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Discrete-time delay systems: part 1. Global fully actuated case

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Abstract A basic introduction to the fully actuated system (FAS) approaches for discrete-time systems with delays is given. Firstly, general dynamical discrete-time FAS models with time-varying state delays and constant input delays are proposed. The FAS models are classified into affine ones and non-affine ones, and also ones with and without interconnections. Secondly, controllers for such FASs are designed, which result in constant linear closed-loop systems with arbitrarily assignable eigenstructure. Different from the case of FAS with state delays only, the controller for a discrete-time FAS with an input delay involves a prediction scheme which is constructed based on the open-loop system. The contribution of this paper has laid a fundamental basis for FAS approaches to discrete-time delay systems, and further specific analysis and design problems can be established similar to the continuous-time system case.

Keywords discrete-time systems, time-delay systems, time-varying delays, fully actuated systems, control designs

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1 Introduction

1.1 Discrete-time delay systems

Time delays are often encountered in various practical engineering systems, such as long transmission lines in pneumatic systems, nuclear reactors, rolling mills, hydraulic systems, and manufacturing processes [1]. Besides, due to the vigorous development of computers and microprocessors, the control of discrete-time systems with delays has attracted more and more attention from the control theory community.

Nonlinear time-delay system control has remained to be a difficult problem for decades. As a common fact about dynamical system control using state-space approaches, nonlinearities have always been a vital problem, and the control of a discrete-time nonlinear delay system does not stand as an exception. Nonlinear control system theories based on state-space approaches have appeared for more than half a century, and yet systematic and effective methods are still lacking for the various analysis and design problems. Nonlinear control problems can only be solved with state-space approaches under very strict assumptions, either on the structure of the system or on the magnitude of the nonlinear terms in the system. Furthermore, time-varying delays also add real difficulties, and make the general control problem truly a harder one.

Indeed, a discrete-time time-delay system can be transformed into a delay-free one by state/input/output augmentation (see [2]). However, such a method may lead to complicated operations, or even fail, in some cases that the delays are uncertain, time-varying or very large ([2, 3]). In 2003, Gu and Niculescu [4] claimed that many important issues in this field are far from being satisfactorily solved, and "there are still very few strong results on nonlinear systems". It is pointed out in the monograph [5] that "discrete-time interconnected uncertain time-delay systems are still at their infancy", and "there is

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still an ample amount of technical problems open for scientific research". In [6], the control of nonlinear time-delay systems is argued to be "quite challenging due to complicated dynamical properties".

Owing to the great difficulties in nonlinear time-delay systems, most reported results on the control of discrete-time systems with time delays are obtained for linear systems (see [7–15]). Nevertheless, today even the control of a discrete-time linear system with time-varying delays still remains to be a problem generally unsolvable. The problem of stabilization via state feedback is investigated in detail for both the certain systems [7, 10] and the uncertain systems [8]. The output feedback case is further considered in [9, 11, 14]. Some other important issues are also investigated, such as H-infinity control [12], stability analysis for stochastic systems [13], and consensus problems [15].

In fact, even if the open-loop system is linear, the control of discrete time-delay systems is also of great difficulty, especially for the case that the delays are time-varying or uncertain. This is because in such a case, the commonly used state augmentation technique is usually difficult to apply, or even fails [2, 3]. Moreover, most existing results based on various types of methods are only obtained in a local sense (see [7–9, 11, 12, 15]). It is claimed in [16] that, although the considered time-delay system is linear, "there still are some important issues that are not completely solved".

Let us point out that there exists a typical route in control system design using state-space approaches, that is, first to establish the stability results for the type of systems, and then to perform the control design based on the obtained stability conditions. The design problem is difficult because the stability analysis of a nonlinear system, especially a nonlinear delay system, is difficult. For example, it is pointed out in 2008 by Xu and Lam [1] that "how to develop new methods in order to further reduce the conservatism in existing stability results while keeping a reasonably low computational complexity is an important issue to be investigated in the future". In our typical thinking, stability is a problem which is insurmountable. The better the stability results are, the better the design results are. It is hard to imagine an effective nonlinear control method which is not dependent on the stability analysis of the open-loop systems.

1.2 Fully actuated system (FAS) approaches

Like the case of dynamical systems without delays, state-space approaches are also the dominant ones to handle the analysis and design of discrete-time delay systems. As we have seen above, great difficulties are encountered with state-space approaches in dealing with control of nonlinear systems, particularly those with time-varying delays.

Very recently, a different approach is proposed in parallel to the state-space one for dynamical control system design, especially for nonlinear control systems. The approach is termed as the fully actuated system (FAS) approach, and was firstly proposed for continuous-time dynamical systems (see [17–28]), and then generalized into the case of discrete-time systems [29].

A state-space model concentrates on the state variables, and integrates all the independent state variables together. As a consequence, it is convenient to solve all the state variables, and is therefore effective for the problems of state solution (response analysis) and estimation (observation, filtering and prediction). Although control problems are also tackled with state-space approaches, the results are not as satisfactory as desired in the nonlinear system case.

On the contrary, FAS models, proposed in [17, 20, 26] for continuous-time systems, in [29] for discrete-time systems, and in this paper for discrete-time delay systems, concentrate on the control variables, and have the ability to solve out the control vectors. Hence it allows one to cancel the nonlinearities in the system, and eventually produces a constant linear system with an arbitrarily assignable eigenstructure. The FAS approach is convenient for control problems, and has been sufficiently shown to be extremely effective and simple in dealing with the control problems of general dynamical nonlinear systems, such as robust control [22, 24], adaptive control [23, 24], disturbance rejection [25], optimal control [27], and signal tracking control [28].

1.3 Contribution of the paper

In this paper, a basic introduction to the FAS approach for discrete-time delay systems is presented. Two main topics are covered, namely, (1) FAS models for general discrete-time delay systems, and (2) control of discrete-time FASs with delays.

For FAS models of general dynamical discrete-time delay systems, the linear system case is firstly discussed. The FAS model of a general linear system with time delay is introduced by proving that a state-space model with both constant state delay and input delay can be converted into a linear FAS with constant time delays. Motivated by this fact, a general FAS model for a linear system with time-varying state delays and a constant input delay is proposed. For the nonlinear system case, it is firstly proven that a discrete-time state-space strict-feedback system with an input delay can be converted into a nonlinear time-delay FAS. This conversion inspires us to propose the general FAS model of a general nonlinear system with time delays. Parallel to the linear system case, a general FAS model for a nonlinear system with time-varying state delays and a constant input delay is proposed. Corresponding to the multiple order FAS for continuous-time systems, both linear and nonlinear FAS models with interconnections are proposed. Further generalized forms of FASs for discrete-time delay systems are also given.

For the control of discrete-time FAS models with time-varying state delays only, controllers in state feedback form can be conveniently designed, which result in constant linear closed-loop systems with arbitrary system orders and arbitrarily assignable eigenstructures. While for the control of discrete-time FAS models with both time-varying state delays and a constant input delay, the problem of controller design turns out to be a little more complicated. In order to make the controller realizable, the states of certain steps ahead have to be predicted, and the prediction is easily realized based on the open-loop system. In this case, the controller eventually turns out to be a dynamical one.

This paper is organized into 6 sections. Section 2 gives certain symbols and notions used in the paper. In Sections 3 and 4, the discrete-time delay FAS models for linear systems and nonlinear systems are proposed, respectively. Section 5 presents the designs of controllers for the proposed discrete-time FASs with time delays, followed by a brief concluding remark in Section 6.

The sub-fully actuated case will be addressed in the part two of the paper.

2 Notations

In this section, certain notations used in the paper are explained.

2.1 General notations

In this paper, I_n denotes the identity matrix of order n, $\emptyset$ denotes the null set, $\mathbb{N}$ is the set of natural numbers, and $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ denote the spaces of n dimensional vectors and $m\times n$ dimensional matrices, respectively. Furthermore, $\det(A)$ and A^{-1} denote the determinant and the inverse of a matrix A, respectively.

For $x_i \in \mathbb{R}^m$, $i = 1, 2, \ldots, n$, let

$$x_{i\sim j}(k) = \begin{bmatrix} x_i(k) \\ x_{i+1}(k) \\ \vdots \\ x_j(k) \end{bmatrix}, \quad i \leqslant j \leqslant n.$$

For $A_i \in \mathbb{R}^{m\times m}$, $i = 0, 1, \ldots, n$ as in [22], the following symbols are used:

$$A_{0\sim n} = \begin{bmatrix} A_0 & A_1 & \cdots & A_n \end{bmatrix}, \quad \Psi(A_{0\sim n}) = \begin{bmatrix} A_0 & A_1 & \cdots & A_n \\ I & & & \\ & \ddots & & \\ & & I & 0 \end{bmatrix}.$$

2.2 Step backward operations

For $x(k) \in \mathbb{R}^m$, it is well-known that the one-step backward operator is usually defined as q^{-1}, which operates in the following way:

$$q^{-1}x(k) = x(k-1), \quad q^{-i}x(k) = x(k-i).$$

For convenience, in this paper we represent the above operation by the following notation:

$$x^{\lceil i \rceil}(k) = x(k-i).$$

Based on this notation, for $x \in \mathbb{R}^m$, $n_i \in \mathbb{N}$, $i = 1, 2, \ldots, n$, the following symbols are used in the paper:

$$x^{\lceil 0\sim n\rceil}(k) = \begin{bmatrix} x(k) \\ x(k-1) \\ \vdots \\ x(k-n) \end{bmatrix}, \quad x^{\lceil n_1\sim n_2\rceil}(k) = \begin{bmatrix} x^{\lceil n_1\rceil}(k) \\ x^{\lceil n_1+1\rceil}(k) \\ \vdots \\ x^{\lceil n_2\rceil}(k) \end{bmatrix}, n_1 \leqslant n_2, \quad x^{\lceil n_{1\sim p}\rceil}(k) = \begin{bmatrix} x^{\lceil n_1\rceil}(k) \\ x^{\lceil n_2\rceil}(k) \\ \vdots \\ x^{\lceil n_p\rceil}(k) \end{bmatrix}, p \geqslant 1.$$

Further for $x_i \in \mathbb{R}^m$, $n_0 \in \mathbb{N}$, $n_0 < n_i$, $i = 1, 2, \ldots, n$, the following symbols are used:

$$x_{i\sim j}^{\lceil n_1\sim n_2\rceil}(k) = \begin{bmatrix} x_i^{\lceil n_1\sim n_2\rceil}(k) \\ x_{i+1}^{\lceil n_1\sim n_2\rceil}(k) \\ \vdots \\ x_j^{\lceil n_1\sim n_2\rceil}(k) \end{bmatrix}, i \leqslant j,\ n_1 \leqslant n_2, \quad x_p^{\lceil n_p\rceil}(k)\,|_{p=i\sim j} = \begin{bmatrix} x_i^{\lceil n_i\rceil}(k) \\ x_{i+1}^{\lceil n_{i+1}\rceil}(k) \\ \vdots \\ x_j^{\lceil n_j\rceil}(k) \end{bmatrix}, i \leqslant j,$$

$$x_p^{\lceil n_0\sim n_p\rceil}(k)\,|_{p=i\sim j} = \begin{bmatrix} x_i^{\lceil n_0\sim n_i\rceil}(k) \\ x_{i+1}^{\lceil n_0\sim n_{i+1}\rceil}(k) \\ \vdots \\ x_j^{\lceil n_0\sim n_j\rceil}(k) \end{bmatrix}, i \leqslant j.$$

3 Linear time-delay FASs

3.1 Motivation

In this subsection, it is shown that a discete-time system with both state delays and an input delay can be converted into the form of a discrete-time FAS.

Consider the following discrete-time linear delay system in state-space form:

$$x(k+1) = \sum_{i=0}^{m} A_i x(k-i) + B_0 u(k-h), \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^r$ are the state vector and input vector, respectively, h and m are two nonnegative integers which represent the time delays in the control vector and the state vector, respectively, and $A_i \in \mathbb{R}^{n\times n}, i = 1, 2, \ldots, m$, and $B_0 \in \mathbb{R}^{n\times r}$ are the coefficient matrices.

For the above state-space system (1), it is generally not realistic to require the matrix B_0 to be a square nonsingular matrix. However, we can show that the system (1) can be indeed converted into a form with such a similar condition satisfied.

Let

$$X(k) = x^{\lceil 0\sim m\rceil}(k) = \begin{bmatrix} x(k) \\ x(k-1) \\ \vdots \\ x(k-m) \end{bmatrix}.$$

Then the system (1) is equivalent to

$$X(k+1) = \Psi(A_{0\sim m}) X(k) + B_0 \Gamma_c u(k-h) \tag{2}$$

with

$$\Gamma_c = \begin{bmatrix} I_r \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{3}$$

Therefore, the delay system (1) is controllable if and only if the system (2) and (3) is controllable. Further note that the controllability of the system (2) and (3) with $h \neq 0$ is equivalent to that of the system with $h = 0$. Applying the Theorem 3.1 in [29] to the above system (2) and (3), we immediately have the following result.

Proposition 1. The linear time-delay system (1) is controllable if and only if it can be converted equivalently into a system in the form of

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_\eta(k+1) \end{bmatrix} = \begin{bmatrix} L_1\left(x_p^{\lceil 0\sim\mu_p-1\rceil}(k)\,|_{p=1\sim\eta}\right) \\ L_2\left(x_p^{\lceil 0\sim\mu_p-1\rceil}(k)\,|_{p=1\sim\eta}\right) \\ \vdots \\ L_\eta\left(x_p^{\lceil 0\sim\mu_p-1\rceil}(k)\,|_{p=1\sim\eta}\right) \end{bmatrix} + Bu(k-h), \tag{4}$$

where $\eta \geqslant 1$ is an integer, μ_p, $p = 1, 2, \ldots, \eta$ are a set of integers, $x_p \in \mathbb{R}^{r_p}$, $p = 1, 2, \ldots, \eta$ are a set of state vectors, with r_p, $p = 1, 2, \ldots, \eta$, being a set of distinct integers satisfying

$$r_1 + r_2 + \cdots + r_\eta = r. \tag{5}$$

Further, $L_p(\cdot)$, $p = 1, 2, \ldots, \eta$, are a set of linear functions, and $B \in \mathbb{R}^{r\times r}$ is a square upper-triangular matrix with diagonal elements all being 1.

Generally, a system in the form of (4) is called a linear time-delay FAS if $B \in \mathbb{R}^{r\times r}$ is nonsingular, that is,

$$\mathrm{rank} B = r. \tag{6}$$

As we will see in Section 5, such a full-actuation condition provides great convenience in the control of the system.

Since the system (4) has a set of interconnected sub-states, it is often called a time-delay FAS with interconnections for short.

With our notations, the above FAS (4) can be also compactly written in the form of

$$x_{1\sim\eta}(k+1) = L\left(x_p^{\lceil 0\sim\mu_p-1\rceil}(k)\,|_{p=1\sim\eta}\right) + Bu(k-h), \tag{7}$$

where

$$L(\cdot) = \begin{bmatrix} L_1(\cdot) \\ L_2(\cdot) \\ \vdots \\ L_\eta(\cdot) \end{bmatrix}.$$

A typical special form of the above linear discrete-time interconnected FAS (4) with delays is the following linear time-delay FAS:

$$x(k+1) = L\left(x^{\lceil 0\sim n-1\rceil}(k)\right) + Bu(k-h), \tag{8}$$

where $L(\cdot)$ is a linear function, and the matrix $B \in \mathbb{R}^{r\times r}$ is nonsingular.

As we will show later, or as demonstrated with the non-delay FAS approaches ([17, 18, 20, 29]), a great advantage of a FAS lies in the control of the system.

Based on the above description, we can now further propose more general linear FASs with time delays.

3.2 FAS without interconnections

Inspired by the above Proposition 1, we can now introduce a general linear discrete-time FAS with time delays by (1) adding the time variable k, and (2) changing the constant time delays into time-varying ones. Eventually, the following discrete-time FAS with time-varying state delays and a constant input delay can be proposed as follows:

$$x(k+1) = L\left(x^{\lceil \tau_{1\sim\varsigma}(k)\rceil}(k), k\right) + Bu(k-h), \tag{9}$$

where $B \in \mathbb{R}^{r\times r}$ is a nonsingular matrix, $L(\cdot)$ is a linear function with respect to $x^{\lceil \tau_{1\sim\zeta}(k)\rceil}(k)$, and by our notations,

$$\tau_{1\sim\zeta}(k) = \begin{bmatrix} \tau_1(k) \\ \tau_2(k) \\ \vdots \\ \tau_\zeta(k) \end{bmatrix}. \tag{10}$$

Here $\tau_i(k), i = 1, 2, \ldots, \zeta$, are a group of nonnegative scalar functions which represent the time delays and satisfy

$$0 \leqslant \tau_i(k) \leqslant \bar{\tau}_i, \quad i = 1, 2, \ldots, \zeta, \quad k \geqslant 0,$$

with $\bar{\tau}_i, i = 1, 2, \ldots, \zeta$ being a group of nonnegative constant integers. Furthermore, again by our notation, we have

$$x^{\lceil \tau_{1\sim\zeta}(k)\rceil}(k) = \begin{bmatrix} x(k-\tau_1(k)) \\ x(k-\tau_2(k)) \\ \vdots \\ x(k-\tau_\zeta(k)) \end{bmatrix}.$$

Clearly, a specific type of time-delay FASs in the form of (9) is the following:

$$x(k+1) = \sum_{i=1}^{\zeta} A_i x^{\lceil \tau_i(k)\rceil}(k) + Bu(k-h), \tag{11}$$

which can also be written as

$$x(k+1) = A_{1\sim\zeta} x^{\lceil \tau_{1\sim\zeta}(k)\rceil}(k) + Bu(k-h). \tag{12}$$

3.3 FAS with interconnections

As a direct generalization to the linear discrete-time delay FAS (4), we can also introduce the following more general linear discrete-time delay FASs with interconnections:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_\eta(k+1) \end{bmatrix} = \begin{bmatrix} L_1\left(x_p^{\lceil \tau_{p,1\sim\zeta_p}(k)\rceil}(k)\,|_{p=1\sim\eta}, k\right) \\ L_2\left(x_p^{\lceil \tau_{p,1\sim\zeta_p}(k)\rceil}(k)\,|_{p=1\sim\eta}, k\right) \\ \vdots \\ L_\eta\left(x_p^{\lceil \tau_{p,1\sim\zeta_p}(k)\rceil}(k)\,|_{p=1\sim\eta}, k\right) \end{bmatrix} + Bu(k-h), \tag{13}$$

where η and ζ_p, $x_p \in \mathbb{R}^{r_p}$, r_p and $L_p(\cdot), p = 1, 2, \ldots, \eta$, are all as stated in Proposition 1, $\tau_{p,i}(k), i = 1, 2, \ldots, \zeta_p, p = 1, 2, \ldots, \eta$, are the time-varying time delays, and B is a nonsingular coefficient matrix.

Recalling our notations, we have

$$x_{1\sim\eta}(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_\eta(k) \end{bmatrix}, \quad x^{\lceil \tau_{p,1\sim\zeta_p}(k)\rceil}(k) = \begin{bmatrix} x(k-\tau_{p,1}(k)) \\ x(k-\tau_{p,2}(k)) \\ \vdots \\ x\left(k-\tau_{p,\zeta_p}(k)\right) \end{bmatrix}. \tag{14}$$

Then the above FAS (13) can be compactly written as

$$x_{1\sim\eta}(k+1) = L\left(x_p^{\lceil \tau_{p,1\sim\zeta_p}(k)\rceil}(k)\,|_{p=1\sim\eta}, k\right) + Bu(k-h). \tag{15}$$

3.4 Further generalizations

In linear system theory, the following discrete-time system in self-recursive form:

$$x(k+1)=\sum_{i=1}^{\zeta}A_ix(k-\tau_i(k))+\sum_{i=0}^{m}B_iu(k-h-i) \tag{16}$$

is often encountered. By our notations, it can also be represented in the form of

$$x(k+1)=A_{1\sim\zeta}x^{\lceil\tau_{1\sim\zeta}(k)\rceil}(k)+B_{0\sim m}u^{\lceil 0\sim m\rceil}(k-h). \tag{17}$$

It can be easily reasoned that, when $B=B_0$ is a square nonsingular matrix, the system (17) can be converted equivalently into the following FAS:

$$x(k+1)=A_{1\sim\zeta}x^{\lceil\tau_{1\sim\zeta}(k)\rceil}(k)+Bu'(k-h), \tag{18}$$

where

$$u'(k)=B^{-1}\sum_{i=0}^{m}B_iu(k-i)=u(k)+\sum_{i=1}^{m}B^{-1}B_iu(k-i). \tag{19}$$

Clearly, the original control vector $u(k)$ can be uniquely solved through (19).

More generally, let us consider the following system, represented by the step backward operator,

$$x(k+1)=\sum_{i=1}^{\zeta}A_ix(k-\tau_i(k))+q^{-h}B\left(q^{-1}\right)u(k), \tag{20}$$

where $h\geqslant 0$, and

$$B\left(q^{-1}\right)=\sum_{i=0}^{m}B_iq^{-i},\quad B_0\neq 0.$$

Such a system is a time-delay FAS if $B(q^{-1})$ is a unimodular matrix, that is,

$$\det B(q)\neq 0,\quad \forall q\in\mathbb{C}. \tag{21}$$

In this case, $B^{-1}(q^{-1})$ is also a polynomial. Hence, the new control

$$u'(k)=B\left(q^{-1}\right)u(k) \tag{22}$$

can be introduced, which gives

$$u(k)=B^{-1}\left(q^{-1}\right)u'(k), \tag{23}$$

and the system (20) can be reduced to the form of

$$B^{-1}\left(q^{-1}\right)\left[x(k+1)-\sum_{i=1}^{\zeta}A_ix(k-\tau_i(k))\right]=u'(k-h), \tag{24}$$

which can be easily shown to be a discrete-time FAS with time delays.

4 Nonlinear time-delay FASs

4.1 Motivation

As in the linear case, in this subsection we show that a type of nonlinear state-space models, namely, the discrete-time strict-feedback systems with input delays, can be converted into the form of a discrete-time FAS.

By slightly modifying the well-known discrete-time strict-feedback system (see [21,29–33]), the following discrete-time strict-feedback system with an input time delay can be proposed:

$$\begin{cases} x_1(k+1) = f_1(x_1(k)) + g_1(x_1(k))\, x_2(k), \\ x_2(k+1) = f_2(x_{1\sim 2}(k)) + g_2(x_{1\sim 2}(k))\, x_3(k), \\ \vdots \\ x_{n-1}(k+1) = f_{n-1}(x_{1\sim n-1}(k)) + g_{n-1}(x_{1\sim n-1}(k))\, x_n(k), \\ x_n(k+1) = f_n(x_{1\sim n}(k)) + g_n(x_{1\sim n}(k))\, u(k-h), \end{cases} \tag{25}$$

where u and $x_i \in \mathbb{R}^r$, $i = 1, 2, \ldots, n$, are the system input vector and state vectors, respectively, $f_i(\cdot) \in \mathbb{R}^r$, and $g_i(\cdot) \in \mathbb{R}^{r\times r}, i = 1, 2, \ldots, n$, are nonlinear functions, and particularly, $g_i(\cdot)$ satisfies the following assumption.

Assumption 1. $\det g_i(x_{1\sim i}(k)) \neq 0, \forall x_{1\sim i}(k) \in \mathbb{R}^{ir}$, $i = 1, 2, \ldots, n$, and $k \geqslant h$.

In [29], it is shown that the above strict-feedback system (25), with $h = 0$, can be equivalently converted into a FAS with state delays only. Applying this result to the above strict-feedback system (25), with $h \neq 0$, gives the following result which tells us that a strict-feedback system (25) with input delay can be converted equivalently into a FAS with both state and control delays.

Proposition 2. The above system (25), with Assumption 1 satisfied, can be equivalently converted into the following form:

$$x(k+1) = f\left(x^{\lceil 0\sim n-1\rceil}(k)\right) + B\left(x^{\lceil 0\sim n-1\rceil}(k)\right) u(k-h), \tag{26}$$

where $f(\cdot) \in \mathbb{R}^r$ and $B(\cdot) \in \mathbb{R}^{r\times r}$ are nonlinear functions, and particularly, $B(\cdot)$ satisfies the following condition:

$$\det B\left(x^{\lceil 0\sim n-1\rceil}(k)\right) \neq 0, \quad \forall x^{\lceil 0\sim n-1\rceil}(k) \in \mathbb{R}^{nr} \text{ and } k \geqslant h. \tag{27}$$

The above condition (27) is called a full-actuation condition, while the system (26) is called a discrete-time FAS with delays.

The full-actuation condition (27) makes the above time-delay FAS (26) totally different from a state-space time-delay system. Without loss of generality, let us choose to consider the following nonlinear (affine) discrete-time delay system in a state-space representation:

$$x(k+1) = f(x(k), x(k-\tau), k) + B(x(k), x(k-\tau), k)\, u(k-h), \tag{28}$$

where $f(\cdot) \in \mathbb{R}^n$ and $B(\cdot) \in \mathbb{R}^{n\times r}$ are nonlinear functions. Although in appearance the state-space system (28) looks like a specific form of the FAS (26), it differs from the FAS (26) in the system dimensions. For the FAS (26), we have $r = n$, and the full-actuation condition (27) holds, while for the state-space system (28), we have $r < n$, not to mention the nonsingularity of the matrix $B(x(k), x(k-\tau), k)$. It is this difference that really makes the control of a nonlinear discrete-time delay system in a state-space approach still a very difficult problem today. Nevertheless, the above Proposition 2 has shown a very important fact that a discrete-time delay system in a state-space representation may be converted into a discrete-time FAS with time delays. As we will see in Section 5, this brings about great convenience in the control of time-delay dynamical systems.

4.2 Affine FASs

Motivated by Propositions 1 and 2, we can now introduce general nonlinear discrete-time FAS with time delays, on the basis of system (26), by (1) adding the time variable k, and (2) changing the constant time delays into time-varying ones. Hence the system appears as

$$x(k+1) = f\left(x^{\lceil \tau_{1\sim\varsigma}(k)\rceil}(k), k\right) + B\left(x^{\lceil \tau_{1\sim\varsigma}(k)\rceil}(k), k\right) u(k-h), \tag{29}$$

where $\tau_{1\sim\varsigma}(k)$ is the time delay vector defined in (10), $f(\cdot, k) \in \mathbb{R}^r$ and $B(\cdot, k) \in \mathbb{R}^{r\times r}$ are nonlinear functions, and the matrix $B(\cdot, k)$ satisfies the following full-actuation assumption.

Assumption 2. $\det B\left(x^{\lceil\tau_{1\sim\varsigma(k)}\rceil}\left(k\right),k\right)\neq 0,\forall x^{\lceil\tau_{1\sim\varsigma(k)}\rceil}\left(k\right)$ and $k\geqslant h$.

Corresponding to the general linear discrete-time delay FAS (13) with interconnections, we can also propose the following nonlinear discrete-time delay FAS with interconnections:

$$\begin{bmatrix} x_1\left(k+1\right) \\ x_2\left(k+1\right) \\ \vdots \\ x_\eta\left(k+1\right) \end{bmatrix} = \begin{bmatrix} f_1\left(x_p^{\left\lceil\tau_{p,1\sim\zeta_p(k)}\right\rceil}\left(k\right)|_{p=1\sim\eta},k\right) \\ f_2\left(x_p^{\left\lceil\tau_{p,1\sim\zeta_p(k)}\right\rceil}\left(k\right)|_{p=1\sim\eta},k\right) \\ \vdots \\ f_\eta\left(x_p^{\left\lceil\tau_{p,1\sim\zeta_p(k)}\right\rceil}\left(k\right)|_{p=1\sim\eta},k\right) \end{bmatrix} + B\left(x_p^{\left\lceil\tau_{p,1\sim\zeta_p(k)}\right\rceil}\left(k\right)|_{p=1\sim\eta},k\right)u\left(k-h\right), \quad (30)$$

where, again, $\zeta_p, p=1,2,\ldots,\eta$, are a set of distinct integers, $x_p\in\mathbb{R}^{r_p}, p=1,2,\ldots,\eta$, are a set of state vectors, with $r_p, p=1,2,\ldots,\eta$, being a set of distinct integers satisfying (5). $\tau_{p,1\sim\zeta_p}\left(k\right), p=1,2,\ldots,\eta$, are the delay vectors defined in (14). Further, $f_p\left(\cdot\right)\in\mathbb{R}^{r_p}, p=1,2,\ldots,\eta$, are a set of nonlinear vector functions, and $B\left(\cdot\right)\in\mathbb{R}^{r\times r}$ is a nonlinear matrix function satisfying the following full-actuation assumption.

Assumption 3. $\det B(x_p^{\left\lceil\tau_{p,1\sim\zeta_p(k)}\right\rceil}\left(k\right)|_{p=1\sim\eta},k)\neq 0$, $\forall x_p^{\left\lceil\tau_{p,1\sim\zeta_p(k)}\right\rceil}\left(k\right)|_{p=1\sim\eta}$, and $k\geqslant h$.

Let

$$f\left(\cdot,k\right)=\begin{bmatrix} f_1\left(\cdot,k\right) \\ f_2\left(\cdot,k\right) \\ \vdots \\ f_\eta\left(\cdot,k\right) \end{bmatrix}, \quad (31)$$

and then the FAS (30) can be compactly written as

$$x_{1\sim\eta}\left(k+1\right)=f\left(x_p^{\left\lceil\tau_{p,1\sim\zeta_p(k)}\right\rceil}\left(k\right)|_{p=1\sim\eta},k\right)+B\left(x_p^{\left\lceil\tau_{p,1\sim\zeta_p(k)}\right\rceil}\left(k\right)|_{p=1\sim\eta},k\right)u\left(k-h\right). \quad (32)$$

In fact, under Assumption 3, we can introduce a new control vector in the following way:

$$\tilde{u}\left(k-h\right)=B\left(x_p^{\left\lceil\tau_{p,1\sim\zeta_p(k)}\right\rceil}\left(k\right)|_{p=1\sim\eta},k\right)u\left(k-h\right),$$

and then the system (32) can be written into the following standard form:

$$x_{1\sim\eta}\left(k+1\right)=f\left(x_p^{\left\lceil\tau_{p,1\sim\zeta_p(k)}\right\rceil}\left(k\right)|_{p=1\sim\eta},k\right)+\tilde{u}\left(k-h\right). \quad (33)$$

Furthermore, this system can also be written in the following decoupled form:

$$x_i\left(k+1\right)=f_i\left(x_p^{\left\lceil\tau_{p,1\sim\zeta_p(k)}\right\rceil}\left(k\right)|_{p=1\sim\eta},k\right)+\tilde{u}_i\left(k-h\right), \quad i=1,2,\ldots,\eta, \quad (34)$$

where $\tilde{u}_i, i=1,2,\ldots,\eta$ are defined as

$$\tilde{u}=\begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \vdots \\ \tilde{u}_\eta \end{bmatrix}, \quad \tilde{u}_i\in\mathbb{R}^{r_i}. \quad (35)$$

4.3 Non-affine FASs

Parallel to the general nonlinear affine discrete-time delay FAS (29), we can also introduce the following non-affine one:

$$x\left(k+1\right)=f\left(x^{\lceil\tau_{1\sim\varsigma(k)}\rceil}\left(k\right),k\right)+g\left(x^{\lceil\tau_{1\sim\varsigma(k)}\rceil}\left(k\right),k,u\left(k-h\right)\right), \quad (36)$$

where $g(\cdot,k,u)\in\mathbb{R}^r$ is a nonlinear function satisfying the following full-actuation assumption.

Assumption 4. $g\left(x^{\lceil\tau_{1\sim\varsigma}(k)\rceil}(k), k, u\right) = \tilde{u}$ forms a differential homeomorphism from u to $\tilde{u}$ for all $x^{\lceil\tau_{1\sim\varsigma}(k)\rceil}(k)$ and $k \geqslant h$.

Corresponding to the general nonlinear affine discrete-time delay FAS (32) with interconnections, we can also propose the following non-affine discrete-time delay FAS with interconnections:

$$x_{1\sim\eta}(k+1) = f\left(x_p^{\lceil\tau_{p,1\sim\varsigma_p}(k)\rceil}(k)\,|_{p=1\sim\eta}, k\right) + g\left(x_p^{\lceil\tau_{p,1\sim\varsigma_p}(k)\rceil}(k)\,|_{p=1\sim\eta}, k, u(k-h)\right), \tag{37}$$

where $g(\cdot) \in \mathbb{R}^r$ is a nonlinear function satisfying the following full-actuation assumption.

Assumption 5. $g(x_p^{\lceil\tau_{p,1\sim\varsigma_p}(k)\rceil}(k)\,|_{p=1\sim\eta}, k, u) = \tilde{u}$ forms a differential homeomorphism from u to $\tilde{u}$ for all $x_p^{\lceil\tau_{p,1\sim\varsigma_p}(k)\rceil}(k)\,|_{p=1\sim\eta}$ and $k \geqslant h$.

Under Assumption 5, the following new control can be also introduced:

$$\tilde{u}(k-h) = g\left(x_p^{\lceil\tau_{p,1\sim\varsigma_p}(k)\rceil}(k)\,|_{p=1\sim\eta}, k, u(k-h)\right),$$

and with this control variable transformation, the FAS (37) can be also turned into the standard form of (33), as well as the decoupled form of (34).

4.4 Further generation

As a further generalization of the discrete-time time-delay FAS (30), the following discrete-time FAS with time-varying multiple input delays is proposed:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_\eta(k+1) \end{bmatrix} = \begin{bmatrix} f_1\left(x_p^{\lceil\tau_{p,1\sim\varsigma_p}(k)\rceil}(k)\,|_{p=1\sim\eta}, u^{\lceil\sigma_{1\sim\xi}(k)\rceil}(k-h), k\right) \\ f_2\left(x_p^{\lceil\tau_{p,1\sim\varsigma_p}(k)\rceil}(k)\,|_{p=1\sim\eta}, u^{\lceil\sigma_{1\sim\xi}(k)\rceil}(k-h), k\right) \\ \vdots \\ f_\eta\left(x_p^{\lceil\tau_{p,1\sim\varsigma_p}(k)\rceil}(k)\,|_{p=1\sim\eta}, u^{\lceil\sigma_{1\sim\xi}(k)\rceil}(k-h), k\right) \end{bmatrix}$$
$$+B\left(x_p^{\lceil\tau_{p,1\sim\varsigma_p}(k)\rceil}(k)\,|_{p=1\sim\eta}, u^{\lceil\sigma_{1\sim\xi}(k)\rceil}(k-h), k\right) u(k-h), \tag{38}$$

where $\sigma_i(k)$, $i = 1, 2, \ldots, \xi$ are a set of positive integer functions, while the other variables are as stated before. The matrix $B(\cdot) \in \mathbb{R}^{r\times r}$ is a nonlinear function satisfying the following assumption.

Assumption 6. $B(x_p^{\lceil\tau_{p,1\sim\varsigma_p}(k)\rceil}(k)\,|_{p=1\sim\eta}, u^{\lceil\sigma_{1\sim\xi}(k)\rceil}(k-h), k)$ is nonsingular for all its variables when $k \geqslant h$.

Let

$$f(\cdot,\cdot,k) = \begin{bmatrix} f_1(\cdot,\cdot,k) \\ f_2(\cdot,\cdot,k) \\ \vdots \\ f_\eta(\cdot,\cdot,k) \end{bmatrix}, \tag{39}$$

and then the time-delay FAS (38) can be compactly written as

$$\begin{aligned} x_{1\sim\eta}(k+1) = {} & f\left(x_p^{\lceil\tau_{p,1\sim\varsigma_p}(k)\rceil}(k)\,|_{p=1\sim\eta}, u^{\lceil\sigma_{1\sim\xi}(k)\rceil}(k-h), k\right) \\ & +B\left(x_p^{\lceil\tau_{p,1\sim\varsigma_p}(k)\rceil}(k)\,|_{p=1\sim\eta}, u^{\lceil\sigma_{1\sim\xi}(k)\rceil}(k-h), k\right) u(k-h). \end{aligned} \tag{40}$$

The above FAS can be further simply written in the form of

$$x_{1\sim\eta}(k+1) = f_{x,u}(k) + B_{x,u}(k)\, u(k-h) \tag{41}$$

with the following notations:

$$\begin{aligned} f_{x,u}(k) &\triangleq f\left(x_p^{\lceil \tau_{p,1\sim\zeta_p(k)}\rceil}(k)\,|_{p=1\sim\eta}, u^{\lceil \sigma_{1\sim\xi}(k)\rceil}(k-h), k\right), \\ B_{x,u}(k) &\triangleq B\left(x_p^{\lceil \tau_{p,1\sim\zeta_p(k)}\rceil}(k)\,|_{p=1\sim\eta}, u^{\lceil \sigma_{1\sim\xi}(k)\rceil}(k-h), k\right). \end{aligned} \tag{42}$$

Parallel to the extended affine FAS (38), the following extended non-affine one can also be defined:

$$x_{1\sim\eta}(k+1) = f_{x,u}(k) + g\left(x_p^{\lceil \tau_{p,1\sim\zeta_p(k)}\rceil}(k)\,|_{p=1\sim\eta}, u^{\lceil \sigma_{1\sim\xi}(k)\rceil}(k-h), k, u(k-h)\right), \tag{43}$$

where $\tilde{u} = g(\cdot,\cdot,k,u)$ forms a differential homeomorphism from u to $\tilde{u}$.

5 Controller designs

For simplicity, let us first treat the FAS with state delays only.

5.1 FASs with state delays only

Consider the following discrete-time interconnected FAS with state delays only:

$$x_{1\sim\eta}(k+1) = f\left(x_p^{\lceil \tau_{p,1\sim\zeta_p(k)}\rceil}(k)\,|_{p=1\sim\eta}, k\right) + B\left(x_p^{\lceil \tau_{p,1\sim\zeta_p(k)}\rceil}(k)\,|_{p=1\sim\eta}, k\right)u(k). \tag{44}$$

The one without interconnections appears as follows:

$$x(k+1) = f\left(x^{\lceil \tau_{1\sim\zeta}(k)\rceil}, k\right) + B\left(x^{\lceil \tau_{1\sim\zeta}(k)\rceil}, k\right)u(k). \tag{45}$$

Owing to the full-actuation feature, the following important fact can be easily verified, which reveals a great advantage about the control of the FAS (44).

Theorem 1. Let Assumption 3 be met. Further, let $\varpi_i \geqslant 1,\ i = 1,2,\ldots,\eta$ be a set of integers, and $[A_i]_{0\sim\varpi_i-1} \in \mathbb{R}^{r_i\times\varpi_i r_i},\ i = 1,2,\ldots,\eta$ be a set of arbitrarily given matrices. Then the following controller:

$$\begin{cases} u(k) = B^{-1}\left(x_p^{\lceil \tau_{p,1\sim\zeta_p(k)}\rceil}(k)\,|_{p=1\sim\eta}, k\right)\left[-f\left(x_p^{\lceil \tau_{p,1\sim\zeta_p(k)}\rceil}(k)\,|_{p=1\sim\eta}, k\right) + u^*(k)\right], \\ u^*(k) = \begin{bmatrix} [A_1]_{0\sim\varpi_1-1}\, x_1^{\lceil 0\sim\varpi_1-1\rceil}(k) \\ [A_2]_{0\sim\varpi_2-1}\, x_2^{\lceil 0\sim\varpi_2-1\rceil}(k) \\ \vdots \\ [A_\eta]_{0\sim\varpi_\eta-1}\, x_\eta^{\lceil 0\sim\varpi_\eta-1\rceil}(k) \end{bmatrix} + v(k) \end{cases} \tag{46}$$

for system (44) produces the following constant linear closed-loop system:

$$x_p(k+1) = [A_p]_{0\sim\varpi_p-1} x_p^{\lceil 0\sim\varpi_p-1\rceil}(k) + v_p(k), \quad p = 1,2,\ldots,\eta, \tag{47}$$

with v_p, $p = 1,2,\ldots,\eta$ being defined as

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_\eta \end{bmatrix}, \quad v_p \in \mathbb{R}^{r_p}. \tag{48}$$

If we define

$$A_E = \text{blockdiag}\left(\left[A_p\right]_{0\sim\varpi_p-1}, p=1,2,\ldots,\eta\right), \tag{49}$$

then, obviously, the controller (46) can be more compactly written as

$$\begin{cases} u(k) = B^{-1}\left(x_p^{\lceil \tau_{p,1\sim\zeta_p(k)} \rceil}(k)\,|_{p=1\sim\eta}, k\right)\left[-f\left(x_p^{\lceil \tau_{p,1\sim\zeta_p(k)} \rceil}(k)\,|_{p=1\sim\eta}, k\right)+u^*(k)\right], \\ u^*(k) = A_E x_p^{\lceil 0\sim\varpi_p-1 \rceil}(k)\,|_{p=1\sim\eta} + v(k). \end{cases} \tag{50}$$

Furthermore, if written in a state-space form, the closed-loop system (47) appears as

$$x_p^{\lceil 0\sim\varpi_p-1 \rceil}(k+1) = \Psi\left(\left[A_p\right]_{0\sim\varpi_p-1}\right) x_p^{\lceil 0\sim\varpi_p-1 \rceil}(k) + \Gamma_{cp} v_p(k), \quad p=1,2,\ldots,\eta, \tag{51}$$

where

$$\Gamma_{cp} = \begin{bmatrix} I_{r_p} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad p=1,2,\ldots,\eta. \tag{52}$$

Please note that in this interconnected case, the set of integers $\varpi_i,\ i=1,2,\ldots,\eta$ may be chosen to be equal to each other, that is,

$$\varpi_1 = \varpi_2 = \cdots = \varpi_\eta = \varpi.$$

Specially, letting $\eta = 1$ in the above theorem, we have the following corollary.

Corollary 1. Let Assumption 2 be met. Further, let $\varpi \geqslant 1$ be an integer, and $A_{0\sim\varpi-1} \in \mathbb{R}^{\varpi\times r}$ be an arbitrarily given matrix. Then the following controller:

$$\begin{cases} u(k) = B^{-1}\left(x^{\lceil \tau_{1\sim\zeta(k)} \rceil}(k), k\right)\left[-f\left(x^{\lceil \tau_{1\sim\zeta(k)} \rceil}(k), k\right)+u^*(k)\right], \\ u^*(k) = A_{0\sim\varpi-1} x^{\lceil 0\sim\varpi-1 \rceil}(k) + v(k) \end{cases} \tag{53}$$

for system (45) produces the following constant linear closed-loop system:

$$x(k+1) = A_{0\sim\varpi-1} x^{\lceil 0\sim\varpi-1 \rceil}(k) + v(k). \tag{54}$$

For the control of the system (45), in the case of $\varpi = 1$, we have

$$u^*(k) = A_0 x(k) + v(k),$$

and the closed-loop system is

$$x(k+1) = A_0 x(k) + v(k).$$

While in the case of $\varpi = n$, we have

$$u^*(k) = A_{0\sim n-1} x^{\lceil 0\sim n-1 \rceil}(k) + v(k),$$

and the closed-loop system is

$$x(k+1) = A_{0\sim n-1} x^{\lceil 0\sim n-1 \rceil}(k) + v(k).$$

In practical applications, the integer ϖ may be chosen according to the requirement on the closed-loop system. For example, if there is a requirement on both $x(k)$ and $x(k-1)$, then ϖ should be chosen greater than 2.

Remark 1. For the more general non-affine time-delay FAS (37), a result similar to Theorem 1 still holds. The controller can be obtained using the differential homeomorphism property of the mapping

$\tilde{u} = g(x_p^{\lceil \tau_{p,1\sim\zeta_p(k)} \rceil}(k)|_{p=1\sim\eta}, k, u)$ as

$$\begin{cases} u(k) = g^{-1}\left(x_p^{\lceil \tau_{p,1\sim\zeta_p(k)} \rceil}(k)|_{p=1\sim\eta}, k, \tilde{u}(k)\right), \\ \tilde{u}(k) = -f\left(x_p^{\lceil \tau_{p,1\sim\zeta_p(k)} \rceil}(k)|_{p=1\sim\eta}, k\right) + u^*(k), \\ u^*(k) = \begin{bmatrix} [A_1]_{0\sim\varpi_1-1} x_1^{\lceil 0\sim\varpi_1-1 \rceil}(k) \\ [A_2]_{0\sim\varpi_2-1} x_2^{\lceil 0\sim\varpi_2-1 \rceil}(k) \\ \vdots \\ [A_\eta]_{0\sim\varpi_\eta-1} x_\eta^{\lceil 0\sim\varpi_\eta-1 \rceil}(k) \end{bmatrix} + v(k), \end{cases} \tag{55}$$

and the same closed-loop system as in (47) is obtained.

Remark 2. The design given in Theorem 1 is a decoupled one, and $u^*(k)$ given in the controller (46) is in fact a decentralized one. This is adequate to show the advantage of the FAS approach. However, a coupled design can also be carried out, with the expression of $u^*(k)$ in (46) replaced by

$$u^*(k) = Kx_p^{\lceil 0\sim\varpi_p-1 \rceil}(k)|_{p=1\sim\eta} + v(k),$$

where $K \in \mathbb{R}^{r\times\varkappa}$ with

$$\varkappa = r_1\varpi_1 + r_2\varpi_2 + \cdots + r_\eta\varpi_\eta$$

is a feedback gain. For details, one can refer to the coupled design given in [26] for the continuous-time case.

Remark 3. Another aspect associated with the control design of a discrete-time delay FAS is the complete parameterization of the designs. Since constant linear closed-loop systems are obtained, as done in the continuous-time case, complete parametric expressions for the feedback gains in the controllers as well as the closed-loop systems can be established, which provide all the design degrees of freedom to be used for further improving the system performance.

5.2 FASs with input delays

For simplicity, let us start with time-delay FASs without interconnections.

5.2.1 *FASs without interconnections*

Owing to the existence of the input delay, the nonlinear discrete-time delay FAS (29) is free of control, and runs the first h steps by itself in the following way.

When the system initial values $x^{\lceil \tau_{1\sim\zeta(0)} \rceil}(0)$ and $u(-h)$ are available, it follows from the system equation (29) that

$$x(1) = f\left(x^{\lceil \tau_{1\sim\zeta(0)} \rceil}(0), 0\right) + B\left(x^{\lceil \tau_{1\sim\zeta(0)} \rceil}(0), 0\right) u(-h). \tag{56}$$

By now, $x^{\lceil \tau_{1\sim\zeta(0)} \rceil}(1)$ is available. Further, when $u(-h+1)$ is available (often set to zero), we have

$$x(2) = f\left(x^{\lceil \tau_{1\sim\zeta(1)} \rceil}(1), 1\right) + B\left(x^{\lceil \tau_{1\sim\zeta(1)} \rceil}(1), 1\right) u(-h+1). \tag{57}$$

Carrying on with this process, when $u(-1)$ is given, we finally have

$$x(h) = f\left(x^{\lceil \tau_{1\sim\zeta(h-1)} \rceil}(h-1), h-1\right) + B\left(x^{\lceil \tau_{1\sim\zeta(h-1)} \rceil}(h-1), h-1\right) u(-1). \tag{58}$$

Therefore, by now,

$$x^{\lceil \tau_{1\sim\zeta(k)} \rceil}(k), \quad 0 \leqslant k \leqslant h \tag{59}$$

are all available. From there we can add the effect of control.

Similar to the control of the nonlinear discrete-time FAS (45) with state delay only, for control of the nonlinear discrete-time FAS (29) with time delays, we naturally design the following controller utilizing the full-actuation feature of the system:

$$\begin{cases} u(k-h) = B^{-1}\left(x^{\lceil \tau_{1\sim\zeta(k)} \rceil}(k), k\right)\left[-f\left(x^{\lceil \tau_{1\sim\zeta(k)} \rceil}(k), k\right) + u^*(k-h)\right], \\ u^*(k-h) = [A]_{0\sim\varpi-1} x^{\lceil 0\sim\varpi-1 \rceil}(k) + v(k), \ k \geqslant h. \end{cases} \tag{60}$$

With the starting values in (59), we can get

$$\begin{cases} u(0) = B^{-1}\left(x^{\lceil \tau_1 \sim \varsigma(h) \rceil}(h), h\right)\left[-f\left(x^{\lceil \tau_1 \sim \varsigma(h) \rceil}(h), h\right) + u^*(0)\right], \\ u^*(0) = [A]_{0\sim\varpi-1} x^{\lceil 0\sim\varpi-1 \rceil}(h) + v(h). \end{cases} \tag{61}$$

To further get $u(1)$, the value of $x(h+1)$ may be needed, and can be obtained via the open-loop system (29) as

$$x(h+1) = f\left(x^{\lceil \tau_1 \sim \varsigma(h) \rceil}(h), h\right) + B\left(x^{\lceil \tau_1 \sim \varsigma(h) \rceil}(h), h\right) u(0). \tag{62}$$

Hence $x^{\lceil \tau_1 \sim \varsigma(h) \rceil}(h+1)$ is now available, and we can then further get $u(1)$ using (60) again.

With such a process, we can realize the control of the system.

Based on the above analysis, we can now give the following result.

Theorem 2. Let Assumption 4 be met. Further, let ϖ be an integer satisfying $1 \leqslant \varpi \leqslant n$, and $[A]_{0\sim\varpi-1} \in \mathbb{R}^{r\times\varpi r}$ be an arbitrarily given matrix. Then the following controller:

$$\begin{cases} u(k-h) = B^{-1}\left(x^{\lceil \tau_1 \sim \varsigma(k) \rceil}(k), k\right)\left[-f\left(x^{\lceil \tau_1 \sim \varsigma(k) \rceil}(k), k\right) + u^*(k-h)\right], \\ u^*(k-h) = A_{0\sim\varpi-1} x^{\lceil 0\sim\varpi-1 \rceil}(k) + v(k), \\ x(k) = f\left(x^{\lceil \tau_1 \sim \varsigma(k-1) \rceil}(k-1), k-1\right) + B\left(x^{\lceil \tau_1 \sim \varsigma(k-1) \rceil}(k-1), k-1\right) u(k-h-1), \\ k \geqslant h, \end{cases} \tag{63}$$

for system (29) produces the following constant linear closed-loop system:

$$x(k+1) = A_{0\sim\varpi-1} x^{\lceil 0\sim\varpi-1 \rceil}(k) + v(k), \quad k \geqslant h. \tag{64}$$

Please note that the initial value of the closed-loop system is $x^{\lceil 0\sim\varpi-1 \rceil}(h)$, which is provided by the first h steps self-running of the system.

Example 1. Consider a system in the form of

$$x(k+1) = f(x(k-\tau(k))) + u(k-1),$$

where $x, u \in \mathbb{R}^r$, and the state time delay is given by

$$\tau(k) = \begin{cases} 2, \text{ if } \operatorname{mod}(k,3) = 0, \\ 1, \text{ if } \operatorname{mod}(k,3) = 1, \\ 0, \text{ if } \operatorname{mod}(k,3) = 2, \end{cases}$$

where $\operatorname{mod}(k,3)$ represents the remainder operation.

By Theorem 2, the controller is designed as, when $\varpi = 2$,

$$\begin{cases} u(k-1) = -f(x(k-\tau(k))) + A_0 x(k) + A_1 x(k-1), \\ x(k) = f(x(k-1-\tau(k-1))) + u(k-2), \quad k \geqslant 1. \end{cases}$$

With the following initial values:

$$x(-2), \ x(-1), \ x(0), \ \text{and} \ u(-1),$$

we can obtain, for $k = 1$,

$$\begin{cases} u(0) = -f(x(0)) + A_0 x(1) + A_1 x(0), \\ x(1) = f(x(-2)) + u(-1), \end{cases}$$

and, further, for $k = 2$,

$$\begin{cases} u(1) = -f(x(2)) + A_0 x(2) + A_1 x(1), \\ x(2) = f(x(0)) + u(0). \end{cases}$$

Carrying on this process, we can derive $(u(k), x(k+1)), k = 0, 1, \ldots.$

It can be easily verified that the closed-loop system is given by

$$x(k+1) = A_0 x(k) + A_1 x(k-1), \quad k \geqslant 1$$

whose stability is determined by that of

$$\Psi(A_{0\sim 1}) = \begin{bmatrix} A_0 & A_1 \\ I_r & 0 \end{bmatrix}.$$

5.2.2 *FASs with interconnections*

In this subsection, let us consider the control of the nonlinear discrete-time delay FAS (30) with interconnections.

Here we skip the description of the first h steps of self-running, and directly assume the availability of $x_p^{\lceil \tau_{1\sim\zeta_p(h)} \rceil}(h)\,|_{p=1\sim\eta}$. Hence the following controller:

$$\begin{cases} u(k-h) = B^{-1}\left(x_p^{\left\lceil \tau_{p,1\sim\zeta_p(k)} \right\rceil}(k)\,|_{p=1\sim\eta}, k\right)\left[-f\left(x_p^{\left\lceil \tau_{p,1\sim\zeta_p(k)} \right\rceil}(k)\,|_{p=1\sim\eta}, k\right) + u^*(k-h)\right], \\ u^*(k-h) = A_E x_p^{\lceil 0\sim\varpi_p-1 \rceil}(k)\,|_{p=1\sim\eta} + v(k), \\ x_{1\sim\eta}(k) = f\left(x_p^{\left\lceil \tau_{p,1\sim\zeta_p(k)} \right\rceil}(k-1)\,|_{p=1\sim\eta}, k-1\right) \\ \qquad\qquad +B\left(x_p^{\left\lceil \tau_{p,1\sim\zeta_p(k)} \right\rceil}(k-1)\,|_{p=1\sim\eta}, k-1\right) u(k-h-1), \\ k \geqslant h \end{cases} \tag{65}$$

makes sense.

Similar to the above case without interconnections, we can give the following important fact about the control of the FAS (32) with interconnections.

Theorem 3. Let Assumption 5 be met. Further, let $\varpi_i \geqslant 1,\ i = 1, 2, \ldots, \eta$ be a set of integers, and $[A_i]_{0\sim\varpi_i-1} \in \mathbb{R}^{r_i \times \varpi_i r_i},\ i = 1, 2, \ldots, \eta$ be a set of arbitrarily given matrices. Then the controller (65) for system (32) produces the following constant linear closed-loop system:

$$x_p(k+1) = [A_p]_{0\sim\varpi_p-1} x_p^{\lceil 0\sim\varpi_p-1 \rceil}(k) + v_p(k), \quad k \geqslant h,\ p = 1, 2, \ldots, \eta \tag{66}$$

with v_p, $p = 1, 2, \ldots, \eta$ being defined by (48).

Theoretically, any discrete-time delay FAS in the form of (32) with interconnections can be written in the form of (29) for a discrete-time delay FAS without interconnections; just like a discentralized large-scale system can always be written into a centralized system form with a large dimension. As a matter of fact, the discrete-time delay FAS in the form of (32) with interconnections is often used to describe complicated large-scale systems. However, for only the purpose of demonstrating the proposed approach, let us present a simple example.

Example 2. Consider the following FAS:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} \sin(x_2(k-\tau(k))) \\ x_1(k)\, x_2(k-3) \end{bmatrix} + \begin{bmatrix} 1 + x_1^2(k) & 0 \\ 3ke^{-k} & 1 \end{bmatrix} u(k-2),$$

where the time-varying delay is defined as

$$\tau(k) = \begin{cases} 0, \text{ if } |\cos(10k)| \leqslant \frac{\sqrt{2}}{2}, \\ 1, \text{ if } |\cos(10k)| > \frac{\sqrt{2}}{2}. \end{cases}$$

Therefore, we have $\eta = 2$, and

$$f(k) = f(x_1(k), x_2(k-\tau(k)), x_2(k-3)) = \begin{bmatrix} \sin(x_2(k-\tau(k))) \\ x_1(k)\, x_2(k-3) \end{bmatrix},$$

$$B(k) = B(x_1(k), k) = \begin{bmatrix} 1 + x_1^2(k) & 0 \\ 3k\mathrm{e}^{-k} & 1 \end{bmatrix}.$$

Clearly, in this system we have $\zeta_1 = 0, \zeta_2 = 2$; that is, we do not have a delay in the state x_1, and have two delays in the state x_2, namely, $\tau_{2,1}(k) = \tau(k)$ given above with $\bar{\tau} = 1$, and $\tau_{2,2}(k) = 3$. While the input delay is $h = 2$. Therefore, in the design we can choose

$$\varpi_1 = 1, \ \varpi_2 = 4,$$

and the controller can be designed as

$$\begin{cases} u(k-2) = B^{-1}(k)\left[-f(k) + u^*(k-2)\right], \\ u^*(k-2) = \begin{bmatrix} a_{10}x_1(k) \\ \sum\limits_{i=0}^{3} a_{2i}x_2(k-i) \end{bmatrix} + v(k), \\ x_{1\sim 2}(k) = f(k-1) + B(k-1)u(k-3), \\ k \geqslant 2. \end{cases} \tag{67}$$

By two steps of self-running, the open-loop system provides the following initial values for the above controller:

$$x_1(2), \ x_2(2), \ x_2(1), \ x_2(0), \ x_2(-1), \ u(-1).$$

Hence the controller works and a closed-loop system is obtained which contains the following two decoupled linear systems:

$$x_1(k+1) = a_{10}x_1(k) + v_1(k), \quad k \geqslant 2,$$

and

$$x_2(k+1) = \sum_{i=0}^{3} a_{2i}x_2(k-i) + v_2(k), \ k \geqslant 2.$$

Of course, the system parameters a_{10} and $a_{2i}, i = 0, 1, 2, 3$, should be chosen to make the above two linear systems stable.

To end this section, let us consider the controller of the extended FAS (38) or (41). A straight forward extension can be given as follows:

$$\begin{cases} u(k) = B_{x,u}^{-1}(k)\left[-f_{xu}(k) + u^*(k)\right], \\ u^*(k) = K_{1\sim\eta}x_p^{\lceil 0\sim\varpi_p-1\rceil}(k)\left|_{p=1\sim\eta}\right. + v(k), \\ x_{1\sim\eta}(k) = f_{x,u}(k-1) + B_{x,u}(k-1)u(k-h-1), \\ k \geqslant h, \end{cases} \tag{68}$$

which produces the following constant linear closed-loop system:

$$x_p(k+1)\left|_{p=1\sim\eta}\right. = K_{1\sim\eta}x_p^{\lceil 0\sim\varpi_p-1\rceil}(k)\left|_{p=1\sim\eta}\right. + v(k). \tag{69}$$

As a linear system, the closed-loop system (69) can be easily made stable by properly selecting the feedback gain $K_{1\sim\eta}$. For a systematic method, one can refer to [26]. Particularly, when $K_{1\sim\eta}$ is chosen as

$$K_{1\sim\eta} = \mathrm{blockdiag}\left(\left[A_p\right]_{0\sim\varpi_p-1}, p = 1, 2, \ldots, \eta\right),$$

the closed-loop system (69) reduces to (47), or equivalently the state-space form (51).

Parallel to (68), the controller for the more general non-affine FAS (43) can also be easily given.

To finish this section, let us make a few remarks.

Remark 4. Clearly, the matrix $A_{0\sim\varpi-1}$, or the series of matrices $\left[A_p\right]_{0\sim\varpi_p-1}$, $p = 1, 2, \ldots, \eta$, should be chosen such that the state-space systems in (51) are asymptotically stable. Noting that

$$\Psi(A_{0\sim n-1}) = \Psi(0_{0\sim n-1}) - \Gamma_{\mathrm{c}}A_{0\sim n-1}$$

and $\left[\Psi(0), \Gamma_{\mathrm{c}}\right]$ is controllable, the solution to the feedback gains in the proposed controllers really reduces to a problem of eigenstructure assignment, and hence can be easily solved (see [26] for the continuous-time case).

6 Concluding remarks

This paper sufficiently demonstrates that, as a general approach for control system design, the FAS approach can be well extended to the case of discrete-time delay systems.

Firstly, it is shown that a general FAS model representation exists for discrete-time systems with time-varying state delays and a constant input delay, which covers the discrete-time FAS model proposed in [29] as a special case. More general forms of the FAS models also exist. It is shown by two specific classes of systems that certain discrete-time delay systems in a state-space form can indeed be converted into discrete-time FASs with delays.

Secondly, it is proven that the control strategy based on the full-actuation feature of a FAS works as well on the type of proposed discrete-time delay FASs. As expected, in the case that the FASs have input delays, a prediction mechanism is needed and can be easily constructed based on the open-loop system. As an advantage of the FAS approach, the closed-loop system turns out to be constant linear with an arbitrarily assignable eigenstructure.

The results proposed in the paper are very fundamental and are also vitally important since they lay a solid basis for the FAS approaches to discrete-time delay systems. These basic results allow us to further address, as in the continuous-time FAS case, many design problems associated with discrete-time delay systems, such as robust control [22, 24], adaptive control [23, 24], disturbance rejection [25], optimal control [27], and signal tracking control [28]. Furthermore, the results can also be generalized into the time-varying input delay case. Particularly, the sub-fully actuated cases, that is, the cases in which Assumptions 2–5 are not satisfied, will be considered in the second part of the paper.

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