tag{B8}$$

Denote λ_0 as the smallest eigenvalue of $\mathcal{L}_0$. Then we have

$$\begin{aligned}&-\bar{\xi}_2^T[(D_2+\bar{\rho}_2)\mathcal{L}_0(D_2+\bar{\rho}_2)\otimes PBB^TP]\bar{\xi}_2 \\ \leqslant& -\lambda_0\bar{\xi}_2^T[(D_2+\bar{\rho}_2)^2\otimes PBB^TP]\bar{\xi}_2.\end{aligned} \tag{B9}$$

By Young's Inequality, we can get

$$\begin{aligned}&-2\bar{\xi}_2^T[(D_2+\bar{\rho}_2)G\mathcal{L}_{21}(D_1+\bar{\rho}_1)\otimes PBB^TP]\bar{\xi}_1 \\ \leqslant& \frac{\lambda_0}{2}\bar{\xi}_2^T[(D_2+\bar{\rho}_2)^2\otimes PBB^TP]\bar{\xi}_2 + \frac{2\sigma_{\max}^2(G\mathcal{L}_{21})}{\lambda_0}\bar{\xi}_1^T[(D_1+\bar{\rho}_1)^2\otimes PBB^TP]\bar{\xi}_1.\end{aligned} \tag{B10}$$

Substituting (B6)-(B10) into (B5) and choosing $\alpha = \frac{\lambda_2(\hat{\mathcal{L}}_{11})}{k}\big(\alpha_1 + \frac{2\sigma_{\max}^2(G\mathcal{L}_{21})}{\lambda_0}\big)$ yields

$$\begin{aligned}\dot{V} \leqslant& \alpha\bar{\xi}_1^T[(D_1+\bar{\rho}_1)R\otimes(PA+A^TP+PBB^TP)]\bar{\xi}_1 - \bar{\xi}_1^T[(\alpha_1(D_1+\bar{\rho}_1)^2+\alpha\beta_1 R)\otimes PBB^TP]\bar{\xi}_1 \\ &+ \bar{\xi}_2^T[(D_2+\bar{\rho}_2)G\otimes(PA+A^TP+PBB^TP)]\bar{\xi}_2 - \bar{\xi}_2^T[(\frac{\lambda_0}{2}(D_2+\bar{\rho}_2)^2-\beta_2 G)\otimes PBB^TP]\bar{\xi}_2.\end{aligned}$$

By choosing $\beta_1 \geqslant \frac{9\alpha\lambda_{\max}(R)}{4\alpha_1}$ and in light of Young's Inequality, we have

$$\begin{aligned}&-\bar{\xi}_1^T[(\alpha_1(D_1+\bar{\rho}_1)^2+\alpha\beta_1 R)\otimes PBB^TP]\bar{\xi}_1 \\ \leqslant& -3\alpha\bar{\xi}_1^T[(D_1+\bar{\rho}_1)R\otimes PBB^TP]\bar{\xi}_1.\end{aligned} \tag{B11}$$

Similarly, we can derive

$$\begin{aligned}&-\bar{\xi}_2^T[(\frac{\lambda_0}{2}(D_2+\bar{\rho}_2)^2-\beta_2 G)\otimes PBB^TP]\bar{\xi}_2 \\ \leqslant& -3\bar{\xi}_2^T[(D_2+\bar{\rho}_2)G\otimes PBB^TP]\bar{\xi}_2,\end{aligned} \tag{B12}$$

where $\beta_2 \geqslant \frac{9\lambda_{\max}(G)}{2\lambda_0}$. Then, we can derive that

$$\begin{aligned}\dot{V} \leqslant& -\alpha\bar{\xi}_1^T[(D_1+\bar{\rho}_1)R\otimes W]\bar{\xi}_1 - \bar{\xi}_2^T[(D_2+\bar{\rho}_2)G\otimes W]\bar{\xi}_2 \\ \leqslant& 0,\end{aligned} \tag{B13}$$

where $W = -(PA+A^TP-2PBB^TP)$ is a positive definite matrix.

Therefore, V is bounded, and so are $\bar{\xi}_1$, $\bar{\xi}_2$ and d_i. Note that $\dot{V}\equiv 0$ is equivalent to $\bar{\xi}_1\equiv 0, \bar{\xi}_2\equiv 0$, and thereby $\xi\equiv 0$. By LaSalles Invariance principle, we can conclude that the consensus error asymptotically converges to zero.

References

1 Y. Lv, Z. Li, Z. Duan, et al. Novel distributed robust adaptive consensus protocols for linear multi-agent systems with directed graphs and external disturbances. International Journal of Control, 2017, 90: 137–147