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# Robust adaptive $H_\infty$ control for networked uncertain semi-Markov jump nonlinear systems with input quantization

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Dear editor,

Markov jump systems (MJSs), as one kind of switched systems, are a good candidate for modeling physical systems subject to abrupt variation in their structures or parameters caused by many reasons such as subsystem interconnection change and environmental disturbance [1, 2]. Much effort has been put to study the robust control and filter problems for semi-MJSs, where the corresponding sojourn time between two successive jump modes is allowed to follow more general distributions, such as Weibull distribution and phase-type distribution [3]. This study is the first attempt of developing a robust adaptive quantized control law for a class of continuous-time semi-MJSs with parameter uncertainty, matched uncertain nonlinearity, and disturbance.

Consider the following continuous-time uncertain nonlinear semi-MJS:

$$\begin{cases} \dot{x}(t) = (A_{\delta_t} + \Delta A_{\delta_t})x(t) + B_{\delta_t}[u(t) + f(x(t))] \\ \qquad\quad + (C_{\delta_t} + \Delta C_{\delta_t})w(t), \\ z(t) = D_{\delta_t}x(t) + E_{\delta_t}w(t), \end{cases} \quad (1)$$

where $x(t) \in \mathbb{R}^{n_x}$, $u(t) \in \mathbb{R}^{n_u}$, $w(t) \in \mathbb{R}^{n_w}$ and $z(t) \in \mathbb{R}^{n_z}$ are the state, control input, external disturbance belonging to $l_2[0, +\infty)$ (i.e., $\int_0^\infty \|w(t)\|_2^2 < \infty$), and controlled output, respectively; $f(x(t))$ is the matched uncertain nonlinearity subject to $\|f(x(t))\|_2 \leqslant \alpha_1 + \alpha_2\|x(t)\|_2$, in which $\alpha_1$ and $\alpha_2$ are unknown; $\Delta A_{\delta_t}$ and $\Delta C_{\delta_t}$ denote parameter uncertainty, satisfying $\Delta A_{\delta_t} = M_{1\delta_t}F_1(t)N_{1\delta_t}$ and $\Delta C_{\delta_t} = M_{2\delta_t}F_2(t)N_{2\delta_t}$; $F_i(t)$ is time-varying but unknown, subject to $F_i^{\mathrm{T}}(t)F_i(t) \leqslant I$ $(i = 1, 2)$; $A_{\delta_t}$, $B_{\delta_t}$, $C_{\delta_t}$, $D_{\delta_t}$, $E_{\delta_t}$, $M_{i\delta_t}$, and $N_{i\delta_t}$ are given; $\delta_t$ is used to describe a discrete-state, continuous-time semi-Markov process with transition rate matrix $\Pi = [\theta_{sg}(d)]$, and $\delta_t$ takes a value in the set $\mathcal{G} = \{1, 2, \ldots, G\}$ and satisfies $\Pr\{\delta_{t+d} = g|\delta_t = s\} = \begin{cases} \theta_{sg}(d)d + o(d), & s \neq g, \\ 1 + \theta_{ss}(d)d + o(d), & s = g, \end{cases}$ where $\theta_{sg}(d)$ denotes the time-varying jump rate from mode $s$ at time $t$ to mode $g$ at time $t + d$ with $\theta_{sg}(d) \geqslant 0$, $s \neq g$ and $\theta_{ss}(d) = -\sum_{g=1, g\neq s}^{G} \theta_{sg}(d)$. Assume that $0 \leqslant \underline{\theta}_{sg} < \theta_{sg}(d) < \overline{\theta}_{sg}$ for $s \neq g$, where $\underline{\theta}_{sg}$ and $\overline{\theta}_{sg}$ are given. Thus, it follows that $-\sum_{g=1, g\neq s}^{G} \overline{\theta}_{sg} < \theta_{ss}(d) < -\sum_{g=1, g\neq s}^{G} \underline{\theta}_{sg}$.

**Assumption 1.** $(A_{\delta_t}, B_{\delta_t})$ is mean-square stabilizable [4].

Logarithmic quantizers are employed to quantize $u(t)$ with the following dynamics: (i) when $\frac{\omega_r^\vartheta}{1+\sigma_r} \leqslant u_r(t) \leqslant \frac{\omega_r^\vartheta}{1-\sigma_r}$, $\mathcal{Q}(u_r(t)) = \omega_r^\vartheta$, (ii) when $u_r(t) < 0$, $\mathcal{Q}(u_r(t)) = -\mathcal{Q}(-u_r(t))$, and (iii) otherwise $\mathcal{Q}(u_r(t)) = 0$. The set of quantization levels is $\Omega_r = \{\pm\omega_r^\vartheta : \omega_r^\vartheta = \beta_r^\vartheta\omega_r^0, \vartheta = \pm1, \pm2, \ldots\} \cup \{0\}$ and $r \in \{1, \ldots, n_u\}$, where $\beta_r^\vartheta\omega_r^0$ denotes the output of the $r$th quantizer under the quantization level $\vartheta$ and the initial quantization state $\omega_r^0$, $\beta_r$ is the quantization density, satisfying $0 < \beta_r < 1$, and $\sigma_r = \frac{1-\beta_r}{1+\beta_r}$. It follows that $\sigma_r \in (0, 1)$. By using the sector bound approach [5], the quantization error is obtained as $\mathcal{Q}(u_r(t)) - u_r(t) = \mu_r u_r(t)$, $\mu_r \in [-\sigma_r, \sigma_r]$. It further follows that $\mathcal{Q}(u(t)) = (I + \Theta)u(t)$, where $\Theta = \mathrm{diag}\{\mu_1, \ldots, \mu_{n_u}\}$. Let $\bar{\sigma} = \max\{\sigma_1, \ldots, \sigma_{n_u}\}$. Then, $0 < (1 - \bar{\sigma})I \leqslant I + \Theta \leqslant (1 + \bar{\sigma})I < 2I$.

The objective of this study is to design a robust control strategy such that (i) when $w(t) \equiv 0$, system (1) is stochastically stable, and (ii) under the zero initial condition, system (1) ensures a noise attenuation level $\gamma$ in terms of an $H_\infty$ measure: $\int_0^\infty \mathrm{E}\{\|z(t)\|_2^2\}\mathrm{d}t < \gamma^2 \int_0^\infty \|w(t)\|_2^2\mathrm{d}t$.

The following mode-dependent adaptive control protocol is proposed:

$$\begin{aligned} &u(t) = K_{\delta_t}x(t) + (\hat{\alpha}_1 + \hat{\alpha}_2\|x(t)\|_2)\mathrm{sign}(K_{\delta_t}x(t)), \\ &\dot{\hat{\alpha}}_1 = \tau_1\|K_{\delta_t}x(t)\|_2, \ \dot{\hat{\alpha}}_2 = \tau_2\|x(t)\|_2\|K_{\delta_t}x(t)\|_2, \end{aligned} \quad (2)$$

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where $K_{\delta_t}$ is the controller gain to be designed, and $\tau_1 > 0$ and $\tau_2 > 0$ are given constants. The 1-norm and 2-norm of vector $x$ are respectively represented by $\|x\|_1$ and $\|x\|_2$. Function sign$(\cdot)$ satisfies (i) when scalar $a > 0$, sign$(a) = 1$, (ii) when $a<0$, sign$(a)=-1$, and (iii) otherwise, sign$(a)=0$.

**Theorem 1.** Under Assumption 1, system (1) is stochastically stable with a given $H_\infty$ performance $\gamma > 0$ subject to control law (2) with $K_s = -B_s^{\mathrm{T}} Q_s^{-1}$, if there exist $Q_s > 0$, $\varepsilon_1 > 0$, and $\varepsilon_2 > 0$ for all $s \in \mathcal{G}$, such that

$$\Upsilon = \begin{bmatrix} \Upsilon_{11} & C_s & \Upsilon_{13} & \Upsilon_{14} \\ * & -\gamma^2 I & \Upsilon_{23} & 0 \\ * & * & \Upsilon_{33} & 0 \\ * & * & * & \Upsilon_{44} \end{bmatrix} < 0, \tag{3}$$

where $\Upsilon_{11} = A_s Q_s + Q_s A_s^{\mathrm{T}} + \varepsilon_1 M_{1s} M_{1s}^{\mathrm{T}} + \varepsilon_2 M_{2s} M_{2s}^{\mathrm{T}} - 2(1-\bar{\sigma}) B_s B_s^{\mathrm{T}} - \sum_{g=1, g\neq s}^{G} \underline{\theta}_{sg} Q_s$, $\Upsilon_{13} = [Q_s D_s^{\mathrm{T}}\ Q_s N_{1s}^{\mathrm{T}}\ 0]$, $\Upsilon_{23} = [E_s^{\mathrm{T}}\ 0\ N_{2s}^{\mathrm{T}}]$, $\Upsilon_{33} = -\mathrm{diag}\{I, \varepsilon_1 I, \varepsilon_2 I\}$, $\Upsilon_{14} = [\sqrt{\overline{\theta}_{s1}} I, \ldots, \sqrt{\overline{\theta}_{ss-1}} I, \sqrt{\overline{\theta}_{ss+1}} I, \ldots, \sqrt{\overline{\theta}_{sG}} I] Q_s$, $\Upsilon_{44} = -\mathrm{diag}\{Q_1, \ldots, Q_{s-1}, Q_{s+1}, \ldots, Q_G\}$.

*Proof.* Select the following mode-dependent Lyapunov function: $V(t) = x^{\mathrm{T}}(t) P_s x(t) + (1-\overline{\sigma})[\frac{(\hat{\alpha}_1 - \beta_1)^2}{\tau_1} + \frac{(\hat{\alpha}_2 - \beta_2)^2}{\tau_2}]$, where $\delta_t = s$, $\beta_1 > \frac{\alpha_1}{1-\overline{\sigma}}$, and $\beta_2 > \frac{\alpha_2}{1-\overline{\sigma}}$. Along the trajectory of system (1), one obtains $\mathcal{L}V(t) = x^{\mathrm{T}}(t)[2P_s(A_s + \Delta A_s) + \bar{P}_s + 2P_s B_s (I + \Theta) K_s] x(t) + 2(\hat{\alpha}_1 + \hat{\alpha}_2 \|x(t)\|_2) x^{\mathrm{T}}(t) P_s B_s (I + \Theta) \mathrm{sign}(K_s x(t)) + 2x^{\mathrm{T}}(t) P_s B_s f(x(t)) + 2x^{\mathrm{T}}(t) P_s (C_s + \Delta C_s) w(t) + 2(1 - \overline{\sigma}) \|B_s^{\mathrm{T}} P_s x(t)\|_2 [\hat{\alpha}_1 - \beta_1 + (\hat{\alpha}_2 - \beta_2)\|x(t)\|_2] < x^{\mathrm{T}}(t) \Lambda_{11} x(t) + 2x^{\mathrm{T}}(t) P_s C_s w(t) + \frac{1}{\varepsilon_2} w^{\mathrm{T}}(t) N_{2s}^{\mathrm{T}} N_{2s} w(t)$, where $\bar{P}_s = \sum_{g=1}^{G} \theta_{sg}(d) P_g$, $K_s = -B_s^{\mathrm{T}} P_s$, and $\Lambda_{11} = P_s A_s + A_s^{\mathrm{T}} P_s + \bar{P}_s - 2(1-\bar{\sigma}) P_s B_s B_s^{\mathrm{T}} P_s + \frac{1}{\varepsilon_1} N_{1s}^{\mathrm{T}} N_{1s} + \varepsilon_1 P_s M_{1s} M_{1s}^{\mathrm{T}} P_s + \varepsilon_2 P_s M_{2s} M_{2s}^{\mathrm{T}} P_s$. $\mathcal{L}$ is the weak infinitesimal operator of the random process $\{x(t), \delta_t\}$. Set $\Sigma = \mathrm{diag}\{Q_s^{-1}, I, I, I\}$ and $Q_s^{-1} = P_s$. With (3), using the Schur complement to inequality $\Sigma \Upsilon \Sigma^{\mathrm{T}}$ yields $\Gamma = [\begin{smallmatrix} \Gamma_{11} & P_s C_s + D_s^{\mathrm{T}} E_s \\ * & \frac{1}{\varepsilon_2} N_{2s}^{\mathrm{T}} N_{2s} + E_s^{\mathrm{T}} E_s - \gamma^2 I \end{smallmatrix}] < 0$, where $\Gamma_{11} = P_s A_s + A_s^{\mathrm{T}} P_s + \sum_{g=1, g\neq s}^{G} \overline{\theta}_{sg} P_g - \sum_{g=1, g\neq s}^{G} \underline{\theta}_{sg} P_s - 2(1 - \overline{\sigma}) P_s B_s B_s^{\mathrm{T}} P_s + \frac{1}{\varepsilon_1} N_{1s}^{\mathrm{T}} N_{1s} + \varepsilon_1 P_s M_{1s} M_{1s}^{\mathrm{T}} P_s + \varepsilon_2 P_s M_{2s} M_{2s}^{\mathrm{T}} P_s + D_s^{\mathrm{T}} D_s$. Since $0 \leqslant \underline{\theta}_{sg} \leqslant \theta_{sg}(d) \leqslant \overline{\theta}_{sg}$ for $s \neq g$ and $\theta_{ss}(d) = -\sum_{g=1, g\neq s}^{G} \theta_{sg}(d)$, one has $\bar{P}_s = \sum_{g=1, g\neq s}^{G} \theta_{sg}(d) P_g + \theta_{ss}(d) P_s \leqslant \sum_{g=1, g\neq s}^{G} \overline{\theta}_{sg} P_g - \sum_{g=1, g\neq s}^{G} \underline{\theta}_{sg} P_s$. It further follows that $[\begin{smallmatrix} \Lambda_{11} + D_s^{\mathrm{T}} D_s & P_s C_s + D_s^{\mathrm{T}} E_s \\ * & \frac{1}{\varepsilon_2} N_{2s}^{\mathrm{T}} N_{2s} - \gamma^2 I + E_s^{\mathrm{T}} E_s \end{smallmatrix}] < 0$. When $w(t) \equiv 0$, one obtains $\mathcal{L}V(t) < x^{\mathrm{T}}(t) \Lambda_{11} x(t) < 0$, which implies the stochastic stability of system (1). When $w(t) \not\equiv 0$, it follows from $\Lambda < 0$ that $\mathrm{E}\{\|z(t)\|_2^2 - \gamma^2 \|w(t)\|_2^2\} + \mathcal{L}V(t) < 0$. Thus, under the zero initial condition, it is concluded that $H_\infty$ performance $\int_0^\infty \mathrm{E}\{\|z(t)\|_2^2\} \mathrm{d}t < \gamma^2 \int_0^\infty \|w(t)\|_2^2 \mathrm{d}t - \mathrm{E}\{V(\infty)\} < \gamma^2 \int_0^\infty \|w(t)\|_2^2 \mathrm{d}t$ is guaranteed. Then, one has that if there is a solution to (3), controller gains are computed as $K_s = -B_s^{\mathrm{T}} Q_s^{-1}$.

The designed control law (2) can also be used to deal with the robust stabilization problem for nonlinear MJSs subject to actuator failure, studied in [6], where an adaptive sliding mode control (SMC) technique is applied with a rank constraint on $B_{\delta_t}$. The system studied in [6] is

$$\dot{x}(t) = A_{\delta_t} x(t) + B_{\delta_t} [\rho u(t) + f(x(t)) + \kappa(t)], \tag{4}$$

where $\rho = \mathrm{diag}\{\rho_1, \ldots, \rho_{n_u}\}$ denotes actuator failure with upper bound matrix $\overline{\rho} = \mathrm{diag}\{\overline{\rho}_1, \ldots, \overline{\rho}_{n_u}\} < I$ and lower bound matrix $0 < \underline{\rho} = \mathrm{diag}\{\underline{\rho}_1, \ldots, \underline{\rho}_{n_u}\}$. Moreover, $\|f(x(t))\|_2 < \alpha_1 + \alpha_2 \|x(t)\|_2$, and the external bounded matched disturbance $\kappa(t)$ satisfies $\|\kappa(t)\|_2 < \overline{\kappa}$ with $\alpha_1 > 0$, $\alpha_2 > 0$, and $\overline{\kappa} > 0$ being unknown.

**Theorem 2.** Under Assumption 1, system (4) is stochastically stable subject to control law (2) with $K_s = -B_s^{\mathrm{T}} Q_s^{-1}$, if there exist $Q_s > 0$, $\varepsilon_1 > 0$, and $\varepsilon_2 > 0$ for all $s \in \mathcal{G}$, such that

$$\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ * & \Psi_{22} \end{bmatrix} < 0, \tag{5}$$

where $\Psi_{11} = A_s Q_s + Q_s A_s^{\mathrm{T}} - 2\underline{\rho}' B_s B_s^{\mathrm{T}} - \sum_{g=1, g\neq s}^{G} \theta_{sg} Q_s$, $\Psi_{12} = [\sqrt{\theta_{s1}} I, \ldots, \sqrt{\theta_{ss-1}} I, \sqrt{\theta_{ss+1}} I, \ldots, \sqrt{\theta_{sG}} I] Q_s$, $\Psi_{22} = -\mathrm{diag}\{Q_1, \ldots, Q_{s-1}, Q_{s+1}, \ldots, Q_G\}$, $\underline{\rho}' = \min\{\underline{\rho}_1, \ldots, \underline{\rho}_{n_u}\}$.

*Proof.* The Lyapunov function is chosen as $V(t) = x^{\mathrm{T}}(t) P_s x(t) + \frac{\underline{\rho}' (\hat{\alpha}_1 - \beta_1)^2}{\tau_1} + \frac{\underline{\rho}' (\hat{\alpha}_2 - \beta_2)^2}{\tau_2}$, where $\delta_t = s$, $\beta_1 > \frac{\alpha_1 + \overline{\kappa}}{\underline{\rho}'}$, and $\beta_2 > \frac{\alpha_2}{\underline{\rho}'}$. By a similar analysis as in Theorem 1, the result is obtained.

**Remark 1.** In the above proposed control law, only minimal low bound $\underline{\rho}'$ needs to be known in advance and there is no rank constraint on $B_{\delta_t}$. However, the information of both lower and upper bounds of actuator failure must be given with a full column-rank constraint on $B_{\delta_t}$ in [6]. By comparison, the proposed control law (2) has fewer requirements and is much easier to implement.

*Conclusion.* The robust $H_\infty$ stabilization problem has been analyzed for uncertain nonlinear semi-MJSs subject to input quantization and external disturbance. A mode-dependent adaptive control law has been designed. By using a mode-dependent Lyapunov function, sufficient conditions have been developed to ensure the stochastic stability of the closed-loop systems with a given $H_\infty$ performance and to guarantee the existence of controller gains.

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