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Common quantitative characteristics of music melodies — pursuing the constrained entropy maximization casually in composition

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Music with a beautiful melody has been appreciated by humankind for thousands of years. Almost all music is generally tonal, e.g., Western music originated from ancient Greece and developed for more than 300 years from Baroque to classical, romantic, and modern music. Tonal music has a tonal center with a melody named tonic, and it is based on a specific pitch series called mode scale and harmonic progressions. On the contrary, atonal music experimented from around the 1900s, e.g., the twelve-tone system advocated by Schoenberg [1], does not have a tonal center and a melody accordingly. A melody of tonal music consists of the permutation and combination of pitches in series, where the pitch marks the acoustic frequency of the note and the rhythm marks the duration of the note [2]. It is desirable to know if there are some common characteristics, objective and quantitative, embedded in a melody.

Many efforts were made to quantify music, and a general pitch interval representation is proposed to represent pitch structures of a wide variety of music styles (see more details in Supplementary Materials (SM) I Section 1) [3, 4]. Nevertheless, few efforts were made to connect the essence of the melody composition and the quantitative characteristics of melodic intervals, namely melody variations. This study focuses on modeling and analyzing the mathematical characteristics of melody variations based on composition theory.

The melody variations are represented by the number of semitones (distance between adjacent keys on a piano) based on the twelve-tone scale. The variation of a full octave has 12 semitones [2]. To facilitate analysis, the descending interval is taken to be the same as the ascending interval, and the rhythm is not considered. We extensively analyze various styles of tonal music, mainly from Baroque to modern pops [1] (see the database in Supplementary Dataset S1 and the analysis in SM II Subsection 1.4), and find the three mathematical characteristics of music melodies. These characteristics are quantified based on the music composition theories taught in all music schools worldwide. From these three characteristics, a constrained entropy maximization problem can be formulated. We can derive that the melody variations of tonal music observe the power law consistent with the observations difficult to explain in three decades. This study helps reveal the common characteristics of music melodies and promote the application of artificial intelligence (AI) in composition.

Stationary distribution of melody variations. The first characteristic is the stationary distribution of note sequences. That is, if the notes of a music piece are considered as a stochastic sequence, the probability distribution of a particular melodic interval is stationary; in other words, the profile of melodic intervals approximates to constant as the notes move on. The reason behind this is the extensive use of composition techniques such as repetition, sequence, retrograde, and inversion (see some examples of score demonstrations of these repetitive techniques in SM II Figure SMII 1) [5]. The phrases, sentences, and paragraphs of the tonal music often recur. For example, small ternary, minuet, and scherzo are structurally A-B-A′, where A′ denotes a modified part originated from A. Rondo (typically A-B-A-C-A) and sonata have a similar repetitive structure [5]. Such progression tends to the steadiness of melodic interval sequences.

Moreover, a music piece is required to be structurally coherent and perceived as a whole. Motive and theme should be like a germ that keeps growing [5]. To achieve this goal, composers would not use melodic intervals arbitrarily [5, 6]. This results in the probability of a particular melodic interval in semitones to tend to be constant as the note sequence moves on

$$\lim_{N\to+\infty} p(i,N) = p(i), \tag{1}$$

where $p(i,N)$ is the probability of the i semitone with N notes. This implies that, no matter how the probability of a particular melodic interval starts in the melody, it tends to be constant when the melody develops. The evidence of

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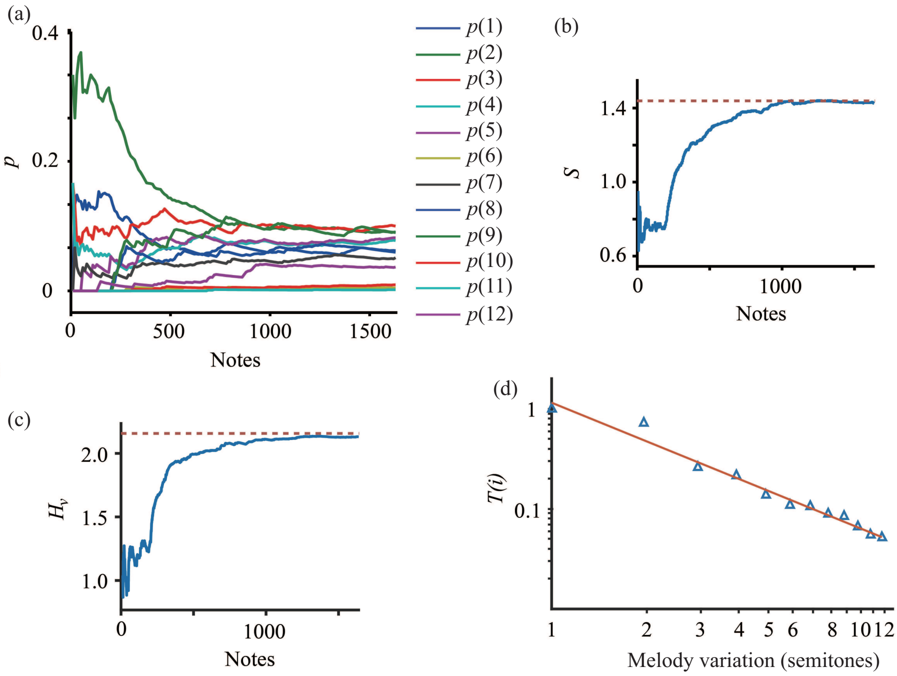

Figure 1 (Color online) (a) Frequencies of melodic intervals (semitones) with the melody developing. (b) The smoothness of the melody curves S tends to a small "smoothness attractor" as the melody develops. (c) The melody variation entropy H_v achieves its maximum with the melody developing. (d) The CCDF of the melody variations (semitones) of the tonal music observes the power law. This example is 'Concerto for flute and harp in C Major' by Mozart. More examples are shown in SM I Section 2 and Supplementary Datasets S2–S5.

this characteristic is clear since the data of various works of different styles supports this characteristic as shown in SM I Subsection 2.1, Supplementary Dataset S2, and a typical example in Figure 1(a).

Smoothness of melody curves. The second characteristic describes the smoothness of melody curves. In composition theory, melodic intervals are classified as wide intervals (i larger than 5) and narrow intervals (i from 1 to 4). The melody develops primarily with narrow intervals, especially step progression (i from 1 to 2) [2]. Second is the building unit of the melody and transitions of the larger melodic connections necessary to pursue a clear melodic line, enhance the structure, and promote the melody progress [6] (see the natural sequence of melodic intervals by Hindemith's theory in SM I Section 1 and Subsection 2.2).

Moreover, nonchord tones, including changing, passing, suspension, neighboring, anticipation, and free tone, play a vital role in melody development. Most nonchord tones form variations of seconds ($i = 1$, 2) versus the previous or next tone. In addition, decorations are very common in a composition to enrich the melody as a moving and colorful unity and generally produce step intervals [2, 5].

The usage of the above techniques causes the curve of melody variation as the degree of pitch variation over note sequences to be smooth [5, 6]. The definition of the curve smoothness S is based on the Hölder exponent h_x of the fractal function $f(x)$ which represents the degree of irregularity around x in the graphs of fractal functions and is defined as [7]

$$h_x = \lim_{\epsilon\to 0} \inf \left\{ \frac{\log|f(x)-f(y)|}{\log|x-y|} : y \in B(x;\epsilon) \right\}, \tag{2}$$

where $B(x;\epsilon)$ denotes the ϵ neighborhood of x. A melody curve with a large number of notes is wiggly and zigzag with melody variations and has the same nature as the graphs of fractal functions. Therefore, based on the analysis in SM II Subsection 1.2, the curve smoothness S is usually quantified as the expectation of $\log i$ [7]:

$$S = \mathrm{E}(\log i) = \sum_i p(i) \log i \to m. \tag{3}$$

Moreover, the curve smoothness should approach a small constant m. The smoothness of the melody curves based on this definition is supported by the curves of various studies in SM I Subsection 2.2, Supplementary Dataset S3, and a typical example in Figure 1(b).

Entropy maximization of melody variations. The third characteristic is the entropy maximization of melody variations. Similar to thermodynamics and information entropy, which describe the randomness of a system or the uncertainty of information sources, we define the entropy of melody variations as follows to measure their diversity:

$$H_v = -\sum_i p(i) \log p(i). \tag{4}$$

Tonal music has the nature of "developing variation". The basic motive is modified or developed by wrapping, filling, transformation, expanding, reducing, etc. [2, 5]. Composers generally try to have diversified variations, and therefore, the entropy H_v must be maximized subject to the first two characteristics, which is supported by the data of various studies in SM I Subsection 2.3, Supplementary Dataset S4, and a typical example in Figure 1(c).

Mathematical model. Based on the above three characteristics, the following constrained entropy maximization problem can be formulated:

$$\max H_v = -\sum_i p(i) \ln p(i), \tag{5}$$

subject to

$$\begin{cases} \sum_i p(i) = 1, \\ \sum_i p(i) \ln i = m, \end{cases} \tag{6}$$

where $p(i)$ is the probability density function of the melodic interval for semitone i. In actual music pieces, the range of melodic interval i is mainly from 1 to 19 and no more than 24 in general.

The constrained entropy maximization problem is similar to the optimal control problems intensively studied in the 1970s to 1980s, and the solution to $p(i)$ is thus obtained by the calculus of variations.

By using the Lagrange multiplier method, we obtain

$$\begin{aligned} L = & -\sum_i [p(i) \ln p(i) + \lambda_0 p(i) + \lambda_1 p(i) \ln i] \\ & + \lambda_0 + \lambda_1 m, \quad \lambda_0, \lambda_1 > 0. \end{aligned} \tag{7}$$

Let

$$J = \sum_i [p(i) \ln p(i) + \lambda_0 p(i) + \lambda_1 p(i) \ln i] = \sum_i L_i. \tag{8}$$

Based on the calculus of variations, we have

$$\delta J = \sum_i \left[\frac{\partial L_i}{\partial p(i)} \delta p(i) \right] = 0, \tag{9}$$

where $\delta p(i)$ is the first-order variation of $p(i)$.

If i is continuous with $i \in (0, \infty]$, the complementary cumulative distribution function (CCDF) is

$$T(i) = \int_i^\infty p(s) \mathrm{d}s = c i^{-D}, \tag{10}$$

where $c = \frac{C_0}{\lambda_1 - 1} > 0$ and $D = \lambda_1 - 1 > 0$. This is exactly the power law function. However, the melody variations in semitone i are positive integers, using the approximate integration method (see details in SM II Subsection 1.3) to compute the CCDF of melody variations i as follows:

$$\begin{aligned} T(i) = P(I \geqslant i) = \sum_{s=i}^{I_b} p(s) = \sum_{s=i}^{I_b} C_0 s^{-\lambda_1} \\ = c i^{-D} + q_{I_b} + e(s, i), \end{aligned} \tag{11}$$

where $q_{I_b} = -\frac{C_0}{\lambda_1 - 1} I_b^{-\lambda_1 + 1} < 0$, $e(s, i) \leqslant \frac{-C_0 \lambda_1}{2} \frac{(I_b - i)^{-\lambda_1 + 1}}{I_b - i + 1}$, $i \in \{I_a, I_a + 1, \ldots, I_T\}$, I_b is the upper bound of i, $I_T \leqslant I_b$ generally equals 12, and integer I_a is the lower bound of i in the whole piece. The numerical results show that the error term $e(s, i)$ and the constant term q_{I_b} can be ignored with the actual parameters in (11) (see details in SM II Subsection 1.3). Thus, we derive that the CCDF $T(i)$ of melodic intervals follows the power law as many studies have found [8, 9]:

$$T(i) = P(I \geqslant i) = 1 - F(i) = c i^{-D}, \tag{12}$$

where $F(i)$ is the probability distribution function and c and D are constants. Clearly, $\log T(i)$ versus $\log i$ is an affine function or a straight line geographically. Therefore, we state that melody variations or melodic intervals of tonal music observe the power law (a typical example is in Figure 1(d), and more examples are in SM I Section 3 and Supplementary Dataset S5). The reason why some contemporary atonal music, such as the twelve-tone system, does not observe the power law is given in SM I.

In conclusion, the three mathematical characteristics of a melody, which are the stationary distribution of note sequences, the smoothness of the melody curves, and the entropy maximization of melody variations, are embedded in the composition theory and techniques taught in music schools worldwide. Based on these characteristics, it turns out that composers of tonal music in the past 200–300 years all pursued the constrained entropy maximization of melody variations. The power law of music melody variations can thus be derived based on these three characteristics, consistent with the observations in the past three decades [8, 9].

The results in this study fill the research gap of quantitative analysis on music melodies. A new model is developed to explain why melody variations of tonal music observe the power law, which is quite universal for many natural and artificial systems (see details in SM I Section 5). The finding can help us comprehend music with scientific quantification, can serve as a basis for investigating if there is any physiological reason why the public without music training likes tonal music, and provides a necessity for music composition aided by AI since a mechanism built in the AI aided composition methods should exclude the note series that do not observe the power law. This study provides a bridge between information science and music art.

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Supporting information Videos and other supplemental documents. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

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