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Adaptive output-feedback tracking for nonlinear systems with unknown control direction and generic inverse dynamics

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Abstract This paper studies adaptive output-feedback tracking for a class of typical uncertain nonlinear systems. In the context of unknown control direction, a generic uncertainty encountered inevitably in practice is adequately taken into consideration, i.e., the ISpS (input-to-state practically stable) inverse dynamics acting as the dynamic differences between the real plants and the models. Besides, the systems in question also permit two critical ingredients, i.e., unmeasured-state dependent nonlinearities and arbitrary function-of-output growth on unknown system nonlinearities. The three ingredients together largely challenge the feasibility/availability of practical tracking by means of output feedback. Nevertheless, a new control strategy is proposed by flexibly integrating the dynamic compensation based on Nussbaum-type gain, backstepping design technique together with the refined pseudo-sign and pseudo-dead-zone functions that were introduced for the first time in our previous studies. The two refined functions, which are sufficiently smooth, can moderately avoid the use of smooth domination/treatment in control design and can potentially render the attained control strategy tighter and less conservative. Moreover, to keep the order of the closed-loop system at a low level, an n-dimensional filter with a dynamic high gain is delicately devised instead of a $2n$-dimensional one used in the relevant literature. It turns out that the proposed adaptive output-feedback controller is capable of guaranteeing the global boundedness of all states of the resulting closed-loop system, while steering the system tracking error to enter, in finite time, a prescribed λ-neighborhood of the origin and keeping it inside thereafter. A simulation example is provided to demonstrate the proposed approach.

Keywords nonlinear systems, ISpS inverse dynamics, unknown control direction, practical output tracking, global output feedback, adaptive compensation

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1 Introduction

Ubiquitous uncertainties/noises have been seriously challenging the feasibility/availability of control and the underlying system performance, and great efforts have been made to counteract/eliminate their negative influence [1–3]. Despite considerable progress, certain rather stringent constraints are still necessarily imposed on the system uncertainties as well as on the system nonlinearities to achieve the core objectives which are theoretically perfect, such as asymptotic stabilization/consensus, output regulation, and asymptotic/perfect tracking (e.g., [1, 4–7]). Recognize that theoretically perfect objectives potentially render scarce applicability, and often come along with unexpected high cost. Certain imperfectness or certain error tolerance after careful trade-off is preferred from a practical perspective, which inspires one to pursue degenerated objectives for practical purposes, such as, satisfactory stabilization, practical output regulation, and practical tracking.

Prominently, practical tracking (also termed approximate tracking or λ-tracking) aims to find a feedback law capable of driving the tracking error to a prescribed vicinity around zero in finite time and

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keeping it inside thereafter. For this purpose, it needs to cope not only with inherent system nonlinearities and/or uncertainties, but also with extra substantial time variation and/or new types of uncertainties coming from the reference signals to be tracked. The latter could drastically change some of the intrinsic attributes of the system, such as ISS (input-to-state stable) property, equilibria, and essentially distinguishes practical tracking from asymptotic stabilization/consensus (e.g., [8–12]). While viewed as a degenerated version of asymptotic tracking (e.g., [5]), practical tracking owes its importance to two aspects: (1) It is typically of low expense and often adequate in practical applications; (2) its realization needs fewer/weaker constraints on reference signals and systems; therefore more sorts of uncertainties are allowed and a wider range of nonlinear systems can be studied. Particularly, for some highly nonlinear systems, practical tracking is solvable while asymptotical tracking is not [13].

Hitherto, numerous results have been reported on practical tracking for nonlinear systems with various uncertainties/unknowns [8–11, 13–18], but only a few are in the setting of output feedback and unknown control directions. In such a setting, practical tracking would become rather ambitious to be achieved and requires rather stringent restrictions on the system structure, nonlinearities/uncertainties or reference signals (e.g., [8, 9, 18]). In detail, Ref. [8] specialized system relative degree to 1 (which avoided the dynamic compensation, i.e., the observer/filter), and the input unmatched nonlinearities were merely linearly dependent on the (uncorrupted) output. As an essential extension of [8], Ref. [9] allowed for arbitrary known relative degree and, more importantly, it for the first time addressed practical tracking for systems having polynomial-of-output growth. Nevertheless, Ref. [9] required a priori information on the polynomial power that plays an indispensable part in control design and system performance analysis. Once it is unknown, the feasibility of control would be definitely violated, let alone the cases where no such polynomial power exists at all (for instance, where function e^y cannot be upper bounded by any polynomial of y on the whole $\mathbb{R}$). Overcoming this drawback, our recent work [18] successfully loosened unknown system nonlinearities from polynomials to arbitrary functions of output. Particularly, in [18], the pseudo-sign and pseudo-dead-zone functions, both with sufficient smoothness, were introduced for the first time for the direct utilization in the recursive control design; thereby, an ingenious estimation was performed on unknown system nonlinearities, which, together with a novel Lyapunov function and the elegant system performance analysis, successfully removed the substantial polynomial constraint.

Nevertheless, two types of typical uncertainties/unknowns are not considered in [18]: unmeasured-state dependent nonlinearities and inverse dynamics. The former fulfilling the fundamental limit given in [19] has been frequently involved in systems with known control direction. For instance, to achieve global practical tracking, Refs. [10, 11, 13, 14, 16, 17] concentrated on overcoming the essential influence arising from the nonlinearities of this type with known or unknown growth rates of constant or polynomial-of-output. But when unknown control direction presents, no substantial work on practical tracking has yet been achieved by output feedback. The latter, i.e., inverse dynamics, are the internal unobservable dynamics of the system (for details, please refer to [20] and the references therein). Interpreted as unmodeled dynamics (or dynamic uncertainties), they can also be used to account for the dynamic difference between (reduced-order) models and real plants; thus, their presence broadens, to some extent, the types of real plants that can be represented mathematically. It is also worth pointing out that under the input that holds the output constantly at zero, the inverse dynamics are referred to as zero dynamics. The asymptotically stable zero dynamics means the minimum-phase property of the nonlinear systems and the property plays a key role in the design of controllers for nonlinear systems (see [20] for details).

In this paper, we confine ourselves on global practical tracking for a class of typical nonlinear systems in the context of output feedback and unknown control direction. Particularly, the two aforementioned uncertainties/unknowns are taken into account seriously, which in conjunction with arbitrary function-of-output growth distinguish this paper markedly from our recent one [18] as well as the related ones on practical tracking (e.g., [8,9]). For unmeasured-state dependent nonlinearities and the arbitrary function-of-output growth, both of them can bring high nonlinearities which invalidate the control scheme based on polynomial of output in [9]. Thus far, the latter is allowed only in [18], while no substantial work on practical tracking has involved the former. With regard to inverse dynamics, mild restriction (see Assumption 1 below) makes them be of generic ISpS (input-to-state practically stable) in this paper. Whereas in [16,17], despite known control direction, inverse dynamics are of a special ISpS type (the input is in a quadratic form instead of being an arbitrary function of output) and are even specialized to be of an ISS type respectively. The rather stringent constraints imposed on inverse dynamics for practical tracking contrast sharply with those of other control objectives (for instance, stabilization and practical output regulation) for which the IS(p)S/iISS conditions can be highly nonlinear (parallel Assumption 1) even if control direction is unknown [21, 22]. This, in turn, indicates that the high nonlinearities and unwanted new uncertainties stemming from inverse dynamics may directly undermines the control feasibility of practical tracking, especially in such a severe context.

The tracking problem, obstructed by diverse uncertainties/unknowns, calls for integration of multiple advanced techniques and subtle system performance analysis routes, typically such as, the flexible use of the refined design functions [15, 18], the delicate scaling of observer error and the elegant choice of Lyapunov functions. In fact, for the systems under investigation without inverse dynamics, asymptotic stabilization and asymptotic tracking have been achieved in [5] under restrictive assumptions on the reference signal. But, as stated above, since practical tracking requires fewer restrictions on systems and reference signals, more types of uncertainty are involved in control design and performance analysis; therefore, it cannot be simply transformed into stabilization as asymptotic one does and more importantly, the presence of generic inverse dynamics brings extra essential difficulties for system performance analysis and calls for specific treatment. (i) In contrast to inverse dynamics in special forms in relevant literature (e.g., [16, 17]), the generic ISpS ones considered in this paper would result in extra high nonlinearities, which, in conjunction with the intrinsic nonlinearities (i.e., $\psi_i^y(y)$'s in Assumption 2 below), would directly undermine the commonly used controller design routes for polynomial growth and importantly, certain implications between the high nonlinearities and the Lyapunov functions are required exploring to establish the tracking performance. (ii) An instructive pattern of control design and analysis is achieved with multiple critical functions/variables that are delicately defined. Due to the special scaled observer error, its dynamics are in a nice form to facilitate system performance analysis. Since inverse dynamics give rise to high nonlinearities and unwanted new uncertainties, the function $q(\cdot)$ is especially devised (i.e., (23)) to eliminate their negative influence. Also, to circumvent the obstruction in utilizing a fundamental lemma (i.e., Lemma 3), the functions $W_1(\cdot)$ and $W_2(\cdot)$ are typically proposed (to satisfy (37)). (iii) The dimension of the closed-loop system is kept at a low level. To this end, an n-dimensional filter with a dynamic high gain is particularly proposed to rebuild unmeasured states, instead of the $2n$-dimensional one given in [5], which is instructive to related studies. It is also noteworthy that the ingenious dynamic high gain plays an indispensable part in compensating for unmeasured-state dependent nonlinearities.

The remainder of this paper is organized as follows. Section 2 introduces some notations and preliminaries. Section 3 presents the system model and the control objective. In Section 4, the adaptive control scheme is proposed via output feedback, and the main results are summarized in Section 5. A simulation example is provided in Section 6 and concluding remarks are given in Section 7.

2 Notations and preliminaries

Some notations adopted throughout the paper are collected below, in conjunction with the several important lemmas to be used frequently in later development.

For a vector or matrix $W = (w_{ij})_{n\times m}$, we use $\|W\|$ to denote the Euclidean norm; we also introduce the Frobenius norm of W defined by $\|W\|_F = \sqrt{\sum_{i=1}^n \sum_{j=1}^m w_{ij}^2}$ which owns the properties: $\|W\| \leqslant \|W\|_F$ and $\|WV\|_F \leqslant \|W\| \cdot \|V\|_F$ for any matrix V with appropriate dimension. For a vector $x \in \mathbb{R}^n$, let x_i be its i-th element, and $x_{[i]} = [x_1, \ldots, x_i]^{\mathrm{T}}$ and define $\mathrm{diag}\{a_1, \ldots, a_n\}$ as the diagonal matrix.

$\mathcal{C}^i$ denotes the set of i-th continuously differentiable functions, and $\mathcal{C}$ and $\mathcal{C}^\infty$ are the sets of continuous and smooth functions, respectively. By a $\mathcal{C}^i$ function, we mean that the function belongs to $\mathcal{C}^i$. A function $\alpha(x) : [0, a) \to [0, +\infty)$ is a class $\mathcal{K}$ function if it is $\mathcal{C}$, strictly increasing and $\alpha(0) = 0$; it is a class $\mathcal{K}_\infty$ function if it is a class $\mathcal{K}$ function on $[0, +\infty)$ and $\lim_{x\to+\infty} \alpha(x) = +\infty$. Moreover, for brevity, a function $\Gamma(x)$ is simplified by $\Gamma(\cdot)$ or Γ if no confusion arises.

Let $\mathrm{sign}(\cdot)$ be the sign function; i.e., $\mathrm{sign}(x) = -1$ if $x < 0$, $\mathrm{sign}(x) = 0$ if $x = 0$, and $\mathrm{sign}(x) = 1$ if $x > 0$. We use $N(\cdot)$ to denote the Nussbaum function enjoying the property: $\limsup_{k\to\infty} \frac{1}{k}\int_0^k N(s)\mathrm{d}s = +\infty$ and $\liminf_{k\to\infty} \frac{1}{k}\int_0^k N(s)\mathrm{d}s = -\infty$, which has proven effective to handel unknown control direction [8, 15, 23] and works as the sole way to realize continuous feedback.

Next, we would like to give five important lemmas. While having been presented in our recent work [18], they are still collected here for the later frequent use in control design and system performance analysis, also for the sake of self-containedness and high readability of the paper. In detail, Lemmas 1 and 2 are drawn from [15] and [18], while the last three from [23], [8] (or [9]), and [22].

Lemma 1 ([15])**.** There exists a $\mathcal{C}^n$ function $P_n(\cdot)$ ensuring the following $\mathcal{C}^n$ pseudosign function (which is in line with the sign function as μ goes to zero):

$$\mathrm{sgn}_{\mu,n}(x) = \begin{cases} \mathrm{sign}(x), & |x| \geqslant \mu, \\ P_n\left(\dfrac{x}{\mu}\right), & |x| < \mu. \end{cases}$$

Lemma 2 ([18])**.** There exists a $\mathcal{C}^n$ function $Q_{\frac{\mu}{2},n}(\cdot)$ guaranteeing the following $\mathcal{C}^n$ pseudo-dead-zone function (which can be regarded as an approximation of the ideal dead-zone function):

$$D_{\mu,n}(x) = \begin{cases} (|x|-\mu)\mathrm{sign}(x), & |x| \geqslant \dfrac{3}{2}\mu, \\ Q_{\frac{\mu}{2},n}(|x|-\mu)\mathrm{sign}(x), & \dfrac{1}{2}\mu < |x| < \dfrac{3}{2}\mu, \\ 0, & |x| \leqslant \dfrac{1}{2}\mu. \end{cases}$$

Remark 1. Observing from the definitions of $\mathrm{sgn}_{\frac{\mu}{2},n}(\cdot)$ and $D_{\mu,n}(\cdot)$ that there holds $|D_{\mu,n}(\cdot)| = D_{\mu,n}(\cdot)\,\mathrm{sgn}_{\frac{\mu}{2},n}(\cdot)$, we can readily know $|D_{\mu,n}(\cdot)|$ is $\mathcal{C}^n$ on $\mathbb{R}$ as well. Since they belong to $\mathcal{C}^n$, pseudosign function $\mathrm{sgn}_{\frac{\mu}{2},n}(\cdot)$ and pseudo-dead-zone function $D_{\mu,n}(\cdot)$ can be used to construct the output-feedback controller (see (17)–(20) below) iteratively. The direct utilization would moderately avoid the use of domination based on completing squares in control design, which would render the to-be-pursued output-feedback controller tighter and less conservative. Actually, in [9], more smooth treatments need taking to find $\mathcal{C}^\infty$ dominations since the discontinuous sign$(\cdot)$ and the $\mathcal{C}$ dead-zone function were both adopted.

Lemma 3 ([23])**.** Suppose for a $\mathcal{C}^\infty$ even Nussbaum function $N(\cdot)$, there exists a $\mathcal{C}^1$ nonnegative functions $V(\cdot)$ and a $\mathcal{C}^1$ function $k(\cdot)$, both defined on $[0,\, t_f)$ $(0 < t_f \leqslant +\infty)$, such that

$$V(t) \leqslant a_1 + \int_0^t (gN(k(\tau)) + a_2)\mathrm{d}k(\tau), \quad \forall t \in [0,\, t_f)$$

with constants a_i's and $g \neq 0$. Then $V(t)$, $k(t)$ and $\int_0^t N(k(\tau))\mathrm{d}k(\tau)$ are all bounded on $[0,\, t_f)$.

Lemma 4 ([8,9])**.** Suppose nonnegative functions $V_1(\cdot)$ and $V_2(\cdot)$ which are $\mathcal{C}^1$ and $\mathcal{C}$ respectively satisfy $\dot{V}_1(t) \leqslant -a_1V_1(t) + a_2(1 + V_2(t))$, $\forall t \in [0,\, t_f)$ for constants $a_1 > 0$ and $a_2 > 0$. Then there holds, for $a_3 > 0$

$$\int_0^t \sqrt{V_1(\tau)} \cdot \sqrt{V_2(\tau)}\mathrm{d}\tau \;\leqslant\; a_3 \int_0^t \sqrt{V_2(\tau)}\big(1 + \sqrt{V_2(\tau)}\big)\mathrm{d}\tau, \quad \forall t \in [0,\, t_f).$$

Lemma 5 ([22])**.** For any $\mathcal{C}$ nonnegative function $h(\cdot)$ on $\mathbb{R}^m \times \mathbb{R}^n$, there are $\mathcal{C}^\infty$ positive functions $h_1(\cdot)$ and $h_2(\cdot)$ such that $h(x,y) \leqslant h_1(x)h_2(y)$, for any $(x,y) \in \mathbb{R}^m \times \mathbb{R}^n$.

3 Problem formulation

As aforementioned, the related studies addressed global practical tracking for uncertain nonlinear system with various types of uncertainties and/or nonlinearities by means of adaptive output feedback. Quite typically, we confine ourselves on the following rather generic system:

$$\begin{cases} \dot{\eta} = \psi_0(\eta, y), \\ \dot{x}_i = x_{i+1} + \phi_i^{\mathrm{T}}(y)x_{[i]} + \psi_i(\eta, y), \; i = 1, \ldots, n-1, \\ \dot{x}_n = gu + \phi_n^{\mathrm{T}}(y)x_{[n]} + \psi_n(\eta, y), \\ y = x_1, \end{cases} \tag{1}$$

where $\eta \in \mathbb{R}^m$ and $x = [x_1, \ldots, x_n]^{\mathrm{T}} \in \mathbb{R}^n$ are the system state vectors with the initial conditions $\eta(0) = \eta_0$ and $x(0) = x_0$; $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are the control input and system output, respectively; $\phi_i(\cdot) = [\phi_{i1}, \ldots, \phi_{ii}]^{\mathrm{T}}$ with each entry ϕ_{ij} a known $\mathcal{C}^\infty$ function; $\psi_i(\cdot)$'s, called system nonlinearities, are unknown locally Lipschitz functions; unknown nonzero constant g is termed the control coefficient, whose unknown sign acts as unknown control direction of the system.

Remark that system (1) and its variants have been thoroughly studied during the past few decades, e.g., [4, 14, 16–18]. Diverse practical plants can be taken as special cases of system (1), such as robot systems, ship steering, and motor drive [24, 25]. In system (1), inverse dynamics are taken into account in particular (in contrast to related studies, e.g., [5]) which represent the internal unobservable dynamics of system (1). Interpreted as the unmodeled dynamics (or dynamic uncertainties), inverse dynamics can be used to account for the dynamic difference between the (reduced-order) model (i.e., x-subsystem) and the real plant (i.e., system (1)). Thus, the existence of inverse dynamics η broadens the types of systems that (1) can represent to a certain extent.

When control direction is known precisely, system (1) has been investigated in [4] where the unmeasured-state dependent nonlinearities "$\phi_i^{\mathrm{T}}(y)x_{[i]}$" were tackled successfully by a sophisticated reduced-order observer. However, in the setting of output feedback and unknown control direction, the presence of this ingredient (i.e., $\phi_i^{\mathrm{T}}(y)x_{[i]}$) would result in the observer in [4] no longer applicable which in turn undermines the foundation of the control design therein.

When control direction is unknown, asymptotic tracking has been achieved in [5] for system (1) without inverse dynamics in virtue of stabilization. Whereas, since practical tracking requires fewer restrictions on systems and reference signals, more types of uncertainties would involve in control design and performance analysis; therefore, it cannot be converted into stabilization as the the asymptotic one does. Therefore, up to now no substantial work has been achieved to solve practical tracking.

The control objective is to pursue an adaptive controller by output feedback for system (1) such that: (i) for any initial values $\eta_0 \in \mathbb{R}^m$ and $x_0 \in \mathbb{R}^n$, all the resulting closed-loop system states are globally bounded on $[0, +\infty)$, and that (ii) the system tracking error, i.e., $z_1(t) = y(t) - y_r(t)$, enters the prescribed strip $-\lambda < z_1 < \lambda$ in finite time and lies inside thereafter. Moveover, to keep the order of the closed-loop system at a low level, an n-dimensional filter with a dynamic high gain is delicately devised to rebuild the unmeasured states rather than the two n-dimensional K-filters in [5].

Assumption 1. Inverse dynamics η in system (1) has an ISpS Lyapunov function $U_0(\eta)$, i.e., there exist known class $\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, $\alpha_3(\cdot)$, and $\gamma(\cdot)$, and an unknown constant $\theta_\eta > 0$ such that

$$\alpha_1(\|\eta\|) \leqslant U_0(\eta) \leqslant \alpha_2(\|\eta\|), \quad \frac{\partial U_0}{\partial \eta}\psi_0(\eta, y) \leqslant -\alpha_3(\|\eta\|) + \theta_\eta \gamma(|y|) + \theta_\eta.$$

Assumption 2. There are known $\mathcal{C}^\infty$ nonnegative functions $\psi_i^\eta(\cdot)$ with $\psi_i^\eta(0) = 0$ and $\psi_i^y(\cdot)$, $i = 1, \ldots, n$, and an unknown nonnegative constant θ_ψ, such that

$$|\psi_i(\eta, y)| \leqslant \theta_\psi \psi_i^\eta(\|\eta\|) + \theta_\psi \psi_i^y(y).$$

Assumption 3. For reference signal y_r which is $\mathcal{C}^1$, there holds for an unknown constant $M_r \geqslant 0$,

$$\sup_{t \geqslant 0}(|y_r(t)| + |\dot{y}_r(t)|) \leqslant M_r.$$

Assumption 1, in terms of arbitrary class $\mathcal{K}_\infty$ functions $\alpha_i(\cdot)$'s and $\gamma(\cdot)$, presents rather generic ISpS inverse dynamics for global practical tracking, while in [16] the inverse dynamics are of a special ISpS type and in [17] they are even specialized to be of ISS type (despite known control direction). Assumption 3 is a standing one for global practical tracking which shows merely crude knowledge is additionally needed on the reference signal besides its current value (e.g., [10, 15]). Owing to the presence of arbitrary function of output "$\psi_i^y(y)$", Assumption 2 distinguishes practical tracking in this paper from the existing ones in [8, 9] where only functions of polynomial-of-output is considered in the context of output feedback and unknown control direction. Besides, as explicated in our recent work [18], Assumption 2 also enables the system in question to cover the scenario with power uncertainty in polynomial of output which can be referred to as a new type of uncertainty.

We now would like to present an important proposition under Assumption 1, which implies that an exp-ISpS Lyapunov function can be derived by any ISpS one (inspired by [21, 26, 27]). This proposition essentially facilitates the analysis of system performance later on, whose proof is not detailed here due to the space limitation.

Proposition 1. Under Assumption 1, there is an exp-ISpS Lyapunov function $\mathscr{V}(\eta)$ for any given constant $\mu > 0$, such that for any $(\eta, y) \in \mathbb{R}^m \times \mathbb{R}$,

$$\underline{\alpha}(\|\eta\|) \leqslant \mathscr{V}(\eta) \leqslant \overline{\alpha}(\|\eta\|), \quad \frac{\partial \mathscr{V}}{\partial \eta}\psi_0(\eta, y) \leqslant -\frac{\mu}{2}\mathscr{V}(\eta) + \bar{\theta}_\eta \bar{\gamma}(y), \tag{2}$$

where $\overline{\alpha}(\cdot)$ and $\underline{\alpha}(\cdot)$ are known class $\mathcal{K}_\infty$ functions, $\bar{\gamma}(\cdot)$ is a known $\mathcal{C}^\infty$ function satisfying $\bar{\gamma}(y) \geqslant 1$, $\forall y \in \mathbb{R}$ and $\bar{\theta}_\eta$ is an unknown nonnegative constant.

4 Adaptive output-feedback controller design

To realize global practical tracking for system (1) with Assumptions 1–3, this section is partitioned into two parts, consisting of dynamic high-gain filter design and adaptive output-feedback controller design.

4.1 Dynamic high-gain observer design

In this subsection, a dynamic high-gain filter is proposed to rebuild the unmeasured states. Subsequently, three key propositions are given to largely serve the controller design in Subsection 4.2.

We construct the high-gain filter as follows:

$$\begin{cases} \dot{\xi}_i = \xi_{i+1} + \phi_i^{\mathrm{T}}(y)\xi_{[i]} - L^i a_i \xi_1, \ i = 1, \ldots, n-1, \\ \dot{\xi}_n = u + \phi_n^{\mathrm{T}}(y)\xi_{[n]} - L^n a_n \xi_1, \end{cases} \tag{3}$$

where L is a dynamic high gain and a_i's are positive design parameters.

With the filter in hand, define $\hat{x}_i = g\xi_i$, $i = 1, \ldots, n$ as the estimate states of the unmeasured states of system (1). Due to the unknown g, neither the dynamics of $\hat{x}_i$'s are implementable nor the states themselves are available for feedback. However, the importance of such estimate states does not lie in rebuilding unmeasured states x_i's, but in well serving system performance analysis afterwards.

Define the scaled observer error $e = [e_1, \ldots, e_n]^{\mathrm{T}}$:

$$e_i = \frac{x_i - \hat{x}_i}{L^i \theta^*}, \quad i = 1, \ldots, n, \tag{4}$$

where $\theta^* = \max\{1, \theta_\psi, \theta_\psi \bar{\theta}_\psi\}$ is an unknown constant and $\bar{\theta}_\psi$ is defined in Proposition 2. Note that such an unknown weight θ^* is merely for later system performance analysis, not for later control design.

Taking time derivative of e_i's and combining the original system (1) with the dynamic high-gain filter (3), after some simple computations, we obtain

$$\dot{e} = LAe + \frac{a}{\theta^*}x_1 + \Lambda_L^{-1}\Phi(y)\Lambda_L e + \frac{\Lambda_L^{-1}}{\theta^*}\Psi(\eta, y) - \frac{\dot{L}}{L}\Lambda e, \tag{5}$$

where $\Lambda_L = \mathrm{diag}\{L, L^2, \ldots, L^n\}$, $\Lambda = \mathrm{diag}\{1, 2, \ldots, n\}$, and $\Psi(\cdot) = [\psi_1, \ldots, \psi_n]^{\mathrm{T}}$.

Remark 2. It is noteworthy that implementable filter (3) is in a special form, compared with the commonly used ones in related literature (e.g., [14, 18]); that is, it incorporates particularly with the available system nonlinearities $\phi_i^{\mathrm{T}}(y)$'s. These specialized terms are designed to alleviate the negative impacts coming from unmeasured states on dynamics of observer error. Hence, with their help, besides rebuilding unmeasured states, filter (3) in conjunction with the delicately scaled observer error e_i's in (4) leads to a nice error dynamics (5) in the presence of unknown control direction. In addition, it is of n-dimension, which keeps the closed-loop system dimension at a low level in comparison with the two n-dimensional filters in [5].

In light of [28], $a = [a_1, \ldots, a_n]^{\mathrm{T}}$ is chosen to guarantee that matrix $A = [-a, [I_{n-1}, 0_{n-1}]^{\mathrm{T}}] \in \mathbb{R}^{n \times n}$ is Hurwitz (with I_{n-1} an $(n-1)$-dimensional identity matrix and 0_{n-1} a zero column vector of $n-1$ dimension) and meanwhile to ensure there is a positive definite matrix P satisfying

$$A^{\mathrm{T}}P + PA \leqslant -2I_n, \quad c_1 I_n \leqslant \Lambda P + P\Lambda \leqslant c_2 I_n \tag{6}$$

with positive design parameters c_1 and c_2 fulfilling $c_1 < c_2$.

In order to (at least) compensate for the unmeasured-state dependent nonlinearities $\phi_i^{\mathrm{T}}(y)$ in system (1), design the dynamics of high gain L in the following form:

$$\dot{L} = -\frac{L^2}{2c_2} + \frac{L}{c_1}\sqrt{1+4\|P\|^2\|\Phi(y)\|_F^2}, \quad L(0) \geqslant 1, \tag{7}$$

where $\Phi(y) = (\phi_{ij}(y))_{n\times n}$ with $\phi_{ij}(y) = 0$, $j > i$.

We now would like to present three importt propositions on system nonlinearities $\psi_i^y(y)$'s, observer error e and inverse dynamics η. In detail, Proposition 2 provides an appropriate estimate for $\psi_i^y(y)$'s (in Assumption 2); Proposition 3 shows e enjoys ISpS property with respect to inputs (η, z_1, L), while Proposition 4 gives ISpS property of η with respect to tracking error z_1 (or equivalently, with respect to y, since $y = z_1 + y_r$ with bounded y_r). Since Proposition 2 is much similar to Proposition 1 in [18], we omit its proof here.

Proposition 2. Under Assumption 3, for any prescribed tracking accuracy $\lambda > 0$, the $\mathcal{C}^\infty$ functions ψ_i^y's in Assumption 2 satisfy

$$\psi_i^y(y) \leqslant \bar{\theta}_\psi\big(1 + |D_{\frac{\lambda}{2},n}(z_1)|\bar{\psi}_i^y(z_1)\big), \tag{8}$$

where $\bar{\psi}_i^y$'s are known $\mathcal{C}^\infty$ nonnegative functions and $\bar{\theta}_\psi$ is an unknown positive constant.

Proposition 3. Let $V_e = e^{\mathrm{T}}Pe$ with P fulfilling (6). Then, along the solutions of system (5), there holds

$$\dot{V}_e \leqslant -L\|e\|^2 + 12n\|P\|^2\|\psi^\eta(\|\eta\|)\|^2 + \tilde{\psi}^y(z_1, L)D^2_{\frac{\lambda}{2},n}(z_1) + \theta_e, \tag{9}$$

where $\psi^\eta = [\psi_1^\eta, \dots, \psi_n^\eta]^{\mathrm{T}}$ with ψ_i^η's defined in Assumption 2, $\tilde{\psi}^y(\cdot)$ is a known $\mathcal{C}^\infty$ positive function, and θ_e is an unknown positive constant.

Proposition 4. Let $V_\eta = \int_0^{\mathscr{V}(\eta)} q(\tau)\mathrm{d}\tau$ with $q(\cdot)$ a known $\mathcal{C}^\infty$ nonnegative strictly increasing function. Then, along the trajectories of inverse dynamics (η-subsystem), there holds

$$\dot{V}_\eta \leqslant -\frac{\mu}{4}q(\mathscr{V}(\eta))\mathscr{V}(\eta) + \tilde{\theta}_\eta(1 + |D_{\frac{\lambda}{2},n}(z_1)|\tilde{\gamma}(z_1)), \tag{10}$$

where $\tilde{\gamma}(\cdot)$ is a $\mathcal{C}^\infty$ nonnegative function, and $\tilde{\theta}_\eta$ is an unknown positive constant.

Remark 3. The high nonlinearities coming from scaled error e and inverse dynamics η (see the $\tilde{\psi}^y(\cdot)$ in (9) and $\tilde{\gamma}(\cdot)$ in (10) respectively) would directly undermines the feasibility of the control strategies for polynomial growth given in [9]. Hence, we assign them the refined pseudo-dead-zone function "$D_{\frac{\lambda}{2},n}(z_1)$" as a factor, as done in [18]. With the help of such treatment, the unwanted nonlinearities can be eliminated directly by the devised controller afterwards.

Proof of Proposition 3. Taking time derivative of V_e and invoking error dynamics (5) and (6), we find

$$\dot{V}_e \leqslant -2L\|e\|^2 + \frac{2}{\theta^*}e^{\mathrm{T}}P\Lambda_L^{-1}\Psi(\eta, y) + \frac{2}{\theta^*}e^{\mathrm{T}}Pax_1 + 2e^{\mathrm{T}}P\Lambda_L^{-1}\Phi(y)\Lambda_L e - e^{\mathrm{T}}(\Lambda P + P\Lambda)e\frac{\dot{L}}{L}. \tag{11}$$

We next give an appropriate estimate of (11). For later convenience, the 2nd, 3rd, 4th, and 5th terms on the right-hand side of (11) are denoted simply as ②, ③, ④ and ⑤, respectively.

First, by Assumption 2 and Proposition 2, there holds

$$\|\Psi(\eta,y)\| \leqslant \|\Psi(\eta,y)\|_1 \leqslant \theta_\psi \sum_{i=1}^n \big(\psi_i^\eta(\|\eta\|) + \psi_i^y(y)\big) \leqslant \sqrt{n}\theta_\psi\big(\|\psi^\eta(\|\eta\|)\| + \sqrt{n}\bar{\theta}_\psi + \bar{\theta}_\psi|D_{\frac{\lambda}{2},n}(z_1)|\cdot\|\bar{\psi}^y(z_1)\|\big),$$

where $\bar{\psi}^y = [\bar{\psi}_1^y, \dots, \bar{\psi}_n^y]^{\mathrm{T}}$ with $\bar{\psi}_i^y$'s defined in Proposition 2. Then, noting $L \geqslant 1$ and $\theta^* \triangleq \max\{1, \theta_\psi, \theta_\psi\bar{\theta}_\psi\}$, we obtain (by completing the square)

$$② \leqslant \frac{2}{\theta^*}\|e\|\cdot\|P\|\cdot\|\Psi(\eta,y)\|_1 \leqslant \frac{L}{4}\|e\|^2 + \frac{12n\|P\|^2}{L}\big(n + \|\psi^\eta(\|\eta\|)\|^2 + D^2_{\frac{\lambda}{2},n}(z_1)\|\bar{\psi}^y(z_1)\|^2\big).$$

Keeping in mind the definition of $D_{\frac{\lambda}{2},n}(\cdot)$ and the boundedness of y_r, we can derive

$$③ \leqslant 2\|e\|\cdot\|Pa\|\cdot|x_1| \leqslant \frac{L}{4}\|e\|^2 + \frac{4}{L}\|Pa\|^2\left(|z_1| - \frac{\lambda}{2} + |y_r| + \frac{\lambda}{2}\right)^2 \leqslant \frac{L}{4}\|e\|^2 + \frac{8\|Pa\|^2}{L}\left(D^2_{\frac{\lambda}{2},n} + \left(M_r + \frac{\lambda}{2}\right)^2\right).$$

Note from the definitions of Λ_L and $\Phi(y)$ that $\Lambda_L^{-1}\Phi(y)\Lambda_L = (\frac{1}{L^{i-j}}\phi_{ij}(y))_{n\times n},\ \phi_{ij}(y) = 0,\ j > i$. Then, by $L \geqslant 1$ and the definition and properties of Frobenius norm of matrix, we have $\|\Lambda_L^{-1}\Phi(y)\Lambda_L\| \leqslant \|\Lambda_L^{-1}\Phi(y)\Lambda_L\|_F \leqslant \|\Phi(y)\|_F$. Hence there holds

$$④ = 2e^{\mathrm{T}}P\Lambda_L^{-1}\Phi(y)\Lambda_L e \leqslant 2\|P\| \cdot \|\Phi(y)\|_F\|e\|^2. \tag{12}$$

Substituting (6) and (7) into ⑤ immediately yields

$$⑤ = \left(\frac{L}{2c_2} - \frac{1}{c_1}\sqrt{1+4\|P\|^2\|\Phi(y)\|_F^2}\right) e^{\mathrm{T}}(\Lambda P + P\Lambda)e \leqslant \frac{L}{2}\|e\|^2 - \|e\|^2\sqrt{1+4\|P\|^2\|\Phi(y)\|_F^2}. \tag{13}$$

By plugging the estimates on the four terms ②–⑤ into (11) and noting $L \geqslant 1$, we see

$$\dot{V}_e \leqslant -L\|e\|^2 + 12n\|P\|^2\|\psi^\eta(\|\eta\|)\|^2 + \left(\frac{12n\|P\|^2}{L}\left(n + \|\bar{\psi}^y(z_1)\|^2\right) + \frac{8}{L}\|Pa\|^2\right) D^2_{\frac{\lambda}{2},n} + 8\|Pa\|^2\left(M_r + \frac{\lambda}{2}\right)^2,$$

which leads to (9) with $\tilde{\psi}^y = \frac{12n\|P\|^2}{L}\|\bar{\psi}^y(z_1)\|^2 + \frac{8}{L}\|Pa\|^2$ and $\theta_e = 12n^2\|P\|^2 + 8\|Pa\|^2(M_r + \frac{\lambda}{2})^2$.

Proof of Proposition 4. In virtue of Proposition 1, we see $\dot{V}_\eta$ satisfies

$$\dot{V}_\eta = q(\mathscr{V}(\eta))\dot{\mathscr{V}}(\eta) \leqslant q(\mathscr{V}(\eta))\left(-\frac{\mu}{2}\mathscr{V}(\eta) + \bar{\theta}_\eta\bar{\gamma}(y)\right). \tag{14}$$

Further treatment of (14) proceeds with the two scenarios: (i) $\bar{\theta}_\eta\bar{\gamma}(y) \leqslant \frac{\mu}{4}\mathscr{V}(\eta)$; (ii) $\bar{\theta}_\eta\bar{\gamma}(y) > \frac{\mu}{4}\mathscr{V}(\eta)$.

When scenario (i) holds, it directly follows from (14) that $\dot{V}_\eta \leqslant -\frac{\mu}{4}q(\mathscr{V}(\eta))\mathscr{V}(\eta)$.

When scenario (ii) holds, by means of Proposition 1, we have $\underline{\alpha}(\|\eta\|) \leqslant \mathscr{V}(\eta) \leqslant \frac{4}{\mu}\bar{\theta}_\eta\bar{\gamma}(y)$, and hence $\|\eta\| \leqslant \underline{\alpha}^{-1}(\frac{4}{\mu}\bar{\theta}_\eta\bar{\gamma}(y))$. Then, using the increasing property of $q(\cdot)$ and $\mathscr{V}(\eta) \leqslant \overline{\alpha}(\|\eta\|)$, we have

$$q(\mathscr{V}(\eta)) \leqslant q(\overline{\alpha}(\|\eta\|)) \leqslant q \circ \overline{\alpha} \circ \underline{\alpha}^{-1}\left(\frac{4}{\mu}\bar{\theta}_\eta\bar{\gamma}(y)\right). \tag{15}$$

Recall that, by taking advantage of Lemma 5, the term on the right-hand side of (15) can be bounded by an unknown constant multiplying a known function of y. Then by Proposition 2, the known function of y can be estimated similarly to (8). Thus, there is a known $\mathcal{C}^\infty$ nonnegative function $\tilde{\gamma}(z_1)$ and an unknown positive constant $\tilde{\theta}_\eta$, such that $\dot{V}_\eta$ satisfies (noting (14))

$$\dot{V}_\eta \leqslant -\frac{\mu}{2}q(\mathscr{V})\mathscr{V} + \bar{\theta}_\eta\bar{\gamma}(y)q \circ \overline{\alpha} \circ \underline{\alpha}^{-1}\left(\frac{4}{\mu}\bar{\theta}_\eta\bar{\gamma}(y)\right) \leqslant -\frac{\mu}{2}q(\mathscr{V})\mathscr{V} + \tilde{\theta}_\eta\left(1 + |D_{\frac{\lambda}{2},n}(z_1)|\tilde{\gamma}(z_1)\right). \tag{16}$$

Synthesizing these two scenarios immediately arrives at (10).

We would like to end this subsection by clarifying the philosophy behind dynamics (7) that $L(t)$ should fulfill. ① It is key to possess the high gain property (i.e., there should hold $\dot{L}(t) \geqslant 0$ when $L(t) = 1$). This is ensured by $-\frac{1}{2c_2} + \frac{1}{c_1}\sqrt{1+4\|P\|^2\|\Phi(y)\|_F^2} > 0,\ \forall y \in \mathbb{R}$ since $c_1 < c_2$. Actually, by the inequality, and particularly by (6) and (13) above, we find the parameter "$2c_2$" in term $-\frac{L^2}{2c_2}$ can be further diminished but not less than "$\frac{c_2}{2}$". ② It is key to establish the implications between the boundedness of $L(t)$ and $y(t)$. This is guaranteed by the higher power of $L(t)$ in the negative term $-\frac{L^2}{2c_2}$ of dynamics (7). ③ It is key to compensate for the essential impacts coming from the unmeasured-state dependent nonlinearities. This can be seen from the treatment for the 4th term (of (11)) which satisfies (12) and can be dominated by the 5th term satisfying (13) (adopting "$\sqrt{1+4\|P\|^2\|\Phi(y)\|_F^2}$" not "$2\|P\| \cdot \|\Phi\|_F$" is to ensure the smoothness of the dynamics).

4.2 Controller design

This subsection is concerned with the design of an adaptive controller by output-feedback for system (1) under Assumptions 1–3, to achieve global practical tracking.

For any prescribed tracking accuracy $\lambda > 0$, taking advantage of the propositions above and the refined pseudo-sign and pseudo-dead-zone functions "$\mathrm{sgn}_{\frac{\lambda}{4},n}(\cdot)$" and "$D_{\frac{\lambda}{2},n}(\cdot)$", we devise the following adaptive output-feedback controller, built on filter (3) and dynamic high gain (7):

$$\begin{cases} u = x_n^*(\xi, y, y_r, k, L), \\ \dot{k} = D_{\frac{\lambda}{2},n}(z_1)\zeta(y, y_r, L), \quad k(0) > 0, \\ \dot{L} = \dfrac{-L^2}{2c_2} + \dfrac{L}{c_1}\sqrt{1 + 4\|P\|^2\|\Phi(y)\|_F^2}, \quad L(0) \geqslant 1, \end{cases} \tag{17}$$

where $\zeta(\cdot)$ is a known $\mathcal{C}^n$ function defined as

$$\begin{aligned} \zeta = \mathrm{sgn}_{\frac{\lambda}{4},n}(z_1)\Bigg(&\frac{3}{2} + \tilde{\gamma}(z_1) + \sqrt{1 + \phi_{11}^2(y)y^2} + \psi_1^y(y) + \varpi^{\frac{1}{2}}(z_1, L)L^2 \\ &+ |D_{\frac{\lambda}{2},n}|\left(\frac{5}{4} + L^3 + \tilde{\psi}^y(\cdot) + \varpi(\cdot)L^4\right)\Bigg), \end{aligned} \tag{18}$$

and $x_n^*(\cdot)$ is generated recursively by

$$\begin{cases} \begin{cases} x_1^* = N(k)\cdot\zeta(y, y_r, L), \quad x_2^* = \beta_2(\xi_{[2]}, y, y_r, k, L) - z_2, \\ x_i^* = \beta_i(\xi_{[i]}, y, y_r, k, L) - \dfrac{1}{2}z_i - z_{i-1}, \quad i = 3, \ldots, n, \end{cases} \\ z_1 = y - y_r, \quad z_i = \xi_i - x_{i-1}^*(\xi_{[i-1]}, y, y_r, k, L), \quad i = 2, \ldots, n. \end{cases} \tag{19}$$

In (18) and (19), design functions $\varpi(\cdot)$ and $\beta_i(\cdot)$'s ($\mathcal{C}^\infty$ and $\mathcal{C}^{n-i+1}$, respectively) are defined as

$$\begin{cases} \varpi(\cdot) = 1 + \tilde{\gamma}^2(z_1) + \tilde{\psi}^y(z_1, L), \\ \beta_i(\cdot) = L^i a_i\xi_1 - \phi_i^{\mathrm{T}}(y)\xi_{[i]} + x_1\phi_{11}(y)\dfrac{\partial x_{i-1}^*}{\partial y} + \dfrac{\partial x_{i-1}^*}{\partial k}D_{\frac{\lambda}{2},n}(z_1)\zeta(y, y_r, L) \\ \qquad + \dfrac{\partial x_{i-1}^*}{\partial L}\left(\dfrac{-L^2}{2c_2} + \dfrac{L}{c_1}\sqrt{1 + 4\|P\|^2\|\Phi(y)\|_F^2}\right) + \displaystyle\sum_{j=1}^{i-1}\dfrac{\partial x_{i-1}^*}{\partial \xi_j}\left(\xi_{j+1} + \phi_j^{\mathrm{T}}(y)\xi_{[j]} - L^j a_j\xi_1\right) \\ \qquad - \dfrac{z_i^3}{4}\left(1 + (n+1)L^6\right)\left(\dfrac{\partial x_{i-1}^*}{\partial y}\right)^4 - \dfrac{z_i}{2}\left(\dfrac{\partial x_{i-1}^*}{\partial y_r}\right)^2 - \dfrac{z_i}{2}\left(\xi_2^2 + (\psi_1^y(y))^2\right)\left(\dfrac{\partial x_{i-1}^*}{\partial y}\right)^2, \quad i = 2, \ldots, n. \end{cases} \tag{20}$$

Remark 4. The $\mathcal{C}^n$ property of pseudo-sign function $\mathrm{sgn}_{\mu,n}(\cdot)$ in (18) makes $x_1^*(\cdot)$ in (19) belong to $\mathcal{C}^n$ and, in turn, so do the rest virtual controllers x_i^*'s. With $\mathrm{sgn}_{\mu,n}(\cdot)$ in hand, certain smooth treatment, for instance, using completing squares or finding larger smooth functions to get rid of discontinuous functions (as done in [9]), in obtaining a sufficiently smooth controller is no longer needed. Owing to the same reason, a pseudo-dead-zone function $D_{\mu,n}(\cdot)$ is pursued which is smoothed at two breakpoints of the ideal dead zone in small vicinities.

Remark 5. The design function $\zeta(\cdot)$ defined in (18) plays a significant role in handling system nonlinearities such as "$\phi_{11}(\cdot)x_1$" and "$\psi_i(\cdot)$" in system (1) (see Proposition 5 for details). It is also noteworthy that attributed to the two especially designed terms consisting $\varpi(\cdot)$ or $\varpi^{\frac{1}{2}}(\cdot)$, the boundedness of the related states $k(t)$ and $L(t)$ (see Proposition 6) and system performance analysis can be successfully achieved.

To see the above controller indeed lends itself well to global practical tracking, Proposition 5 is provided first, whose proof also constitutes the derivation procedure for (17) via employing backstepping method for the "entire" system (consisting of (1), (3) and (5)) :

$$\begin{cases} \dot{e} = LAe + \dfrac{ax_1}{\theta^*} + \Lambda_L^{-1}\Phi(\cdot)\Lambda_L e + \dfrac{1}{\theta^*}\Lambda_L^{-1}\Psi(\cdot) - \Lambda e\dfrac{\dot{L}}{L}, \\ \dot{y} = x_2 + \phi_1(y)x_1 + \psi_1(\eta, y), \\ \dot{\xi}_i = \xi_{i+1} + \phi_i^{\mathrm{T}}(y)\xi_{[i]} - L^i a_i\xi_1, \ i = 2, \ldots, n-1, \\ \dot{\xi}_n = u + \phi_n^{\mathrm{T}}(y)\xi_{[n]} - L^n a_n\xi_1. \end{cases} \tag{21}$$

Proposition 5. Under Assumptions 1–3, adaptive output-feedback controller (17) with (18)–(20) guarantees the composite Lyapunov function candidate $V_n = V_e + V_\eta + V_{z_1} + \frac{1}{2}\sum_{i=2}^n z_i^2$ to satisfy

$$\dot{V}_n \leqslant -\frac{1}{n+1}L\|e\|^2 - \frac{\mu}{8}q(\mathscr{V}(\eta))\mathscr{V}(\eta) + (gN(k)+\vartheta)\dot{k} - \frac{1}{2}D^2_{\frac{\lambda}{2},n}(z_1) - \frac{1}{2}\sum_{j=2}^n z_j^2 + \bar{\vartheta}_n \tag{22}$$

with unknown nonnegative constants ϑ and $\bar{\vartheta}_n$ and the increasing function $q(\cdot)$ satisfying

$$q(s) \geqslant \delta\|\psi^\eta(\underline{\alpha}^{-1}(s))\|^2 + \delta\big(\psi_1^\eta \circ \tilde{\alpha}^{-1}(s)\big)^2, \tag{23}$$

where $\tilde{\alpha}(s)$ is a class $\mathcal{K}_\infty$ function satisfying $\tilde{\alpha}(s) \leqslant \underline{\alpha}(s) - \frac{1}{\delta}$ when $\underline{\alpha}(s) \geqslant \frac{2}{\delta}$ for constant $\delta = \frac{8n(1+12\|P\|^2)}{\mu\underline{\alpha}(1)}$.

Proof. It suffices to verify step by step that under controller (17) with (18)–(20), the Lyapunov function candidate V_n indeed meets (22).

Step 1. Consider Lyapunov function candidate $V_1(e,\eta,z_1) = V_e(e) + V_\eta(\eta) + V_{z_1}(z_1)$, with V_e and V_η defined in Propositions 3 and 4 respectively, and V_{z_1} is defined as follows:

$$V_{z_1} = \int_0^{z_1} D_{\frac{\lambda}{2},n}(\tau)\mathrm{d}\tau = \begin{cases} \frac{1}{2}\left(|z_1| - \frac{\lambda}{2}\right)^2 + c_\lambda, & |z_1| \geqslant \frac{3\lambda}{4}, \\ \int_{\frac{\lambda}{4}}^{|z_1|} Q_{\frac{\lambda}{4},n}\left(\tau - \frac{\lambda}{2}\right)\mathrm{d}\tau, & \frac{\lambda}{4} < |z_1| < \frac{3\lambda}{4}, \\ 0, & |z_1| \leqslant \frac{\lambda}{4} \end{cases} \tag{24}$$

with $c_\lambda = \int_{\frac{\lambda}{4}}^{\frac{3\lambda}{4}} Q_{\frac{\lambda}{4},n}(\tau - \frac{\lambda}{2})\mathrm{d}\tau - \frac{\lambda^2}{32}$.

Note that by (4) and (19), there holds $x_2 = \theta^* L^2 e_2 + g(z_2 + x_1^*(\cdot))$. Then by z_1 and $\dot{y}$ (see (19) and (21)), as well as by Assumption 2, $\dot{V}_{z_1}$ satisfies

$$\begin{aligned} \dot{V}_{z_1} &= D_{\frac{\lambda}{2},n}(z_1)\big(x_2 + \phi_{11}(y)x_1 + \psi_1(\eta,y) - \dot{y}_r\big) \\ &\leqslant D_{\frac{\lambda}{2},n}(z_1)\big(\theta^* L^2 e_2 + g z_2 + \phi_{11}(y)x_1 - \dot{y}_r\big) + |D_{\frac{\lambda}{2},n}|\theta_\psi\big(\psi_1^\eta(\|\eta\|) + \psi_1^y(y)\big) + gD_{\frac{\lambda}{2},n}x_1^*. \end{aligned} \tag{25}$$

Estimating the first two terms on the right-hand side of (25) yields

$$\begin{cases} \theta^* L^2 D_{\frac{\lambda}{2},n} e_2 \leqslant \dfrac{L}{n+1}\|e\|^2 + (n+1)\theta^{*2}L^3 D^2_{\frac{\lambda}{2},n}, \\ gD_{\frac{\lambda}{2},n}z_2 + |D_{\frac{\lambda}{2},n}|\theta_\psi\psi_1^\eta(\|\eta\|) \leqslant \dfrac{z_2^2}{2} + (\psi_1^\eta(\|\eta\|))^2 + \dfrac{2g^2+\theta_\psi^2}{4}D^2_{\frac{\lambda}{2},n}, \\ D_{\frac{\lambda}{2},n}\phi_{11}(y)x_1 + |D_{\frac{\lambda}{2},n}|\theta_\psi\psi_1^y(y) - D_{\frac{\lambda}{2},n}\dot{y}_r \leqslant \max\{1,\theta_\psi,M_r\}|D_{\frac{\lambda}{2},n}|\left(1 + \psi_1^y(y) + \sqrt{1+\phi_{11}^2(y)y^2}\right). \end{cases} \tag{26}$$

Substituting this into (25), adding and subtracting "$\frac{1}{2}D^2_{\frac{\lambda}{2},n}(z_1)$", and collecting some terms, we arrive at

$$\dot{V}_{z_1} \leqslant \frac{L}{n+1}\|e\|^2 + \frac{z_2^2}{2} + \bar{\vartheta}D_{\frac{\lambda}{2},n}\bar{\zeta}(y,y_r,L) - \frac{1}{2}D^2_{\frac{\lambda}{2},n} + (\psi_1^\eta(\|\eta\|))^2 + gD_{\frac{\lambda}{2},n}x_1^*, \tag{27}$$

where and whereafter

$$\begin{cases} \bar{\vartheta} = \max\left\{1, M_r, \theta_\psi, \theta_\psi^2, g^2, (n+1)\theta^{*2}\right\}, \\ \bar{\zeta} = \mathrm{sgn}_{\frac{\lambda}{4},n}(\cdot)\left(1 + \psi_1^y(\cdot) + \sqrt{1+\phi_{11}^2 y^2} + |D_{\frac{\lambda}{2},n}|\left(\frac{5}{4} + L^3\right)\right), \\ \tilde{\zeta}(\cdot) = \bar{\zeta} + \mathrm{sgn}_{\frac{\lambda}{4},n}(z_1)\big(\tilde{\gamma}(z_1) + |D_{\frac{\lambda}{2},n}|\tilde{\psi}^y(z_1,L)\big). \end{cases}$$

Remark that when we synthesize (27) and Propositions 3 and 4 to obtain the estimate of $\dot{V}_1$, the term "$\bar{\vartheta}D_{\frac{\lambda}{2},n}(z_1)\bar{\zeta}(\cdot)$" in (27) would increase to "$\vartheta D_{\frac{\lambda}{2},n}(z_1)\tilde{\zeta}(\cdot)$" with $\vartheta = \max\{\bar{\vartheta}, \tilde{\theta}_\eta\} > 1$. From this, by the expressions of $\dot{k}$, $\zeta(\cdot)$ and $x_1^*(\cdot)$ in (17)–(19), respectively, it is clear that $\vartheta D_{\frac{\Delta}{2},n}(z_1)\tilde{\zeta}(\cdot) \leqslant \vartheta D_{\frac{\Delta}{2},n}(z_1)\zeta(y, y_r, L) = \vartheta \cdot \dot{k}$ and $gD_{\frac{\Delta}{2},n}(z_1)x_1^* = gN(k)\dot{k}$. Then, from (27) and Propositions 3 and 4, we find

$$\dot{V}_1 \leqslant -\left(1 - \frac{1}{n+1}\right) L\|e\|^2 + 12n\|P\|^2\|\psi^\eta(\|\eta\|)\|^2 - \frac{\mu}{4}q(\mathscr{V}(\eta))\mathscr{V}(\eta) + \frac{z_2^2}{2} + (\psi_1^\eta(\|\eta\|))^2$$
$$+\big(gN(k) + \vartheta\big)\dot{k} - \frac{1}{2}D^2_{\frac{\Delta}{2},n} + \vartheta_1. \quad (\vartheta = \max\{\bar{\vartheta}, \tilde{\theta}_\eta\},\ \vartheta_1 = \theta_e + \tilde{\theta}_\eta)$$

Step 2. Let $V_2 = V_1 + V_{z_2} := V_1 + \frac{1}{2}z_2^2$. Then, by (19), (21) and Assumption 2, we learn

$$\dot{V}_{z_2} \leqslant z_2\left(z_3 + x_2^* + \phi_2^{\mathrm{T}}(y)\xi_{[2]} - L^2a_2\xi_1 - \frac{\partial x_1^*}{\partial k}\dot{k} - \frac{\partial x_1^*}{\partial L}\dot{L}\right) + \left|z_2\frac{\partial x_1^*}{\partial y}\right|\big(\theta_\psi\psi_1^\eta(\|\eta\|) + \theta_\psi\psi_1^y(y)\big)$$
$$- z_2\left(\frac{\partial x_1^*}{\partial y}\big(\theta^*e_2L^2 + g\xi_2 + \phi_{11}(y)x_1\big) + \frac{\partial x_1^*}{\partial y_r}\dot{y}_r\right). \tag{28}$$

By evaluating the last two terms of (28), it directly follows that (utilizing Assumption 3)

$$\begin{cases} -z_2e_2\theta^*L^2\dfrac{\partial x_1^*}{\partial y} \leqslant \dfrac{L}{n+1}\|e\|^2 + \dfrac{(n+1)}{4}z_2^4L^6\left(\dfrac{\partial x_1^*}{\partial y}\right)^4 + \dfrac{(n+1)\theta^{*4}}{16}, \\ -gz_2\xi_2\dfrac{\partial x_1^*}{\partial y} - z_2\dot{y}_r\dfrac{\partial x_1^*}{\partial y_r} \leqslant \dfrac{1}{2}z_2^2\xi_2^2\left(\dfrac{\partial x_1^*}{\partial y}\right)^2 + \dfrac{1}{2}z_2^2\left(\dfrac{\partial x_1^*}{\partial y_r}\right)^2 + \dfrac{M_r^2 + g^2}{2}, \\ \theta_\psi\big(\psi_1^\eta(\cdot) + \psi_1^y(\cdot)\big)\left|z_2\dfrac{\partial x_1^*}{\partial y}\right| \leqslant \dfrac{1}{4}z_2^4\left(\dfrac{\partial x_1^*}{\partial y}\right)^4 + (\psi_1^\eta(\|\eta\|))^2 + \dfrac{1}{2}z_2^2(\psi_1^y(y))^2\left(\dfrac{\partial x_1^*}{\partial y}\right)^2 + \dfrac{\theta_\psi^4 + 8\theta_\psi^2}{16}. \end{cases}$$

Substituting this into (28) and invoking $x_2^*(\cdot)$ and $\beta_2(\cdot)$ defined in (19) and (20) respectively, yield

$$\dot{V}_2 \leqslant -\left(1 - \frac{2}{n+1}\right) L\|e\|^2 + 12n\|P\|^2\|\psi^\eta(\|\eta\|)\|^2 - \frac{\mu}{4}q(\mathscr{V}(\eta))\mathscr{V}(\eta) + 2(\psi_1^\eta(\|\eta\|))^2 + z_2z_3$$
$$+(gN(k) + \vartheta)\dot{k} - \frac{1}{2}D^2_{\frac{\Delta}{2},n} - \frac{z_2^2}{2} + \vartheta_2. \quad \left(\vartheta_2 = \vartheta_1 + \Theta,\ \Theta = \frac{\theta_\psi^4 + (n+1)\theta^{*4}}{16} + \frac{g^2 + \theta_\psi^2 + M_r^2}{2}\right)$$

Recursive design step $i\,(3 \leqslant i \leqslant n)$**.** Assume that the first $i-1$ steps have been checked and $V_{i-1} = V_1 + \frac{1}{2}\sum_{j=2}^{i-1} z_j^2$ fulfills (for some constant $\vartheta_{i-1} > 0$)

$$\dot{V}_{i-1} \leqslant -\left(1 - \frac{i-1}{n+1}\right) L\|e\|^2 - \frac{\mu}{4}q(\mathscr{V}(\eta))\mathscr{V}(\eta) + (i-1)(\psi_1^\eta(\|\eta\|))^2 + 12n\|P\|^2\|\psi^\eta(\|\eta\|)\|^2 + z_{i-1}z_i$$
$$+\big(gN(k) + \vartheta\big)\dot{k} - \frac{1}{2}D^2_{\frac{\Delta}{2},n} - \frac{1}{2}\sum_{j=2}^{i-1} z_j^2 + \vartheta_{i-1}. \tag{29}$$

Choose the Lyapunov function candidate $V_i = V_{i-1} + V_{z_i} := V_{i-1} + \frac{1}{2}z_i^2$ for step i. Particularly, when $i = n$, we let $z_{n+1} = 0$ and $\xi_{n+1} = u$.

Realizing that the way to estimate $\dot{V}_{z_i}$, $i \geqslant 3$ is similar to that of $\dot{V}_{z_2}$ in (28), we can then check that virtual control law x_i^* and the designed function ρ_i make V_i satisfy the inequality similar to (29) (just by replacing $i-1$ in (29) with i and taking $\vartheta_i = \vartheta_{i-1} + \Theta$).

Now, to obtain (22), we need to tackle the three terms containing argument η on the right-hand side of (29). Keep in mind that the two positive terms satisfy, for $i - 1 = n$,

$$n(\psi_1^\eta(\|\eta\|))^2 + 12n\|P\|^2\|\psi^\eta(\|\eta\|)\|^2 \leqslant n\big(1 + 12\|P\|^2\big)\left(\|\psi^\eta(\|\eta\|)\|^2\frac{\underline{\alpha}(\|\eta\|)}{\underline{\alpha}(1)} + \max_{\|\eta\|\leqslant 1}\{\|\psi^\eta(\|\eta\|)\|^2\}\right).$$

Observing from $\mathscr{V}(\eta) \geqslant \underline{\alpha}(\|\eta\|)$ (see Proposition 1) yields $-\frac{\mu}{4}q(\mathscr{V})\mathscr{V} \leqslant -\frac{\mu}{8}q(\mathscr{V})\mathscr{V} - \frac{\mu}{8}q(\underline{\alpha}(\|\eta\|))\underline{\alpha}(\|\eta\|)$. Since $q(s) \geqslant \delta\|\psi^\eta(\underline{\alpha}^{-1}(s))\|^2$ (see (23)) with $\delta = \frac{8n(1+12\|P\|^2)}{\mu\underline{\alpha}(1)}$, there holds $-\frac{\mu}{8}q(\underline{\alpha}(\|\eta\|))\underline{\alpha}(\|\eta\|) + n(1 + 12\|P\|^2)\|\psi^\eta(\|\eta\|)\|^2\frac{\underline{\alpha}(\|\eta\|)}{\underline{\alpha}(1)} \leqslant 0$. Thus, the three terms satisfy

$$-\frac{\mu}{4}q(\mathscr{V})\mathscr{V} + n(\psi_1^\eta)^2 + 12n\|P\|^2\|\psi^\eta(\|\eta\|)\|^2 \leqslant -\frac{\mu}{8}q(\mathscr{V})\mathscr{V} + n(1 + 12\|P\|^2)\max_{\|\eta\|\leqslant 1}\{\|\psi^\eta(\|\eta\|)\|^2\}. \tag{30}$$

Substituting this into (29) for $i - 1 = n$, we can directly obtain (22) holds with $\bar{\vartheta}_n = \vartheta_n + n(1 + 12\|P\|^2)\max_{\|\eta\|\leqslant 1}\{\|\psi^\eta(\|\eta\|)\|^2\}$.

5 Main results

This section aims to present the main results of this paper. In detail, the boundedness of the closed-loop system is shown first and then the performance of global practical tracking is achieved successfully.

With the above adaptive controller in mind, from system (1) and filter (3), it can be readily seen that the vector field of the resulting closed-loop system is locally Lipschitz in (x, ξ, k, L) in an open neighborhood of the initial values; hence the closed-loop system has a unique solution on a small interval $[0, t_s)$ (see Theorem 3.1 of [29]). Maximizing the interval via the continuation of solutions obtains the maximal existence interval $[0, t_f)$ with $0 < t_f \leqslant +\infty$ on which the unique solution exists (see Theorem 2.1 of [29]). It is noted that when $0 < t_f < +\infty$, at least one state escapes to infinity in finite time or equivalently, $\lim_{t\to t_f}(\|x(t)\| + \|\xi(t)\| + |k(t)| + |L(t)|) = +\infty$, and when $t_f = +\infty$, all closed-loop system states are well-defined on $[0, +\infty)$.

Before embarking on the main result, we would like to show the boundedness of $z_1(t)$, $k(t)$, and $L(t)$ first, with whose help the obstruction deriving from the various non-vanishing uncertainties is circumvented (see Theorem 1) and furthermore system performance is obtained successfully.

Proposition 6. Tracking error $z_1(t)$, variable $k(t)$, and high gain $L(t)$ are all bounded on $[0, t_f)$.

Proof. We proceed with two reconstructed Lyapunov function candidates $W_1 = V_e + V_\eta + \frac{1}{2}\sum_{j=2}^{n} z_j^2$ and $W_2 = V_{z_1}$, for which the verification later on shows

$$\begin{aligned} \dot{W}_1 &\leqslant -\theta_{w1}W_1 + \theta_{w2}\big(1 + L^4 D^2_{\frac{\lambda}{2},n}(z_1)\varpi(z_1, y)\big), \\ \dot{W}_2 &\leqslant (\theta^* + |g| + \theta_\psi)L^2|D_{\frac{\lambda}{2},n}|W_1^{\frac{1}{2}} + \big(gN(k) + \tilde{\vartheta}\big)\dot{k}, \end{aligned} \tag{31}$$

where θ_{wi}'s, θ^*, θ_ψ, and $\tilde{\vartheta}$ are unknown positive constants and $\varpi(\cdot)$ is the positive design function in (20).

Applying Lemma 4 to the first inequality in (31) directly arrives at, for some positive constant θ_{w3}

$$\int_0^t L^2|D_{\frac{\lambda}{2},n}|\varpi^{\frac{1}{2}}W_1^{\frac{1}{2}}\mathrm{d}\tau \leqslant \theta_{w3}\int_0^t L^2|D_{\frac{\lambda}{2},n}|\varpi^{\frac{1}{2}}\big(1 + L^2|D_{\frac{\lambda}{2},n}|\varpi^{\frac{1}{2}}\big)\,\mathrm{d}\tau. \tag{32}$$

Integrating both sides of the second differential inequality in (31) and noting $L^2|D_{\frac{\lambda}{2},n}|W_1^{\frac{1}{2}} \leqslant L^2|D_{\frac{\lambda}{2},n}| \cdot\varpi^{\frac{1}{2}}(\cdot)W_1^{\frac{1}{2}}$ since $\varpi(\cdot) > 1$, we have

$$W_2(t) \leqslant W_2(0) + (\theta^* + |g| + \theta_\psi)\int_0^t L^2|D_{\frac{\lambda}{2},n}|\cdot\varpi^{\frac{1}{2}}W_1^{\frac{1}{2}}\mathrm{d}\tau + \int_0^t \big(gN(k) + \tilde{\vartheta}\big)\dot{k}\mathrm{d}\tau. \tag{33}$$

Substituting (32) into (33) and noting $L^2|D_{\frac{\lambda}{2},n}|\varpi^{\frac{1}{2}}\cdot(1 + L^2|D_{\frac{\lambda}{2},n}|\varpi^{\frac{1}{2}}) \leqslant D_{\frac{\lambda}{2},n}\zeta = \dot{k}$ directly yield

$$W_2(t) \leqslant W_2(0) + \int_0^t \big(gN(k) + \tilde{\vartheta}'\big)\dot{k}\mathrm{d}\tau. \quad \big(\tilde{\vartheta}' = \tilde{\vartheta} + (\theta^* + |g| + \theta_\psi)\theta_{w3}\big) \tag{34}$$

Employing Lemma 3 to (34) straightforwardly shows the boundedness of $k(t)$ and $W_2(t)$ on $[0, t_f)$.

Noting $W_2 = V_{z_1}$, the definition of V_{z_1} in (24) and the boundedness of $y_r(t)$, one can readily obtain the boundedness of $z_1(t)$ and hence that of $y(t) = z_1(t) + y_r(t)$, both on $[0, t_f)$. Since the boundedness of $L(t)$ is implied by that of $y(t)$ as discussed below (16), we ultimately conclude $L(t)$ is bounded on $[0, t_f)$.

Verification of (31). The rest of the proof is restricted to verify that the time derivatives of the two reconstructed Lyapunov function candidates indeed meet (31).

Noting the similarity between $\dot{V}_{z_i}$, $i = 3, \ldots, n$ and $\dot{V}_{z_2}$ in (28) with its estimates, and invoking $x_i^*(\cdot)$'s defined in (19) directly yield

$$\sum_{i=2}^{n} z_i\dot{z}_i \leqslant \frac{n-1}{n+1}L\|e\|^2 - \frac{1}{2}\sum_{i=2}^{n} z_i^2 + (n-1)\big(\psi_1^\eta(\|\eta\|)\big)^2 + (n-1)\Theta.$$

By this and the estimates of the time derivatives $\dot{V}_e$ and $\dot{V}_\eta$ in Propositions 3 and 4 respectively, we can obtain

$$\dot{W}_1 \leqslant -\frac{2}{n+1}L\|e\|^2 - \frac{1}{2}\sum_{i=2}^{n} z_i^2 - \frac{\mu}{4}q(\mathscr{V}(\eta))\mathscr{V}(\eta) + 12n\|P\|^2\|\psi^\eta(\|\eta\|)\|^2 + (n-1)\big(\psi_1^\eta(\|\eta\|)\big)^2$$

$$+\max\{1,\tilde{\theta}_\eta\}D^2_{\frac{\lambda}{2},n}(z_1)\big(\tilde{\psi}^y(z_1,L)+\tilde{\gamma}^2(z_1)\big)+(n-1)\Theta+\theta_e+2\tilde{\theta}_\eta. \tag{35}$$

Actually, the term "$\tilde{\theta}_\eta|D_{\frac{\lambda}{2},n}(\cdot)|\tilde{\gamma}(\cdot)$" in Proposition 4 is further amplified by completing squares for power matching to that of the term "$D^2_{\frac{\lambda}{2},n}(\cdot)\tilde{\psi}^y(\cdot)$" in Proposition 3.

Now, similar to the treatment in Proposition 5, we would like to tackle the three terms having argument η in (35). Noting the increasing property of $q(\cdot)$, we have $V_\eta=\int_0^{\mathscr{V}(\eta)}q(\tau)\mathrm{d}\tau\leqslant q(\mathscr{V}(\eta))\mathscr{V}(\eta)$. With this and (30) in hand, we see these three terms satisfy

$$-\frac{\mu}{4}q(\mathscr{V})\mathscr{V}+12n\|P\|^2\|\psi^\eta(\|\eta\|)\|^2+(n-1)(\psi_1^\eta)^2\leqslant-\frac{\mu}{8}\int_0^{\mathscr{V}(\eta)}q(\tau)\mathrm{d}\tau+n\big(1+12\|P\|^2\big)\max_{\|\eta\|\leqslant1}\{\|\psi^\eta(\|\eta\|)\|^2\}.$$

Substituting this into (35) and noting $L\geqslant1$ and the definition of W_1, we can obtain the first inequality in (31) after collecting terms. In fact, L^4 does not show up in (35) and we extra multiply $D^2_{\frac{\lambda}{2},n}(z_1)\varpi(z_1,y)$ in (31) by L^4 simply to dominate some undesirable terms stemming from (33).

Although the time derivative of $W_2=V_{z_1}$ has already been assessed in (26), a different estimate (i.e., the second differential inequality in (31)) is needed in this proposition to derive the boundedness of $k(t)$.

First, keep the last two inequalities in (26) unchanged and replace the first two with

$$\theta^*D_{\frac{\lambda}{2},n}e_2L^2\leqslant\theta^*|D_{\frac{\lambda}{2},n}|L^2\cdot W_1^{\frac{1}{2}},\quad D_{\frac{\lambda}{2},n}gz_2\leqslant|D_{\frac{\lambda}{2},n}|\cdot|g|\cdot|z_2|\leqslant|D_{\frac{\lambda}{2},n}|\cdot|g|\cdot W_1^{\frac{1}{2}}. \tag{36}$$

Unlike these two terms simply estimated in (36), "$|D_{\frac{\lambda}{2},n}|\theta_\psi\psi_1^\eta(\|\eta\|)$" (see (26)) requires more delicate assessment to obtain its relationship with "$W^{\frac{1}{2}}$". Recall that increasing function $q(\cdot)$ defined in (23) satisfies $q(s)\geqslant\delta(\psi_1^\eta\circ\tilde{\alpha}^{-1}(s))^2$ and that the class $\mathcal{K}_\infty$ function $\tilde{\alpha}(\cdot)$ satisfies $\tilde{\alpha}(\|\eta\|)\leqslant\underline{\alpha}(\|\eta\|)-\frac{1}{\delta}$ when $\underline{\alpha}(\|\eta\|)\geqslant\frac{2}{\delta}$. Thus, we proceed further analysis with two scenarios: (i) $\underline{\alpha}(\|\eta\|)\geqslant\frac{2}{\delta}$; (ii) $\underline{\alpha}(\|\eta\|)<\frac{2}{\delta}$.

For scenario (i), there holds

$$\psi_1^\eta(\|\eta\|)\leqslant\left(q\big(\tilde{\alpha}(\|\eta\|)\big)\cdot\frac{1}{\delta}\right)^{\frac{1}{2}}\leqslant\left(\int_{\tilde{\alpha}(\|\eta\|)}^{\underline{\alpha}(\|\eta\|)}q(\tau)\mathrm{d}\tau\right)^{\frac{1}{2}}\leqslant\left(\int_0^{\mathscr{V}(\eta)}q(\tau)\mathrm{d}\tau\right)^{\frac{1}{2}}\leqslant W_1^{\frac{1}{2}}.$$

For scenario (ii), we have $\|\eta\|\leqslant\underline{\alpha}^{-1}(\frac{2}{\delta})$. Then, using the continuity of $\psi_1^\eta(\cdot)$ directly yields $\psi_1^\eta(\|\eta\|)\leqslant\max_{\|\eta\|\leqslant\underline{\alpha}^{-1}(\frac{2}{\delta})}\{\psi_1^\eta(\|\eta\|)\}$. Therefore, from the arguments on the two scenarios, it follows

$$|D_{\frac{\lambda}{2},n}|\theta_\psi\psi_1^\eta(\|\eta\|)\leqslant\theta_\psi|D_{\frac{\lambda}{2},n}|\left(W_1^{\frac{1}{2}}+\max_{\|\eta\|\leqslant\underline{\alpha}^{-1}(\frac{2}{\delta})}\{\psi_1^\eta(\|\eta\|)\}\right).$$

By this, (36) and the last two unchanged estimates in (26), we have

$$\begin{aligned}\dot{W}_2\leqslant\ &\theta^*|D_{\frac{\lambda}{2},n}|L^2\cdot W_1^{\frac{1}{2}}+|D_{\frac{\lambda}{2},n}|\cdot|g|\cdot W_1^{\frac{1}{2}}+\theta_\psi|D_{\frac{\lambda}{2},n}|\left(W_1^{\frac{1}{2}}+\max_{\|\eta\|\leqslant\underline{\alpha}^{-1}(\frac{2}{\delta})}\{\psi_1^\eta(\|\eta\|)\}\right)\\&+|D_{\frac{\lambda}{2},n}|\sqrt{1+\phi_{11}^2(y)y^2}+\max\{\theta_\psi,M_r\}|D_{\frac{\lambda}{2},n}|\big(1+\psi_1^y(y)\big)+D_{\frac{\lambda}{2},n}gx_1^*.\end{aligned} \tag{37}$$

Note that

$$|D_{\frac{\lambda}{2},n}|\left(\sqrt{1+\phi_{11}^2y^2}+\big(1+\psi_1^y(y)\big)\right)\leqslant D_{\frac{\lambda}{2},n}\zeta=\dot{k},\quad gD_{\frac{\lambda}{2},n}(z_1)x_1^*=gN(k)\dot{k}.$$

Hence, after some simple arrangement, we can directly see $\dot{W}_2$ indeed meets the second inequality in (31) with $\tilde{\vartheta}=\max\{1,\theta_\psi,M_r\}+\theta_\psi\max_{\|\eta\|\leqslant\underline{\alpha}^{-1}(\frac{2}{\delta})}\{\psi_1^\eta(\|\eta\|)\}$.

Theorem 1. Consider system (1) with Assumptions 1–3 and the devised adaptive controller (17) with (18)–(20). Then the resulting closed-loop system is well-defined and globally bounded on $[0,+\infty)$. Furthermore, the global practical tracking can be achieved for prescribed tracking accuracy λ, namely, for any initial values, there holds $\sup_{t\geqslant T_\lambda}|y(t)-y_r(t)|\leqslant\lambda$ for a finite time T_λ.

Proof. As the well-defined property has been addressed above for the closed-loop system, we only need to verify the boundedness of $V_n(t)$ on $[0, t_f)$ to prove the first claim. With the proved boundedness of $k(t)$ and $L(t)$ (see Proposition 6) in mind, we learn the boundedness of the rest closed-loop states on $[0, t_f)$ can be indicated by that of $V_n(t)$. In detail, once V_n is bounded, from its definition we can see the boundedness of η, e_i's and z_i's for $i = 1, \ldots, n$. Then, recalling the boundedness of $k(t)$ and $L(t)$ immediately leads to that of x_1^*. By (19) and (20), we see that ξ_i, x_i, and x_i^* are all bounded on $[0, t_f)$.

Now we would like to show that V_n is indeed bounded. In light of the definitions of $D_{\frac{\lambda}{2},n}(z_1)$ and V_{z_1}, there holds $V_{z_1} \leqslant \frac{1}{2}D^2_{\frac{\lambda}{2},n}(z_1) + \vartheta_\lambda$ with unknown constant $\vartheta_\lambda \geqslant 0$. Thus, we learn from (22) that for $c > 0$,

$$\begin{aligned}\dot{V}_n &\leqslant -\frac{1}{n+1}L\|e\|^2 - \frac{\mu}{8}q(\mathscr{V}(\eta))\mathscr{V}(\eta) + (gN(k) + \vartheta)\dot{k} - \frac{1}{2}D^2_{\frac{\lambda}{2},n}(z_1) - \frac{1}{2}\sum_{j=2}^{n} z_j^2 + \bar{\vartheta}_n \\ &\leqslant -cV_n + \big(gN(k) + \vartheta\big)\dot{k} + \bar{\vartheta}'. \quad \big(\bar{\vartheta}' = \bar{\vartheta}_n + \vartheta_\lambda\big)\end{aligned} \tag{38}$$

Thanks to the boundedness of (y, y_r, k, L) on $[0, t_f)$ (see Proposition 6 and Assumption 3), we directly know that $\dot{k}(t)$ (by its definition in (17)) and $N(k(t))$ (by its continuity) are both bounded, and hence so is the term "$(gN(k) + \vartheta)\dot{k}$" in (38). Thus, there holds $\dot{V}_n \leqslant -cV_n + \tilde{\vartheta}$, for some nonnegative constant $\tilde{\vartheta}$. From this, the boundedness of V_n is immediately obtained, and therefore all states of the closed-loop system are bounded on $[0, +\infty)$.

Next we turn to show the achievement of global practical tracking for the closed-loop system. To this end, we first prove the uniform continuity of $\dot{k}(t)$. From the boundedness of all the closed-loop states, we can readily learn the boundedness of $D_{\frac{\lambda}{2},n}$, $\dot{D}_{\frac{\lambda}{2},n}$, ζ and $\dot{\zeta}$ on $[0, +\infty)$. Thus, it follows from (17) the boundedness of $\ddot{k} = \dot{D}_{\frac{\lambda}{2},n}\zeta + D_{\frac{\lambda}{2},n}\dot{\zeta}$ on $[0, +\infty)$, which immediately concludes the uniform continuity of $\dot{k}(t)$ on $[0, +\infty)$. Then, employing Barbălat lemma, we can assert $\lim_{t\to+\infty} \dot{k} = 0$. This demonstrates that for any $\eta_0 \in \mathbb{R}^m$ and $x_0 \in \mathbb{R}^n$, a finite time $T_\lambda > 0$ exists such that $(|z_1(t)| - \frac{\lambda}{2}) \leqslant D_{\frac{\lambda}{2},n}\zeta = \dot{k}(t) \leqslant \frac{\lambda}{2}$, $\forall t > T_\lambda > 0$, which implies $|z_1(t)| = |y - y_r| \leqslant \lambda$, $\forall t > T_\lambda > 0$.

Remark 6. There seemingly holds an intuitive tendency: a smaller λ results in a larger T_λ. Such a changing tendency is true since we indeed have $T_\lambda = +\infty$ as λ converges to 0 if neglecting some special cases (for example, when $y_r = \eta_0 = x_0 = 0$, we know $T_\lambda = 0$ no matter which λ we pick), from the performance analysis above. But for two specific λ_1 and λ_2, they may not follow the tendency. This is because, in light of the definitions of $D_{\frac{\lambda}{2},n}(\cdot)$ and $\mathrm{sgn}_{\frac{\lambda}{4},n}(\cdot)$ in Lemmas 1 and 2 and $\beta_i(\cdot)$'s in (20), that the prescribed λ can be made arbitrarily small but at the cost of larger control magnitude. If a higher tracking accuracy (or equivalently a smaller λ) is pursued, the resulting stronger control effect and the complicated evolution of the system states may lead to tracking error e entering the anticipated vicinity quicker. This prevents us from concluding that a smaller λ results in a larger T_λ.

Remark 7. It is worth pointing out that it is possible for systems (1) to be extended into multi-agent systems interacting over a network, if the mild assumptions are accordingly strengthened, such as additional global Lipschtizness and the homogeneity on system nonlinearities and system dimensions (e.g., [7, 12]). Such an extension, however, still cannot be treated trivially due to the limited information (from the agent itself and its neighbors) and the high nonlinearities/dimensions in agent dynamics. Although basic tools are developed/adopted for single system (1) in this paper (e.g., nonlinear high-gain observer, dynamic high gain), certain constructive idea can be borrowed for multi-agent systems.

6 A simulation example

We consider a 3-order example to illustrate the effectiveness of our control scheme:

$$\begin{cases}\dot{\eta} = -\eta + |y| + 0.5, \\ \dot{x}_1 = x_2 - x_1 - 2\theta_1\sqrt{|\eta|}(x_1 + \arctan x_1^2), \\ \dot{x}_2 = gu + x_2\cos x_1 + \theta_2 x_1, \\ y = x_1\end{cases} \tag{39}$$

with unknown constants $g \neq 0$, $\theta_1 \geqslant 0$, and $\theta_2 \geqslant 0$.

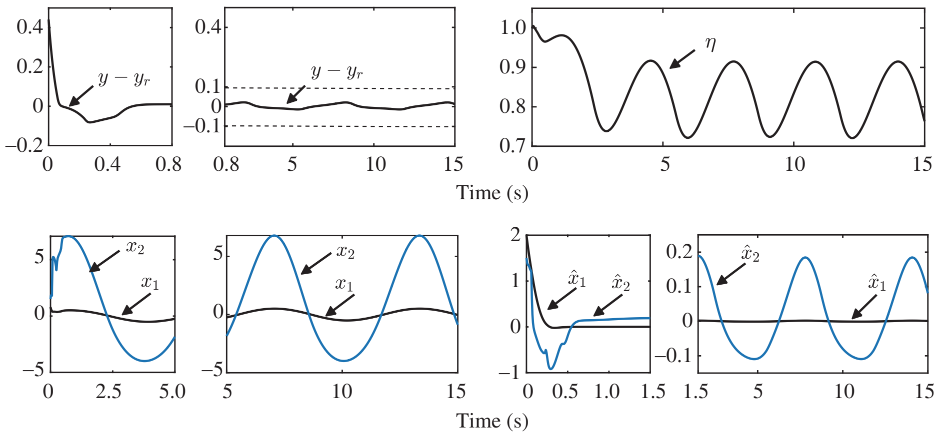

Figure 1 (Color online) Evolution of system tracking error and closed-loop system states.

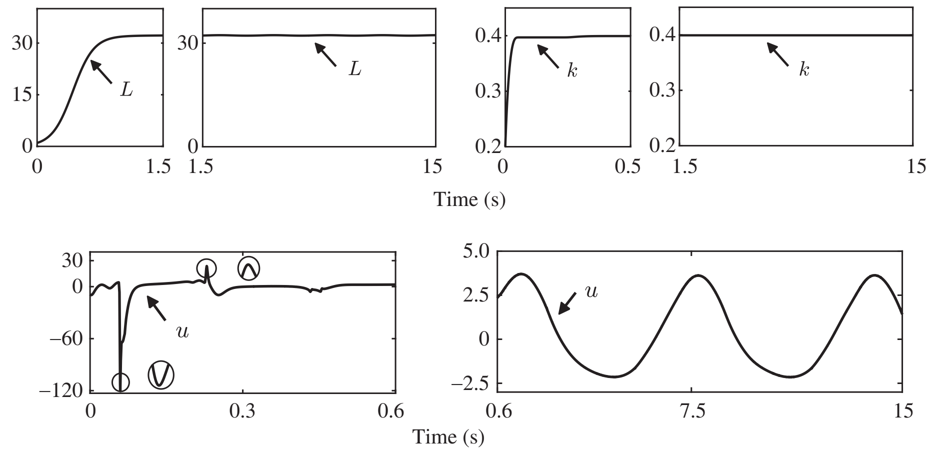

Figure 2 Evolution of system high gains and control input.

Let the desired reference signal $y_r = \frac{1}{2}\sin(t + \frac{4}{5})$ and tracking accuracy $\lambda = 0.2$. Design the critical design functions $P_2(x) = 1.2 \times 10^6 x^5 - 10^4 x^3 - \frac{75}{2}x$ and $Q_{0.05,2}(x) = -500x^4 + 200x^3 - \frac{45}{2}x^2 + x - \frac{1}{64}$. Moreover, select $a = [3, 2]^{\mathrm{T}}$ in the matrix A and $c_1 = 1$, $c_2 = 2$ in (6). Then, a dynamic high-gain filter in the form of (3) with $\dot{L} = \frac{-L^2}{4} + L\sqrt{1 + (1 + 2.4718\cos^2(x_1))}$ is designed to estimate unmeasured state x_2. In order to tackle the unknown control coefficient g, we choose the necessary Nussbaum function $N(k) = \frac{1}{2}k^2\cos k$. Thus in light of the control scheme proposed in Section 4, the adaptive output-feedback controller of system (39) is designed in the form of (17).

Let $g = -2$, $\theta_1 = \theta_2 = 2$ and set the initial conditions $\eta(0) = 1$, $x_1(0) = 0.8$, $x_2(0) = 1.5$, $\xi_1(0) = 2$, $\xi_2(0) = 1.5$, $L(0) = 1$, and $k(0) = 0.2$. Simulation results are displayed in Figures 1 and 2. We find from these figures that all signals of the closed-loop system reach the nice steady state quickly after a short transient stage. Particularly, Figure 1 shows that the tracking error meets the desired accuracy after about 0.45 s, i.e., $|z_1(t)| = |y(t) - \frac{1}{2}\sin(t + \frac{4}{5})| < 0.05 < \lambda = 0.2$, $t \geqslant 0.45$ s, verifying the effectiveness of the proposed control scheme.

7 Concluding remarks

In this paper, global adaptive output-feedback tracking has been achieved for a class of generic uncertain nonlinear systems. Remarkably, the studied systems not merely allow arbitrary functions (of-output) as the growth of inherent nonlinearities and as the growth rate of unmeasured states, but also admit a typical uncertainty encountered inevitably in practice, i.e., the ISpS inverse dynamics, in the context of output feedback and unknown control direction. Moreover, due to the additive non-vanishing uncertainties (e.g., stemming from the unknown system nonlinearities, the reference signals and the generic inverse dynamics), practical tracking cannot be transformed into stabilization as asymptotic one does and calls for specific system performance analysis. To get around the aforementioned challenges, a new control strategy has been put forward by flexibly integrating the dynamic compensation based on Nussbaum-type gain and the refined pseudo-sign and pseudo-dead-zone functions together with backstepping design technique. In addition, with the help of the two critical refined functions, the use of smooth treatment (such as using completing squares) in control design has been moderately avoided, which makes it possible for the tighter and less conservative controller to be established. Actually, unmeasured-state dependent nonlinearities in this paper are required to be affine in unmeasured states and exclude any large parametric uncertainty, or the fundamental tools adopted in this paper would cease to work (for example, a special high-gain filter would be incapable of deriving the anticipated observer error dynamics), making practical tracking quite challenging and worthy of further investigation.

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References

1 Isidori A, Byrnes C I. Output regulation of nonlinear systems. IEEE Trans Autom Control, 1990, 35: 131–140
2 Krstić M, Kanellakopoulos I, Kokotović P V. Nonlinear and Adaptive Control Design. Hoboken: John Wiley & Sons, 1995
3 Astolfi A, Karagiannis D, Ortega R. Nonlinear and Adaptive Control with Applications. Berlin: Springer, 2007
4 Krishnamurthy P, Khorrami F. Dynamic high-gain scaling: state and output feedback with application to systems with ISS appended dynamics driven by all states. IEEE Trans Autom Control, 2004, 49: 2219–2239
5 Zhang X, Lin W. A K-filter-based adaptive control for nonlinear systems with unknown parameters in state and output equations. Automatica, 2019, 105: 186–197
6 Chen Z Y. Nussbaum functions in adaptive control with time-varying unknown control coefficients. Automatica, 2019, 102: 72–79
7 Qin J H, Ma Q C, Yu X H, et al. On synchronization of dynamical systems over directed switching topologies: an algebraic and geometric perspective. IEEE Trans Autom Control, 2020, 65: 5083–5098
8 Ilchmann A, Ryan E P. Universal λ-tracking for nonlinearly-perturbed systems in the presence of noise. Automatica, 1994, 30: 337–346
9 Ye X D. Universal λ-tracking for nonlinearly-perturbed systems without restrictions on the relative degree. Automatica, 1999, 35: 109–119
10 Gong Q, Qian C J. Global practical tracking of a class of nonlinear systems by output feedback. Automatica, 2007, 43: 184–189
11 Liu Y G. Global output-feedback tracking for nonlinear systems with unknown polynomial-of-output growth rate. Control Theory Appl, 2014, 31: 921–933
12 Lv M L, Yu W W, Cao J D, et al. Consensus in high-power multiagent systems with mixed unknown control directions via hybrid nussbaum-based control. IEEE Trans Cybern, 2022, 52: 5184–5196
13 Qian C J, Lin W. Practical output tracking of nonlinear systems with uncontrollable unstable linearization. IEEE Trans Autom Control, 2002, 47: 21–36
14 Bullinger E, Allgöwer F. Adaptive λ-tracking for nonlinear higher relative degree systems. Automatica, 2005, 41: 1191–1200
15 Yan X H, Liu Y G. Global practical tracking for high-order uncertain nonlinear systems with unknown control directions. SIAM J Control Optim, 2010, 48: 4453–4473
16 Yan X H, Liu Y G, Wang Q G. Global output-feedback tracking for nonlinear cascade systems with unknown growth rate and control coefficients. J Syst Sci Complex, 2015, 28: 30–46
17 Yu J B, Zhao Y, Wu Y Q. Global robust output tracking control for a class of uncertain cascaded nonlinear systems. Automatica, 2018, 93: 274–281
18 Wang Y, Liu Y G. Global practical tracking via adaptive output feedback for uncertain nonlinear systems without polynomial constraint. IEEE Trans Autom Control, 2021, 66: 1848–1855
19 Mazenc F, Praly L, Dayawansa W P. Global stabilization by output feedback: examples and counterexamples. Syst Control Lett, 1994, 23: 119–125
20 Isidori A. The zero dynamics of a nonlinear system: from the origin to the latest progresses of a long successful story. Eur J Control, 2013, 19: 369–378
21 Jiang Z P, Praly L. Design of robust adaptive controllers for nonlinear systems with dynamic uncertainties. Automatica, 1998, 34: 825–840
22 Lin W, Qian C J. Adaptive control of nonlinearly parameterized systems: a nonsmooth feedback framework. IEEE Trans Autom Control, 2002, 47: 757–774
23 Ye X D, Jiang J P. Adaptive nonlinear design without a priori knowledge of control directions. IEEE Trans Autom Control, 1998, 43: 1617–1621
24 Fossen T I. Guidance and Control of Ocean Vehicles. Hoboken: John Wiley & Sons, 1994
25 Kokotović P, Arcak M. Constructive nonlinear control: a historical perspective. Automatica, 2001, 37: 637–662
26 Sontag E D, Wang Y. On characterizations of input-to-state stability with respect to compact sets. In: Proceedings of IFAC Nonlinear Control Systems Design Symposium, 1995. 226–231
27 Praly L, Wang Y. Stabilization in spite of matched unmodeled dynamics and an equivalent definition of input-to-state stability. Math Control Signal Syst, 1996, 9: 1–33
28 Praly L, Jiang Z P. Linear output feedback with dynamic high gain for nonlinear systems. Syst Control Lett, 2004, 53: 107–116
29 Hale J K. Ordinary Differential Equations. 2nd ed. Hoboken: John Wiley & Sons, 1980