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Supplementary Materials (SM)
for
Common quantitative characteristics of music melodies – pursuing the constrained entropy maximization casually in composition

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- Concert for Bridging Art and Science - IEEE CASE 2017

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Supplementary Materials I (SM I)

1 Introduction

Music with a beautiful melody has been appreciated by humankind for thousands of years. In ancient China, it was debated whether music was purely spiritual or was related to the physiology of human beings more than 2000 years ago [1]. A fundamental question has been why the public would appreciate the music with beautiful melodies and what in common music composers have been pursuing. To answer this question, it is important to understand how melodies and harmonies are composed and what “a beautiful melody” really means in science [2, 3]. It is desirable to know if there are some common characteristics, objective and quantitative, embedded in a melody [4].

Almost all music is generally tonal, e.g., Western music originated from ancient Greece and developed for more than 300 years from Baroque to Classical, Romantic, and Modern music. Tonal music has a tonal center with a melody named tonic, and it is based on a specific pitch series called mode scale and harmonic progressions. On the contrary, atonal music experimented from around the 1900s, e.g., the twelve-tone system advocated by Schoenberg [5, 6] does not have a tonal center, and a melody accordingly. A melody of tonal music consists of the permutation and combination of pitches in series, where the pitch marks the acoustic frequency of the note and the rhythm marks the duration of the note [5, 7-10]. In music theory, the distance between two pitches is defined as the musical interval measured by scales (the number of semitones) such as minor second (1 semitones), major second (2 semitones), minor third (3 semitones), major third (4 semitones), etc. The interval between two adjacent notes in a melody is called melodic interval [7], namely, melody variation.

In Hindemith’s theory, musical intervals have a natural sequence driven by harmonic and melodic forces like building stones with different strength, hardness, and density [11]. Much of our enjoyment of music is related to the balance of predictability and surprise. Deterministic repetitions and narrow intervals make music predictable and comprehensible, while random fluctuations and diversified intervals would let us feel unexpected refreshing [10]. Therefore quantifying the characteristics of melodic intervals helps us understand the essence melody creation in composition.

Many efforts were made to quantify music, and the elements such as timbre, rhythm, pitch, and harmony are analyzed by mathematical methods [12-17]. A General Pitch Interval Representation is proposed to represent pitch structures of a wide variety of music styles [18]. Melodic intervals are studied on their relations to consonance and predicting music emotion [19, 20]. Nevertheless, few efforts were made to connect the essence of the melody composition and the quantitative characteristics of melodic intervals [13, 14]. This paper focuses on modeling and analyzing the mathematical characteristics of melody variations based on composition theory.

The melody variations are represented by the number of semitones (distance between adjacent keys on a piano) based on the twelve-tone scale [7, 18]. The variation of a full octave has 12 semitones. To facilitate analysis, the descending interval is taken to be the same as the ascending interval, and the rhythm is not considered. We extensively analyze various styles of tonal music, mainly from Baroque to modern pops [6] (see the database in Supplementary Dataset S1 and the analysis in Supplementary Materials II Section 1.4) and find the three mathematical characteristics of music melodies. These characteristics are quantified based on the music composition theories taught in all music schools worldwide. Firstly, if the notes of a music piece are considered as a stochastic sequence, it is stationary or its mean and other moments tend to be time-invariant. Second, the curve of melody in terms of the melodic variations is smooth, and its mathematical smoothness tends to be a small constant. Third, the melodic variations are diversified with their entropy maximized. From these three characteristics, a constrained functional optimization problem or a constrained optimal control problem can be formulated. It is then discovered in this paper that all composers of different styles in the past two, three hundred years were casually pursuing the constrained entropy maximization in melody composition. Based on this formulation, we can derive that the music melody variations observe the Power Law by solving the problem based on the calculus of variation. The result is consistent with the observations difficult to explain in three decades. This study helps reveal the common characteristics of music melodies and promote the application of artificial intelligence (AI) in composition.

2 Three mathematical characteristics in tonal music

2.1 Stationary distribution of melody variations

The first characteristic is the stationary distribution of note sequences. That is, if the notes of a music piece are considered as a stochastic sequence, the probability distribution of a particular melodic interval is stationary; in other words, the profile of melodic intervals approximates to constant as the notes move on. The reason behind this is the extensive use of composition techniques such as repetition, sequence, retrograde, and inversion (see some examples of score demonstrations of these repetitive techniques in Supplementary Materials II Figure SMII 1) [9, 10]. The phrases, sentences, and paragraphs of the tonal music often recur. For example, small ternary, minuet, and scherzo are structurally A-B-A', where A' denotes a modified part originated from A. Rondo (typically A-B-A-C-A) and sonata have a similar repetitive structure. Such progression tends to the steadiness of melodic interval sequences.

Moreover, a music piece is required to be structurally coherent and perceived as a whole [9]. Motive and theme should be like a germ that keeps growing [10]. To achieve this goal, composers would not use melodic intervals arbitrarily [8-10]. This results in the probability of a particular melodic interval in semitones to tend to be constant as the note sequence moves on:

$$\lim_{N\to\infty} p(i,N) = p(i), \qquad \text{(SMI 1)}$$

where $p(i,N)$ is the probability of the i semitone with N notes. This implies that, no matter how the probability of a particular melodic interval starts in the melody, it tends to be constant when the melody develops. The evidence of this characteristic is clear since the data of various works of different styles supports this characteristic, as shown in Supplementary Dataset S2 and the six examples in Figure SMI 1.

2.2 Smoothness of melody curves

The second characteristic describes the smoothness of melody curves. In composition theory, melodic intervals are classified as wide intervals (i larger than 5) and narrow intervals (i from 1 to 4). The melody develops primarily with narrow intervals, especially step-progression (i from 1 to 2) [7-9, 21]. Second is the building unit of the melody and transitions of the larger melodic connections [8, 9, 11, 21] necessary to pursue a clear melodic line, enhance the structure, and promote the melody progress [9] (see the natural sequence of melodic intervals by Hindemith's theory in Figure SMI 2).

Moreover, nonchord tones, including changing, passing, suspension, neighboring, anticipation, and free tone, play a vital role in melody development. Most nonchord tones form variations of seconds ($i = 1$, 2) versus the previous or next tone [9, 11, 21]. In addition, decorations are very common in a composition to enrich the melody as a moving and colorful unity and generally produce step intervals [7, 10].

The usage of the above techniques causes the curve of melody variation as the degree of pitch variation over note sequences to be smooth [8]. The definition of the curve smoothness S is based on the $H\ddot{o}lder$ exponent h_x of fractal function $f(x)$ [22-24] which represents the degree of irregularity around x in the graphs of fractal functions and is defined as

$$h_x = \lim_{\varepsilon \to 0} \inf \left\{ \frac{\log|f(x) - f(y)|}{\log|x - y|} : y \in B(x; \varepsilon) \right\}, \quad \text{(SMI 2)}$$

where $B(x; \varepsilon)$ denotes the ε-neighborhood of x. A melody curve with a large number of notes is wiggly and zigzag with melody variations and has the same nature as the graphs of fractal functions [25]. Therefore, based on the analysis in Supplementary Materials II Section 1.2, the curve smoothness S is usually quantified as the expectation of $\log i$

$$S = E(\log i) = \sum_i p(i) \log i \to m, \quad \text{(SMI 3)}$$

Moreover, the curve smoothness should approach a small constant m. The smoothness of the melody curves based on this definition is supported by the curves of various works (Figure SMI 3) and those in the Supplementary Dataset S3.

2.3 Entropy maximization of melody variations

The third characteristic is the entropy maximization of melody variations. Similar to thermodynamics and information entropy [26], which describe the randomness of a system or the uncertainty of information sources, define the entropy of melody variations as follows to measure their diversity:

$$H_v = -\sum_i p(i) \log p(i) \quad \text{(SMI 4)}$$

Tonal music has the nature of "developing variation". The basic motive is modified or developed by wrapping, filling, transformation, expanding, reducing, etc. [7, 9, 10]. Composers generally try to have diversified variations, and therefore, the entropy H_v must be maximized subject to the first two characteristics, which is supported by the data of various works (Figure SMI 4).

Are the pieces of evidence of the above three characteristics sufficient enough with the supporting data from the tonal music pieces in various styles and periods as shown above and in Supplementary Datasets S2-S4? The pieces of evidence are reinforced by looking into the resulting feature due to the three characteristics. In fact, the following constrained functional optimization problem can be formulated:

$$\max H_v = -\sum_i p(i) \ln p(i), \quad \text{(SMI 5)}$$

subject to

$$\begin{cases} \sum_i p(i) = 1 \\ \sum_i p(i) \ln i = m, \end{cases} \tag{SMI 6}$$

where $p(i)$ is the probability density function of the melodic interval for semitone i. In actual music pieces, the range of melodic interval i is mainly from 1 to 19 and no more than 24 in general.

3 Mathematical model and the Power Law derivation

Based on the above three characteristics, the constrained entropy maximization problem is similar to the optimal control problems intensively studied in the 1970s to 1980s, and the solution to $p(i)$ is thus obtained by the calculus of variations [27, 28].

By using the Lagrange multiplier method, we obtain

$$L = -\sum_i [p(i) \ln p(i) + \lambda_0 p(i) + \lambda_1 p(i) \ln i] + \lambda_0 + \lambda_1 m;\ \lambda_0, \lambda_1 > 0. \tag{SMI 7}$$

Let

$$J = \sum_i [p(i) \ln p(i) + \lambda_0 p(i) + \lambda_1 p(i) \ln i] = \sum_i L_i. \tag{SMI 8}$$

Based on the calculus of variations, we have

$$\delta J = \sum_i \left[\frac{\partial L_i}{\partial p(i)} \delta p(i)\right] = 0, \tag{SMI 9}$$

where $\delta p(i)$ is the first-order variation of $p(i)$.

If i is continuous with $i \in (0, \infty]$, the complementary cumulative distribution function (CCDF) is

$$T(i) = \int_i^\infty p(s) ds = c i^{-D}, \tag{SMI 10}$$

where $c = \frac{C_0}{\lambda_1 - 1} > 0$ and $D = \lambda_1 - 1 > 0$. This is exactly the Power Law function. However, the melody variations in semitone i are positive integers, using the approximate integration method (see details in Supplementary Materials II Section 1.3) to compute the CCDF of melody variations i as follows:

$$T(i) = P(I \geq i) = \sum_{s=i}^{I_b} p(s) \tag{SMI 11}$$

$$= \sum_{s=i}^{I_b} C_0 s^{-\lambda_1} \tag{SMI 12}$$

$$= c i^{-D} + q_{I_b} + e(s, i), \tag{SMI 13}$$

where $q_{I_b} = -\frac{C_0}{\lambda_1 - 1} I_b{}^{-\lambda_1 + 1} < 0, e(s, i) \leq \frac{-C_0 \lambda_1}{2} \frac{(I_b - i)^{-\lambda_1 + 1}}{I_b - i + 1}, i \in \{I_a, I_a + 1, \ldots, I_T\}$, I_b is the upper bound of i, $I_T \leq I_b$ generally equals to 12 and integer I_a is the lower bound of i in the whole piece.

The numerical results show that the error term $e(s, i)$ and the constant term q_{I_b} can be ignored with the actual parameters in (SMI 13) (see details in Supplementary Materials II Section 1.3).

Thus, we derive that the CCDF $T(i)$ of melodic intervals follows the Power Law as many studies have found [29-31]

$$T(i) = P(I \geq i) = 1 - F(i) = ci^{-D}, \quad \text{(SMI 14)}$$

where $F(i)$ is the probability distribution function, and c and D are constants. Clearly $\log T(i)$ versus $\log i$ is an affine function or a straight line geographically. Therefore, we state that melody variations or melodic intervals of tonal music observe the Power Law consistent with our observation. We extensively analyzed the CCDFs of the music melody variations measured by semitones for various styles of tonal music, mainly from Baroque to modern pops [6]. Some typical examples are shown in Figure SMI 5A-D, and the rest observations are in Supplementary Dataset S5.

4 The atonal music does not observe the Power Law

Based on the cases presented here and many others in [29-31], we can conclude that the melody variations of all tonal music observe the Power Law. In other words, those who do not observe the Power Law are not tonal music, some of which are chromatic and atonal pieces.

Around the 1900s, chromaticism and atonalism gradually rose [5]. It is shown that the melody variations of some atonal music promoted by some contemporary musicians since the 30s of the last century, such as the twelve-tone system advocated by Schoenberg DO NOT observe the Power Law [29]. The CCDFs of melody variations of two representative pieces of Stockhausen and Schoenberg are shown in Figure SMI 5E, F, and in fact, it is easy to derive that the CCDF of the twelve-tone system does not observe the Power Law as shown in Figure SMI 5G. The reason why some contemporary atonal music, such as the twelve-tone system, does not observe the Power Law is given as follows.

For the atonal music whose CCDFs do not observe the Power Law, it is seen that not all the three mathematical characteristics hold. Atonal compositions are generally not required to be structurally coherent and perceived as a whole. Therefore, the probability of melody variations P may not be stationary, and the smoothness of the melody S would be changeable without approaching a "smoothness attractor" as the melody develops. What is more, some atonal music pieces do not develop like an organism, so it is possible that the melodic intervals used during composing are not richer than before. That is, the melody variations entropy maybe decrease significantly at some time.

The three mathematical characteristics, namely P, S and H_v, of the atonal music that does not observe the Power Law, are demonstrated in Supplementary Materials II Figure SMII 15.

We note that the twelve-tone system music usually DO NOT observe the Power Law. We illustrate and derive the twelve-tone system as follows.

The core of the twelve-tone system is the tone row (basic set, series), a random and ordered arrangement of the twelve pitch classes (not twelve pitches) selected by composers, with each one occurring once and only once. The row has four basic forms: prime, retrograde, inversion, and retrograde inversion. If we let $\{n_1, n_2, \dots, n_{12}\}$ be the prime tone row consisting of twelve pitch classes, then the retrograde, inversion and retrograde inversion can be described as $\{n_{12}, n_{11}, \dots, n_1\}$, $\{2t - n_1, 2t - n_2, \dots, 2t - n_{12}\}$ and $\{2t - n_{12}, 2t - n_{11}, \dots, 2t - n_1\}$, where t is the mirror plane of inversion (see score demonstrations of the four basic forms in Supplementary Materials II Figure SMII 16). It is obvious that the four basic forms have the same variations. In addition, each of the four basic forms has twelve transpositions; that is, each one may be transposed to begin with any of the twelve pitch classes, so a single row has 48 versions. However, there are several ways in which rows are actually used in compositions. Some rows use the first three, four, or six notes as a pattern from which the rest of the row is derived (such a row is called a derived set) and the segments of the tone row may appear simultaneously in different voice parts [5]. In conclusion, these ways make the pitches of the horizontal voice leading random and uniform.

Based on the above description, we suppose that the pitch sequence generated by the twelve-tone crafts is a uniform random distribution from a_0 to b_0, then the melody variation is the absolute value of the difference between two adjacent pitches. Mathematically, suppose that the stochastic variables A and B are independent and identical uniform random distributed from a_0 to b_0, where $a_0 > b_0 \geq 0$, then determine the distribution of $W = |A - B|$. It is easy to obtain that the PDF of the stochastic variable $W = |A - B|$ is

$$f_W(w) = \begin{cases} \frac{2(-w + b_0 - a_0)}{(b_0 - a_0)^2}, & 0 \leq w \leq b_0 - a_0, \\ 0, & otherwise. \end{cases} \quad \text{(SMI 15)}$$

Therefore, the CCDF of the stochastic variable $W = |A - B|$ is

$$T(i) = P(W \geq i) = \left(1 - \frac{i}{b_0 - a_0}\right)^2, \quad \text{(SMI 16)}$$

where i is the melody variations (semitones). It is not the Power Law. Let $b_0 = 12, a_0 = 1$, then the CCDF of the melody variations of the compositions composed by the twelve-tone system is simulated from (SMI 16) in Figure SMI 5G.

5 Discussions and conclusions

In conclusion, the three mathematical characteristics of a melody, which are the stationary distribution of melody variations, the smoothness of the melody curves, and the entropy maximization of melody variations, are embedded in the composition theory and techniques taught in music schools worldwide. Based on these characteristics, it turns out that the composers of tonal music in the past 200-300 years all pursued the constrained entropy maximization of melody variations. The Power Law of music melody variations can thus be derived based on these three characteristics, consistent with the observations in the past three decades [29-31].

In fact, the Power Law is quite universal for many natural and artificial events such as the earthquake scales, the number of lines connected to a node in a power grid, the times of a web page being browsed, the number of followers for a person in an online social network, the solar flare strength, the protein matching pairs, the number of collaborating actors/actresses in the same movie, the personal wealth in the US, and the frequency of words in a novel [32, 33]. There have been mainly three models: *Yule Process* (Simon Model) [34-37], *Self-Organized Criticality* [38], and *Highly Optimized Tolerance* (HOT) *Theory* [39] for explaining the mechanisms behind the Power Law in the above events. However, these models can not explain why melody variations or melodic intervals of tonal music observe the Power Law but the new model developed in this paper.

The results in this paper fill the research gap of quantitative analysis on music melodies. A new model is developed to explain why melody variations of tonal music observe the Power Law. The finding can help us comprehend music with scientific quantification, can serve as a basis for investigating if there is any physiological reason why the public without music training likes tonal music, and provides a necessity for music composition aided by AI since a mechanism built in the AI aided composition methods to exclude the note series that do not observe the Power Law. This study provides a bridge between information science and music art.

6 Figures

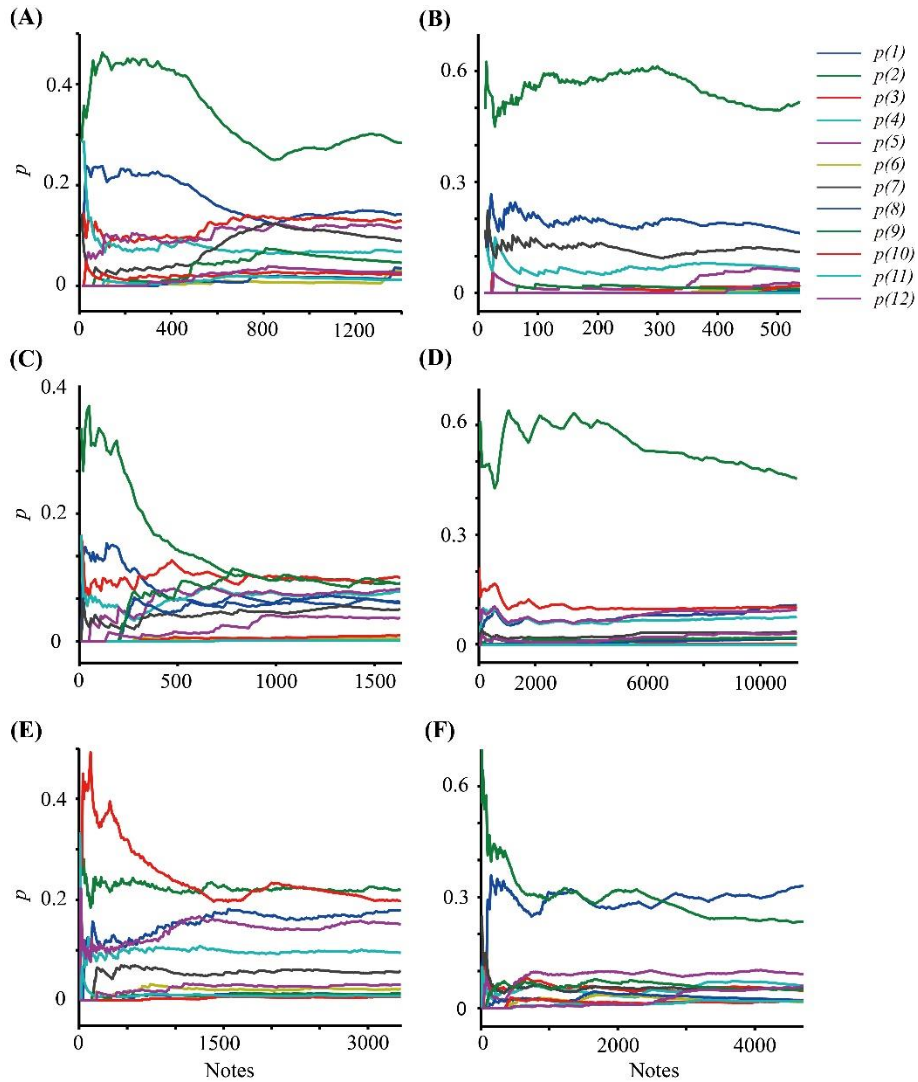

Figure SMI 1. Frequencies of melodic intervals (semitones) with the melody developing. (**A**) J. S. Bach, Concerto for oboe and violin in c minor. (**B**) Beethoven, Symphony No. 5 in c minor. (**C**) Mozart, Concerto for flute and harp in C Major. (**D**) Mendelssohn, Wedding March. (**E**) Tchaikovsky, Swan Lake. (**F**) Webber, Song “Memory” from opera “Cats”. More examples of this characteristic are shown in Supplementary Dataset S2.

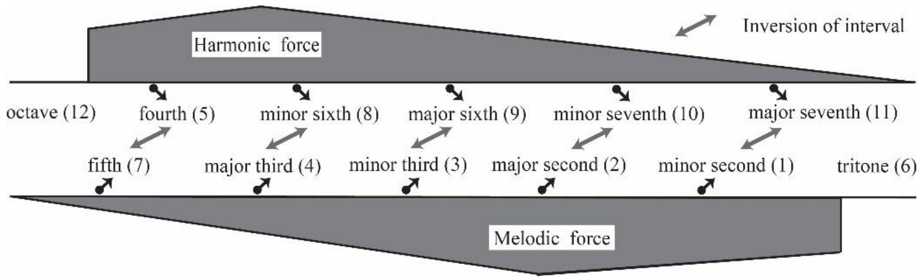

Figure SMI 2. The natural sequence of melodic intervals that driven by harmonic and melodic forces. In this figure, "octave (12)" denotes an octave containing 12 semitones, "fifth (7)" denotes a fifth containing 7 semitones, and so on. The intervals in the upper line are inversions of those in the lower line, and the double arrows show the corresponding inversion relationship. In Hindemith's theory, melodic intervals are like individual building stones according to strength, hardness and density. Harmonic force is the strongest and diminishes towards the end, while melodic force is distributed in the opposite order. The most unambiguous interval is octave (12 semitones), followed by the fifth (7 semitones). Major third (4 semitones) has the strongest harmonic force, and then the harmonic force decreases until it nearly disappears in the minor second (1 semitone) and major seventh (11 semitones). Major second (2 semitones) has the strongest melodic force, followed by the simplest melodic step, the minor second (1 semitone). The tritone (augmented fourth or diminished fifth, 6 semitones) has no melodic force and the weakest harmonic force [11].

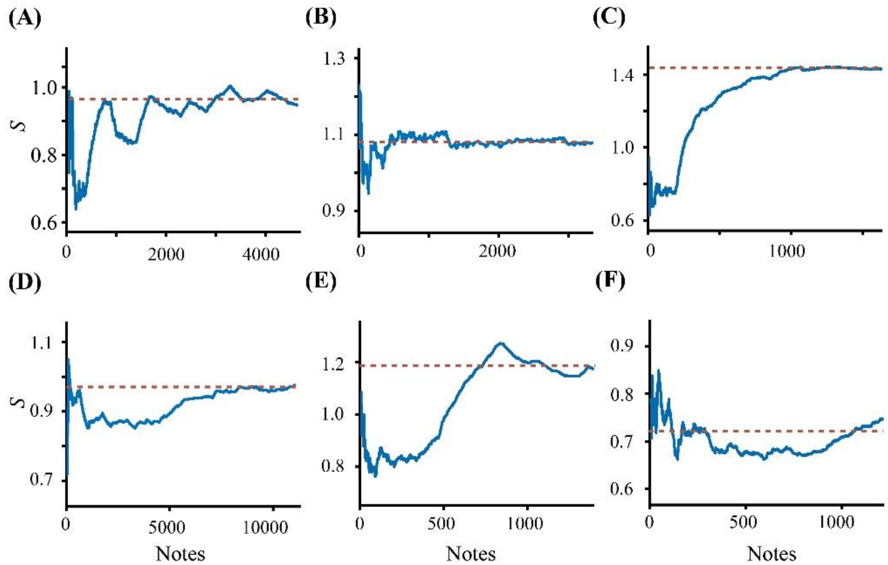

Figure SMI 3. The smoothness of melody curves *S* tends to a small "smoothness attractor" as the melody develops. (**A**) J. S. Bach, Concerto for oboe and violin in c minor. (**B**) Beethoven, Symphony No. 5 in c minor. (**C**) Mozart, Concerto for flute and harp in C Major. (**D**) Mendelssohn, Wedding March. (**E**) Tchaikovsky, Swan Lake. (**F**) Webber, Song "Memory" from opera "Cats". More examples are shown in Supplementary Dataset S3.

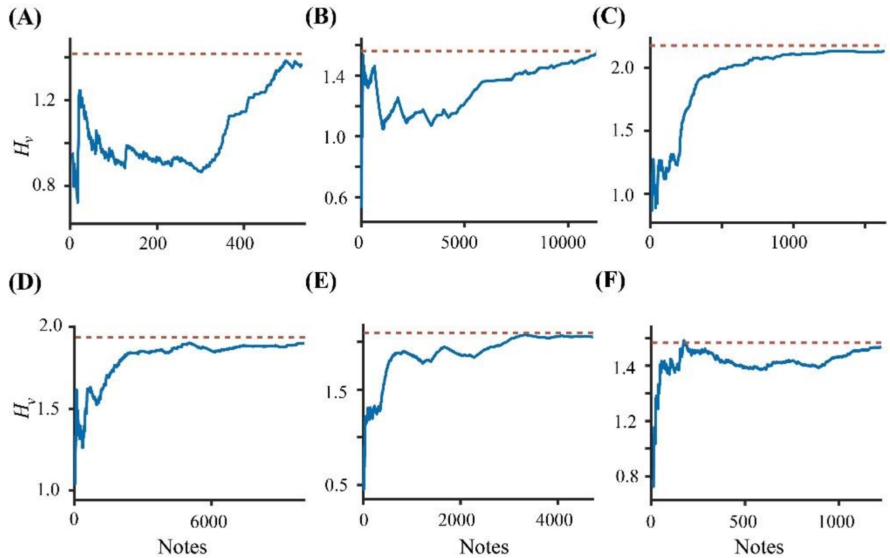

Figure SMI 4. The melody variation entropy H_v achieves its maximum with the melody developing. (**A**) J. S. Bach, Concerto for oboe and violin in c minor. (**B**) Beethoven, Symphony No. 5 in c minor. (**C**) Mozart, Concerto for flute and harp in C Major. (**D**) Mendelssohn, Wedding March. (**E**) Tchaikovsky, Swan Lake. (**F**) Webber, Song "Memory" from opera "Cats". More examples are shown in Supplementary Dataset S4.

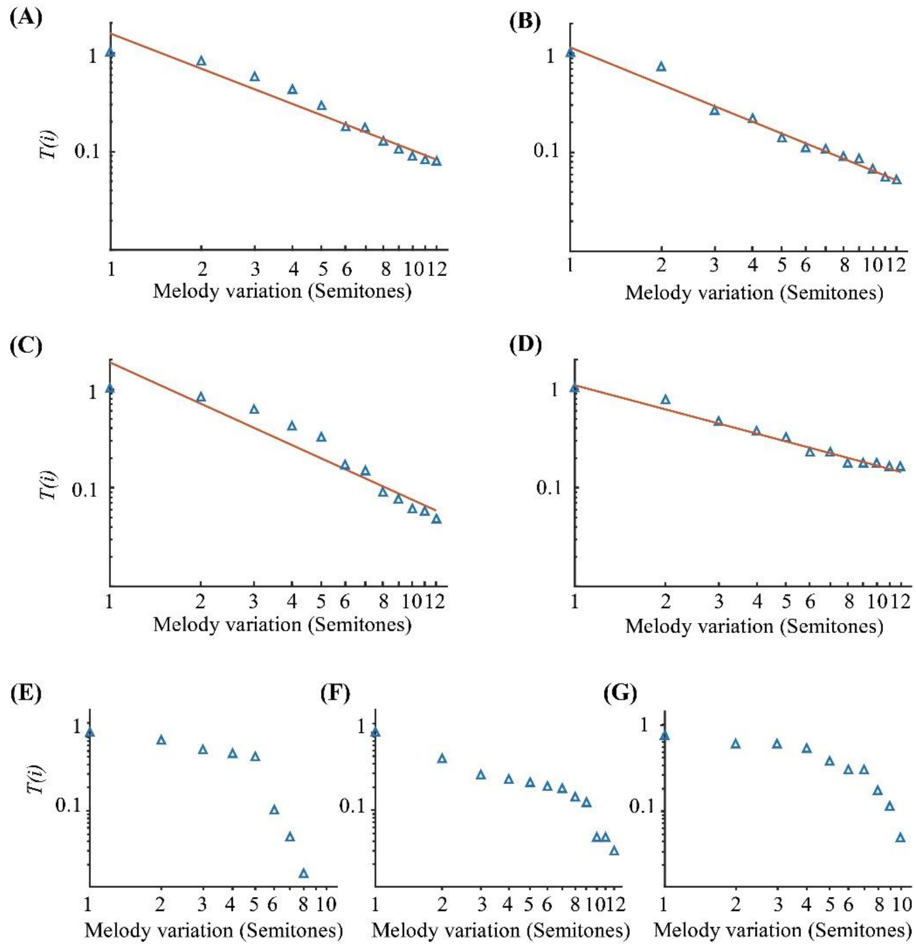

Figure SMI 5. The CCDFs of the melody variations (semitones) of the tonal music observe the Power Law (A, B, C, and D), and the CCDFs of the melody variations (semitones) of the atonal music DO NOT observe the Power Law (E, F, and G). The first four music pieces are respectively (**A**) J. S. Bach, Concerto for oboe and violin in c minor. (**B**) Mozart, Concerto for flute and harp in C Major. (C) Beethoven, Symphony No. 5 in c minor. (**D**) Chopin, Mazurka in g minor. These four musicians are selected as the typical examples from Baroque to Romantic period, while the more general music pieces from other genres and periods are analyzed, of which the CCDFs of the melody variations (semitones) also observe the Power Law (see Supplementary Dataset S5). The three music pieces that Do Not observe the Power Law are respectively (**E**) Stockhausen, Capricorn. (**F**) Schoenberg, Themes and Variations euphonium solo bass. (**G**) Simulation based on Schoenberg's twelve-tone system. More examples are shown in Supplementary Materials II Figure SMII 14.

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Supplementary Materials II (SM II)

1 Materials and methods

1.1 Notation

$\boldsymbol{B(x;\varepsilon)}$	ε-neighborhood of x
$\boldsymbol{D}$	The exponent of Power Low function
$\boldsymbol{D_{lower}}$	The theoretical lower bound of D
$\boldsymbol{D_{upper}}$	The theoretical upper bound of D
$\boldsymbol{D_{fitting}}$	The fitting of D
$\boldsymbol{d}$	The difference of H_v^f
$\boldsymbol{h_x}$	The $H\ddot{o}lder$ exponent at x
$\boldsymbol{h_{n_j}}$	The $H\ddot{o}lder$ exponent at note n_j
$\boldsymbol{\bar{h}}$	The average of all h_{n_j} in a melody
$\boldsymbol{H_v}$	Melody variations entropy
$\boldsymbol{H_v^f}$	The arithmetic average filtering of H_v
$\boldsymbol{i}$	Melody variation (namely, melody interval in semitones)
$\boldsymbol{I_a}$	The lower bound of i
$\boldsymbol{I_b}$	The largest melody variation in an actual composition
$\boldsymbol{I_T}$	The upper bound of i in CCDF
$\boldsymbol{I(t)}$	Stochastic process of melody variation for semitones i
$\boldsymbol{I_t}$	A sample of stochastic process $\{I(t), t \in T\}$
$\boldsymbol{\bar{I}}$	The average of all samples I_t
$\boldsymbol{L}$	Lagrange function
$\boldsymbol{m}$	Smoothness attractor
$\boldsymbol{m_e}$	The actual value of m
$\boldsymbol{m_{lower}}$	The theoretical lower bound of m
$\boldsymbol{m_{upper}}$	The theoretical upper bound of m
$\boldsymbol{N}$	The total number of the notes of a melody
$\boldsymbol{n_j}$	The n_jth note of a melody
$\boldsymbol{P}$	Probability density function of i
$\boldsymbol{p(i)}$	Probability of i
$\boldsymbol{p_f(i)}$	Frequency of i
$\boldsymbol{R^2}$	Goodness of fit (the coefficient of determination)
$\boldsymbol{R_D}$	The ratio of the positive difference sum to the total difference of H_v^f
$\boldsymbol{r(k)}$	Sample autocorrelation function (SAF)
$\boldsymbol{S}$	Smoothness of melody curves
$\boldsymbol{S(n_j)}$	S of the first n_j notes

$\boldsymbol{S_h(i)}$	The definition of Smoothness of melody curve with a simple melody variation i derived from $H\ddot{o}lder$ exponent
$\boldsymbol{SSR}$	Regression sum of squares
$\boldsymbol{SST}$	Total sum of squares
$\boldsymbol{T(i)}$	The complementary cumulative distribution function (CCDF) of melody variation i
$\boldsymbol{u_s}$	The total number of descending intervals between note n_j and n_s of the melody consisting of a simple melody variation i
$\boldsymbol{v_s}$	The total number of ascending intervals between note n_j and n_s of the melody consisting of a simple melody variation i

1.2 Derivation of the melody smoothness

In Eq.3 of the main text, we define S to measure the degree of melody smoothness as the expectation of $\log i$

$$S = E(\log i) = \sum_i p(i) \log i \to m. \quad \text{(SMII 1)}$$

This definition is based on the $Hölder$ function described in [22, 27], where the $Hölder$ exponent of a fractal function $f(x)$ [24]at x is defined as

$$h_x = \lim_{\varepsilon \to 0} \inf \left\{ \frac{\log|f(x) - f(y)|}{\log|x - y|} : y \in B(x; \varepsilon) \right\}. \quad \text{(SMII 2)}$$

For fractal functions, such as continuous interpolation functions and continuous non-differential functions, the $Hölder$ exponent h_x at a point x represents the degree of irregularity around x in the graphs of fractal functions, which demonstrates how oscillatory (or smooth) the graphs are at the neighborhood of point x. Therefore, the $Hölder$ exponent is used to measure the smoothness of melody curves.

In a piece of melody, by linking the pitches in its note sequence with equal rhythm unit since rhythm is not considered, we obtain a melody curve [11, 21] as shown in Figure SMII 2. This melody curve with a large number of notes can be analyzed by fractal theory since it is wiggly and zigzag with melody variations. We assume that (n_j), the pitch of the n_jth note, is sampled from a latent continuous fractal function at $x = n_j$, $j = 1,2, \dots, N$. Then the melody variation i in semitones is $f(n_{j+1}) - f(n_j),\ j = 1,2, \dots, N - 1$. Without loss of generality, we set the distance between two adjunct notes along time to δ, i.e., $n_{j+1} - n_j = \delta \in (0,1), j = 1,2, \dots, N - 1$.

Assume the smallest rhyme unit between consecutive notes in melody curve function is δ corresponding the ε in (SMII 2) [25] and there is only a single melody variation i either descending or ascending randomly (Figure SMII 3). According to (SMII 2), the $Hölder$ exponent of melody curve function $f(n)$ at note n_j is

$$h_{n_j} = \lim_{\varepsilon \to \delta} \inf \left\{ \frac{\log|f(n_j) - f(n_k)|}{\log|n_j - n_k|} : n_k \in B(n_j; \varepsilon) \right\}. \quad \text{(SMII 3)}$$

For h_{n_j}, there must exist a note n_s in $B(n_j; \varepsilon)$ such that

$$h_{n_j} = \frac{\log |u_s - v_s| i}{\log \varepsilon_s} \quad \text{(SMII 4)}$$

$$= a_j \log i + b_j, \quad \text{(SMII 5)}$$

where u_s, v_s are the numbers of the descending and ascending intervals between note n_j and n_s with $u_s \neq v_s$, $\varepsilon_s = |n_j - n_s|$, $a_j = \frac{1}{\log \varepsilon_s} < 0$ and $b_j = \frac{\log |u_s - v_s|}{\log \varepsilon_s} > 0$.

The smoothness of the whole melody curve should be the average value $\bar{h}$ of all h_{n_j}

$$\bar{h} = \frac{1}{N} \sum_{j=1}^{N} h_{n_j} \quad \text{(SMII 6)}$$

$$= \frac{1}{N} \sum_{j=1}^{N} (a_j \log i + b_j) \quad \text{(SMII 7)}$$

$$= \left(\frac{1}{N}\sum_{j=1}^{N} a_j\right)\log i + \left(\frac{1}{N}\sum_{j=1}^{N} b_j\right) \quad \text{(SMII 8)}$$

$$= \bar{a}\log i + \bar{b}, \quad \text{(SMII 9)}$$

where $\bar{a} = \frac{1}{N}\sum_{j=1}^{N} a_j$ and $\bar{b} = \frac{1}{N}\sum_{j=1}^{N} b_j$.

Since we only consider the quantitative relations between the smoothness of melody curves and the melody variation for semitones i, the other elements are constants with respective to i. Thus, we obtain the melody smoothness of a single melody variation i in the form of

$$S_h(i) = \bar{a}\log i + \bar{b}, \quad \text{(SMII 10)}$$

where $\bar{a}, \bar{b}$ are constants, and $\bar{a} < 0, \bar{b} > 0$.

The smoothness of a melody that consists of various melody variations with different occurrences is the weighted average of all melody variations

$$E[S_h(i)] = b + a\sum_{i} p(i)\log i, \quad \text{(SMII 11)}$$

where $p(i)$ is the probability that melody variation i appears and a, b are constants to i. To simplify, we omit these constants in (SMII 11) and the smoothness definition in (SMII 1) is the correct representation of (SMII 2). Due to the stationary process and the extensive use of narrow intervals (especially step-progression) as presented in the main text, S tends to a small constant m called "smoothness attractor" (SMII 1).

1.3 Derivation of the Power Law of the CCDF of melody variations

The probability distribution of music melody variations or melodic intervals of tonal music observes the Power Law consistent with some studies in the past [29-31]. In fact, the Power Law is quite universal for many natural and artificial events such as the earthquake scales, the number of lines connected to a node in a power grid, the times of a web page being browsed, the number of followers for a person in an online social network, the solar flare strength, the protein matching pairs, the number of collaborating actors/actresses in the same movie, the personal wealth in the US, and the frequency of words in a novel [32, 33]. There have been mainly three models: *Yule Process* (Simon Model) [34-37], *Self-Organized Criticality* [38], and *Highly Optimized Tolerance* (HOT) *Theory* [39] for explaining the mechanisms behind the Power Law in the above events. However, these models can not explain why melody variations or melodic intervals of tonal music observe the Power Law.

Based on the three mathematical characteristics in the main text, a constrained functional optimization problem is formulated as follows

$$\max H_v = -\sum_{i=I_a}^{I_b} p(i)\ln p(i), \quad \text{(SMII 12)}$$

subject to

$$\begin{cases} \sum_{i=I_a}^{I_b} p(i) = 1 \\ \sum_{i=I_a}^{I_b} p(i) \ln i = m, \end{cases} \tag{SMII 13}$$

where integers I_a and I_b are the lower and upper bounds of melody variation i in the whole piece with $I_a < I_b$.

We solve the above problem based on the calculus of variations [27, 28]. First, we relax constraints (SMII 13) by Lagrange multipliers and obtain the Lagrange function

$$L = H_v - \lambda_0 f_0\big(i, p(i)\big) - \lambda_1 f_1\big(i, p(i)\big), \tag{SMII 14}$$

where

$$\begin{cases} f_0\big(i, p(i)\big) = \sum_{i=I_a}^{I_b} p(i) - 1 \\ f_1\big(i, p(i)\big) = \sum_{i=I_a}^{I_b} p(i) \ln i - m, \\ H_v = -\sum_{i=I_a}^{I_b} p(i) \ln p(i) \end{cases} \tag{SMII 15}$$

and $\lambda_0, \lambda_1 \geq 0$ are Lagrange multipliers.

Then,

$$L = H_v - \lambda_0 \left(\sum_{i=I_a}^{I_b} p(i) - 1 \right) - \lambda_1 \left(\sum_{i=I_a}^{I_b} p(i) \ln i - m \right) \tag{SMII 16}$$

$$= -\sum_{i=I_a}^{I_b} p(i) \ln p(i) - \lambda_0 \left(\sum_{i=I_a}^{I_b} p(i) - 1 \right) - \lambda_1 \left(\sum_{i=I_a}^{I_b} p(i) \ln i - m \right) \tag{SMII 17}$$

$$= -\sum_{i=I_a}^{I_b} [p(i) \ln p(i) + \lambda_0 p(i) + \lambda_1 p(i) \ln i] + \lambda_0 + \lambda_1 m. \tag{SMII 18}$$

Let

$$J = \sum_{i=I_a}^{I_b} [p(i) \ln p(i) + \lambda_0 p(i) + \lambda_1 p(i) \ln i] = \sum_{i=I_a}^{I_b} L_i. \tag{SMII 19}$$

Suppose the optimal solution is $p^*(i)$, then the admissible trajectory around $p^*(i)$ is

$$p(i) = p^*(i) + \delta p(i), \tag{SMII 20}$$

where $\delta p(i)$ denotes the first-order variation of $p(i)$.

By substituting (SMII 20) into (SMII 19), we obtain

$$J = \sum_{i=I_a}^{I_b} \{[\, p^*(i) + \delta p(i)] \ln[\, p^*(i) + \delta p(i)] + \lambda_0 [\, p^*(i) + \delta p(i)] + \lambda_1 [\, p^*(i) + \delta p(i)] \ln i\}. \tag{SMII 21}$$

Its first-order variation is

$$\delta J = \sum_{i=I_a}^{I_b} \left[\frac{\partial L_i}{\partial p(i)} \delta p(i) \right], \tag{SMII 22}$$

known as the discrete Euler equation [27]. The necessary condition of maximizing the discrete function (SMII 21) is $\delta J = 0$, i.e.

$$\frac{\partial L_i}{\partial p(i)} \delta p(i) = 0. \tag{SMII 23}$$

Since $\delta p(i)$ can be any small non-zero value, then

$$\frac{\partial L_i}{\partial p(i)} = \ln p(i) + 1 + \lambda_0 + \lambda_1 \ln i = 0, \tag{SMII 24}$$

and we get

$$p(i) = C_0 i^{-\lambda_1}, \tag{SMII 25}$$

where $C_0 = e^{-(1+\lambda_0)} > 0$ and $\lambda_1 > 0$.

If i is continuous with $i \in [0, \infty]$, the CCDF is

$$\int_i^{\infty} p(s) ds = c i^{-D}, \tag{SMII 26}$$

where $c = \frac{C_0}{\lambda_1 - 1} > 0$, and $D = \lambda_1 - 1 > 0$ and it is the Power Law function presented in Eq. 12 of the main text.

However, the melody variations in semitone i are positive integers and the complementary cumulative distribution function (CCDF) of melody variation i should

$$T(i) = P(I \geq i) = \sum_{s=i}^{I_b} p(s) \tag{SMII 27}$$

$$= \sum_{s=i}^{I_b} C_0 s^{-\lambda_1}, \tag{SMII 28}$$

where we only consider the range of i from I_a to I_T ($I_T \leq I_b$, and generally equals to 12).

Next, we use the approximate integration method to compute the sum (SMII 28). This method is a reverse process of converting sum to integral in the definition of Riemann integral.

Therefore,

$$T(i) = \sum_{s=i}^{I_b} C_0 s^{-\lambda_1} = \int_i^{I_b} C_0 s^{-\lambda_1} \, ds + e(s,i) \tag{SMII 29}$$

$$= \frac{C_0}{\lambda_1 - 1} \left[i^{-\lambda_1 + 1} - {I_b}^{-\lambda_1 + 1} \right] + e(s,i) \tag{SMII 30}$$

$$= c i^{-D} + q_{I_b} + e(s,i), \tag{SMII 31}$$

where $c = \frac{C_0}{\lambda_1 - 1} > 0, D = \lambda_1 - 1 > 0, q_{I_b} = -\frac{C_0}{\lambda_1 - 1} {I_b}^{-\lambda_1 + 1} < 0, e(s, i) \le \frac{-C_0 \lambda_1}{2} \frac{(I_b - i)^{-\lambda_1 + 1}}{I_b - i + 1}, i \in \{I_a, I_a + 1, \dots, I_T\}$. The CCDF $T(i)$ contains a term of the Power Law function ci^{-D}, a constant term q_{I_b} for i, and an error term.

First, we discuss the common ranges of I_b and I_T in actual music works. In music works, the upper bound of melody variation I_b is usually not smaller than 19 (see the statistical results in Figure SMII 4). Furthermore, the most common intervals are not larger than 12 (Figure SMII 4). Hence, we set the upper bound I_T equal 12 in CCDF in general (sometimes I_T may equal other values in different compositions, e.g. 7-10).

Next, we simulate the disturbance of the error term and the constant term q_{I_b} in $T(i)$ (SMII 31) with the above I_b and the different exponent D (set C_0 as a proper value). Figure SMII 5 shows that the error term hardly disturbs the value in $T(i)$ (SMII 31), then

$$T(i) = \sum_{s=i}^{I_b} C_0 s^{-\lambda_1} \tag{SMII 32}$$

$$= ci^{-D} + q_{I_b} + Err(s, i) \tag{SMII 33}$$

$$\approx ci^{-D} + q_{I_b}. \tag{SMII 34}$$

Figure SMII 6 shows that the usual values of I_b and D conduce to smaller q_{I_b} for $T(i)$, which means that q_{I_b} can generally be omitted and $T(i)$ approximately follows the Power Law ci^{-D}. This conclusion is examined in Figure SMII 7, stating that the actual $T(i)$ calculated from (SMII 27) fit the simulation results from Eq.12 in the main text. From above, q_{I_b} can be ignored for the actual values of parameters in (SMII 31).

Thus, the CCDF of melody variations (semitones i) of the tonal music observes the Power Law:

$$T(i) = P(x \ge i) = ci^{-D}, \tag{SMII 35}$$

where $i = \{I_a, I_a + 1, \dots, I_T\}, I_T \le I_b$. In general, $I_a = 1$ or $2, I_T = 12, I_b \ge 19$.

Furthermore, we can determine the possible values of exponent D and the smoothness attractor m in the Power Law (SMII 35). For simplicity, we consider the continuous function

$$p(i) = C_0 i^{-\lambda_1}, \tag{SMII 36}$$

satisfying

$$\begin{cases} \int_{I_a}^{I_b} p(i) di = 1 \\ \int_{I_a}^{I_b} p(i) \ln i \, di = m. \end{cases} \tag{SMII 37}$$

From (SMII 37), we obtain $m \in (\ln I_a, \ln I_b)$ and $\lambda_1 \in \left(1 - \frac{1}{\ln I_b - m}, 1 + \frac{1}{m - \ln I_a}\right)$. Since m is a small constant satisfying $m < \frac{\ln I_a + \ln I_b}{2}$ and $m < \ln \frac{I_b}{e}$, then $\lambda_1 \in \left(1, 1 + \frac{1}{m - \ln I_a}\right)$ and

$$D = \lambda_1 - 1 \in \left(0, \ \frac{1}{m - \ln I_a}\right). \tag{SMII 38}$$

Thus, we can obtain

$$m \in \left(\ln I_a, \min\left\{\ln I_b - 1, \frac{\ln I_a + \ln I_b}{2}\right\}\right). \tag{SMII 39}$$

Next, we compare the theoretical ranges with the actual results of parameter D and smoothness attractor m (the corresponding 141 compositions are listed in Table SMII 1). On the one hand, we calculate the actual value m_e of the smoothness attractor m from (SMII 1) and the theoretical values m_{lower} and m_{upper} (the lower and upper bounds of m) from (SMII 39) with the actual I_a and I_b of a given music piece. All the values of m_e, m_{lower} and m_{upper} for the 141 music works are shown in Figure SMII 8, which indicates that the actual values m_e are located in the theoretical range (m_{lower}, m_{upper}). On the other hand, we calculate the theoretical upper bound D_{upper} of D from (SMII 38) (the theoretical lower bound is zero) and the corresponding actual values $D_{fitting}$ obtained from fitting CCDF by the Power Law function (SMII 35) (see Section 1.4). From Figure SMII 9, we conclude that $D_{fitting}$ is under the theoretical upper bound D_{upper}.

As a result, the actual data are consistent with the ranges of exponent D and smoothness attractor m analyzed theoretically.

1.4 Music data processing and analysis

We gather 141 tonal compositions of the major and minor mode music as typical examples, ranging in 14 periods and genres in pre-Baroque, Baroque, classicism, romanticism, rationalism, impressionism, neoclassicism, and popular music (some songs with an accompaniment) [6]. Table SMII 2 provides an overview of these pieces. The original MIDI format files of the 141 compositions are obtained [40].

MIDI (Musical Instrument Digital Interface) is a digital communication language and compatible specification that allows multiple hardware and software electronic instruments, performance controllers, computers, and other related devices to communicate with each other over a connected network. It is now the most prevalent representation of music recording the primary attributes of music (pitch, duration, articulation, ornamentation, dynamics, and timbre). An obvious advantage of MIDI is that its data can be input directly into a hardware device or software program (known as a sequencer), edited and transferred to electronic instruments or other devices to create music or control any number of parameters [41].

An overview of all tonal music used in this study is in the file of Dataset S1.xlsx. The table includes three worksheets. The first named "MusicInf" provides the information of all the MIDI files, including music index, group index, genre or period, composer, and the composition title. The second worksheet "Genre&Period" provides the group index, genres, periods, and the typical musicians of the genre groups. The last sheet "MidiPitch" gives the matrix of all channels and pitches of 141 music pieces that convert from MIDI files in MATLAB 2017 by MIDI toolbox 1.1.

1.41 The CCDFs from the actual data: the Power Law

The melody variations for semitone i of the notes sequence is calculated as

$$I(j) = f(n_{j+1}) - f(n_j), j = 1,2, \dots, N-1, \qquad \text{(SMII 40)}$$

where $f(n_j)$ denotes the pitch of the n_jth note, $I(j)$ is the jth melody variation, and N is the total number of the notes.

We count the frequencies $p_f(i)$ of the melody variation i in $[I_a, I_b]$ by (SMII 40) and the complementary cumulative distribution function (CCDF) of the melody variation i from I_a to I_T via

$$T(i) = \sum_{l=i}^{I_b} p_f(l), \qquad \text{(SMII 41)}$$

where $p_f(l)$ is the frequency of melody variation l for the N notes, and $i = \{I_a, I_a + 1, \dots, I_T\}, I_T \leq I_b$. I_a, I_b and I_T refer to the same definitions in Section 2.

We plot the CCDFs of 141 compositions with log-log coordinates, and the straight line is clear in every figure (Dataset S5.xlsx), which is equivalent to the Power Law function $y = cx^{-D}$. We fit the data by an affine function $y = ax + b$ and measure the goodness of fit by the coefficient of determination R-squared [42]

$$R^2 = \frac{SSR}{SST} = \frac{\sum_{j=1}^{N}\left(\hat{y}_j - \bar{y}_j\right)^2}{\sum_{j=1}^{N}\left(y_j - \bar{y}_j\right)^2}, \tag{SMII 42}$$

where $\bar{y}_j$ is the mean value of the sample points and y_j and $\hat{y}_j$ are observations and fitting values. A larger R^2 indicates a better fitting result. Figure SMII 10 shows all R^2 for 141 compositions and more detailed results are provided in Dataset S5.xlsx. The results demonstrate that all CCDFs of the 141 music pieces observe the Power Law and that the parameters are well consistent with the theoretical analysis in Section 2.

1.42 P: A stationary distribution

Consider the frequencies of the melody variation i (generally from 1 to 12) of all 141 compositions from the first 50 notes to the entire notes (see figures in Dataset S2.xlsx). To measure whether the distribution is stationary or not, we consider the stationarity of the stochastic process $\{I(t), t \in T\}$ ($I(t)$ denotes melody variation). The conditions for a stationary stochastic process $\{I(t), t \in T\}$ are as follows [43]:

1. If the mean $M(t) = E[I(t)]$ exists, its quantity must be a constant, $M(t) = M$ for all t.
2. If the second moment $E[I(t)^2]$ is finite, and then the variance $\sigma^2 = E[(I(t) - M)^2]$ is a constant, independent of time.
3. The covariance $E[(I(t) - M)(I(s) - M)]$ that depends only on the time difference $|t - s|$.

We calculate the first two conditions as well as the sample autocorrelation function (SAF) (corresponding the third condition above) to test the stationarity of the stochastic process $\{I(t), t \in T\}$. The sample autocorrelation function is defined by

$$r(k) = \frac{\sum_{t=1}^{n-k}(I_t - \bar{I})(I_{t+k} - \bar{I})}{\sum_{t=1}^{n}(I_t - \bar{I})^2}, k = 1,2,3, \dots, \tag{SMII 43}$$

where I_t is a sample of stochastic process $\{I(t), t \in T\}$, namely the tth melody variation of a music piece and $\bar{I}$ is the mean of all samples.

For a stationary process, the mean and standard deviation are constants over t, and the sample autocorrelation function $r(k)$ decreases to 0 with k increasing (may fluctuates around 0 and gradually converges to 0). Figure SMII 11 demonstrates the SAFs, means, and standard deviations of four music pieces, and the rest results are depicted in Dataset S2.xlsx. It is seen that the stationary nature of all 141 cases is clear.

1.43 S: The smoothness of melody curves

Calculate the smoothness of melody curves S of all 141 compositions from the first 20 notes to the entire notes via Eq.3 in the main text. Then, the difference between S and the smoothness attractor m_e is

$$\Delta S = S(n_j) - m_e, \quad \text{(SMII 44)}$$

where $S(n_j)$ denotes the smoothness of melody curves of the first n_j notes. We plot ΔS of the note sequence (see Figure SMII 12). The rest of empirical data of ΔS are provided in Dataset S3.xlsx, where we can obtain that each S of 141 music pieces tends to the smoothness attractor m_e.

1.44 H_v: The maximum of the melody variations entropy

For all 141 compositions, we calculate the melody variations entropy H_v from the first 20 notes to the entire notes by Eq.4 in the main text. It is obvious that the melody variations entropies H_v increase as the note sequence moves on (see Dataset S4.xlsx). To see this characteristic exactly, we calculate the arithmetic average filtering [44] of H_v in moving windows and the difference d as follows

$$d_j = H_v^f(n_{j+1}) - H_v^f(n_j), j = 1,2,3, \ldots, \quad \text{(SMII 45)}$$

where $H_v^f(n_j)$ is the arithmetic average filtering of $H_v(n_j)$ for the first n_j notes.

Then, we get the ratio of the total difference to the total positive difference defined by

$$R_D = \frac{\sum_j d_j}{\sum_k d_k^+}, \quad \text{(SMII 46)}$$

where d_k^+ denotes the positive difference of H_v^f, and k is the index of all positive differences.

Figure SMII 13 presents all R_D of 141 compositions. We can obtain that almost all R_D are closer to 1, indicating H_v increase significantly as the melodies develop.

2 Figures and Tables

Figure SMII 1. **Repetitive techniques: sequence, retrograde, and inversion.** (A) Sequence. Mozart, 12 Variations on "Ah, vous dirai-je, Maman", K. 265, the first variation. (B) Retrograde. Bach, BWV 1079, Crab Canon in Music Offering. (C) Inversion. Brahms, Symphony No.1 in c Minor, III, the Theme.

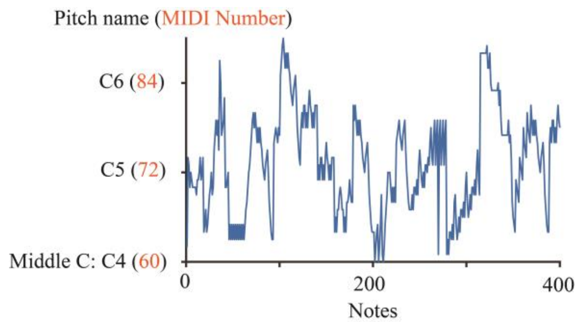

Figure SMII 2. **Melody curve.** Middle C (C4) is assigned the number 60 in MIDI data [5, 41]. The pitch increases per semitone, the corresponding number adds 1. Such as 61 represents $C^{\#}4$, and 72 corresponds C5. Here extract the first 400 notes of the main voice leading from Rhapsody in Blue by Gershwin.

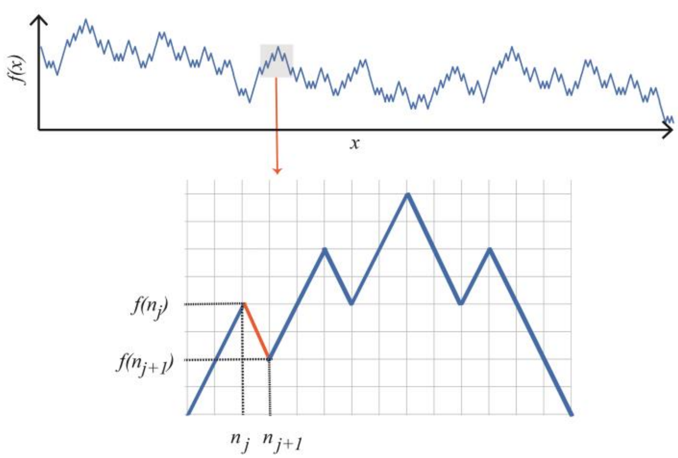

Figure SMII 3. A melody curve consists of a single melody variation $i = 2$. The horizontal and the vertical grid represent note and pitch respectively, where $n_{j+1} - n_j = \delta \in (0,1)$, $\left|f(n_{j+1}) - f(n_j)\right| = i$, $j = 1,2, \dots, N-1$.

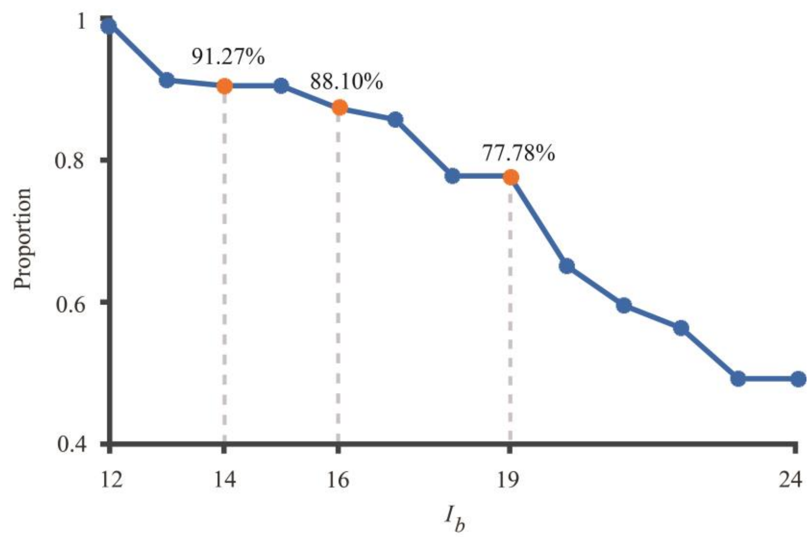

Figure SMII 4. The proportion of compositions with the largest melody variations are more than or equal to I_b in 126 music pieces from Baroque to modern pop music (see Dataset in Table SMII 2). We can see that for most pieces (91.27%) the values of I_b are not smaller than 14. The pieces with I_b more than or equal to 16 also occupy a large proportion, 88.10%. What is more, for about 77.78% of the compositions the largest melody variation I_b is not smaller than 19.

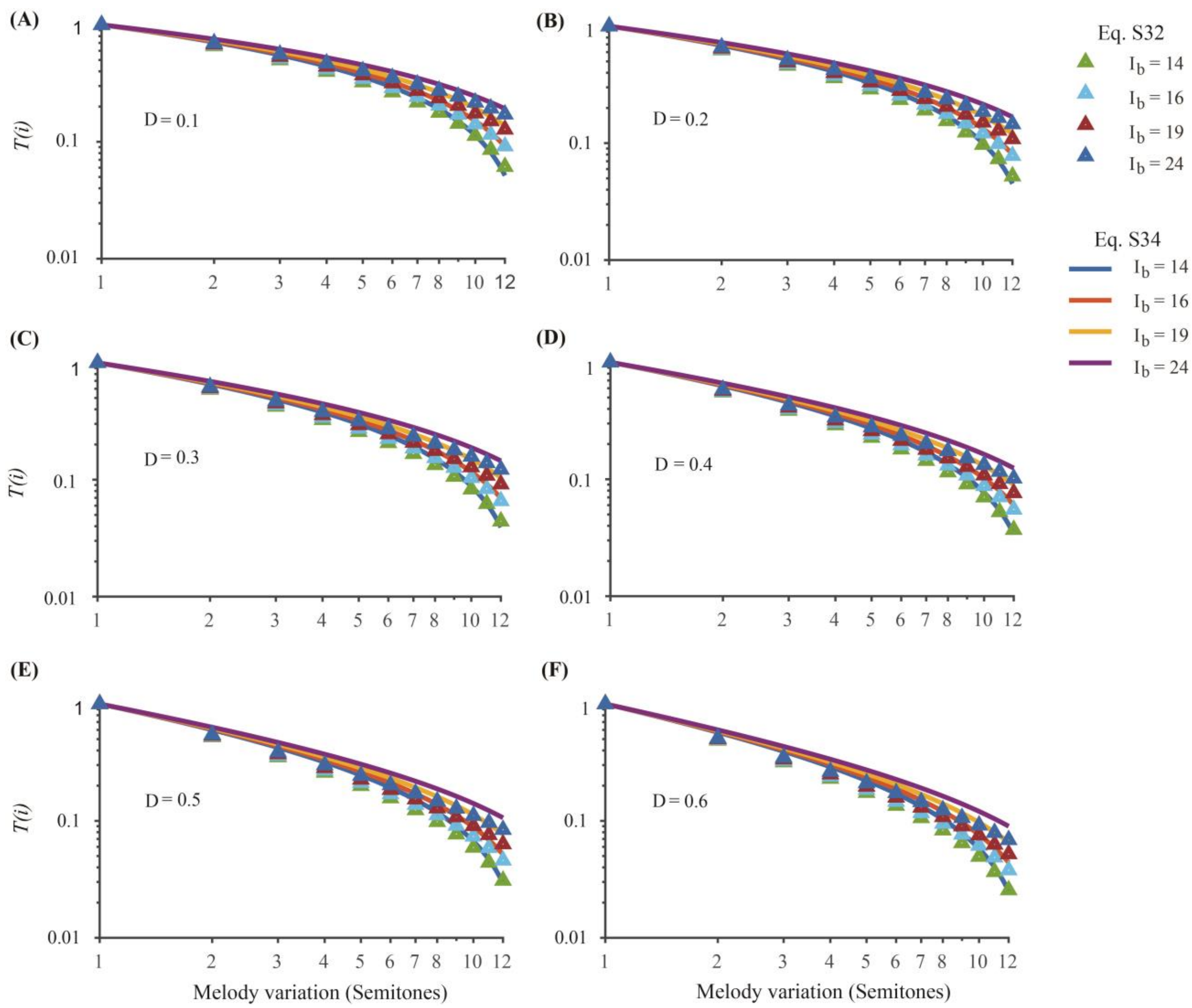

Figure SMII 5. Simulation results for Eq. SMII 32 and Eq. SMII 34. For different exponent D and different upper bound of melody variation I_b in Figure SMII 4, $T(i) = \sum_{s=i}^{I_b} C_0 s^{-\lambda_1}$ (SMII 32) and $T(i) = ci^{-D} + q_{I_b}$ (SMII 34) are simulated above, which indicates the two equations are close, namely, the error term $e(s,i)$ can be ignored.

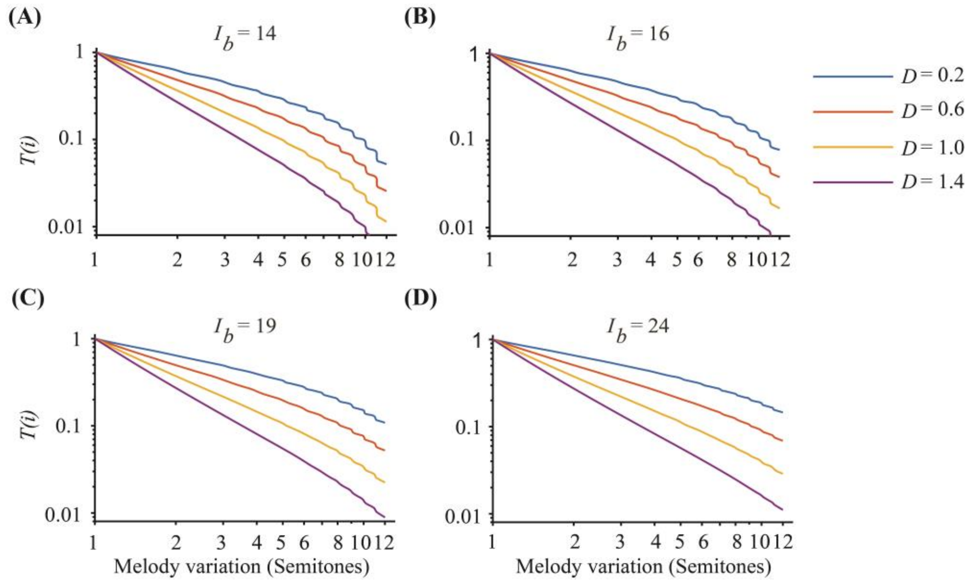

Figure SMII 6. Simulation results. For different exponent D and different upper bound of melody variation I_b in Figure SMII 4, $T(i) = ci^{-D} + q_{I_b}$ is simulated above. (A) $I_b = 14$; (B) $I_b = 16$; (C) $I_b = 19$; (D) $I_b = 24$. If I_b is fixed, the simulation solution is closer to a straight line in the log-log coordinate system when D is larger. If D is fixed, the larger I_b is, the closer the simulation solution is to a straight line in the log-log coordinate system. That is, the parameters D and I_b are large enough the term q_{I_b} could be ignored with regards to the Power Law term ci^{-D} in (SMII 31). Hence, the function $T(i)$ can be approximated as the Power Law function ci^{-D} with I_b equaling to or more than 19 and a fixed D.

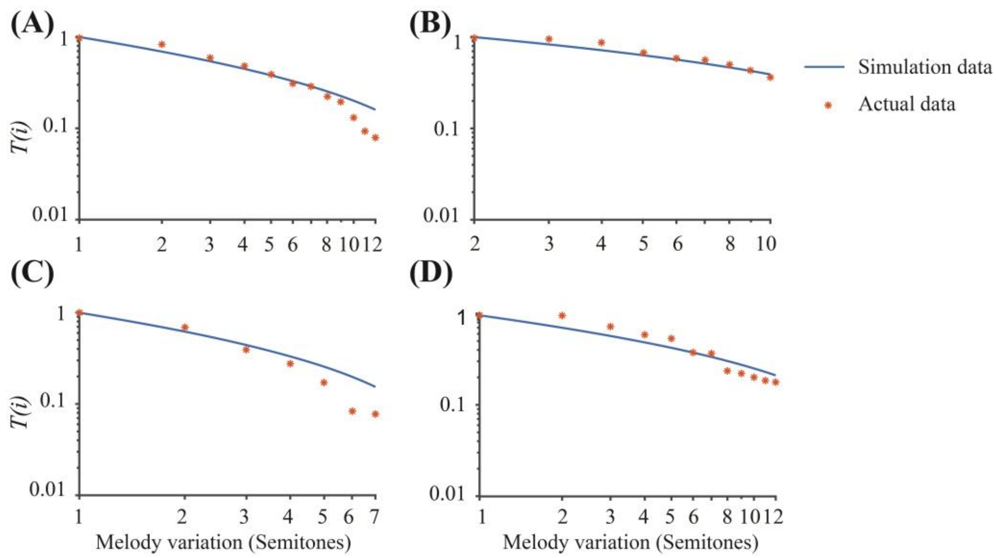

Figure SMII 7. Consistency between the simulated data and the actual data. (A) Prelude and Fugue in C# Major, BWV 848 by J. S. Bach. The largest and smallest melody variation I_a and I_b are 1 and 22 respectively, and the upper bound of melody variation I_T of the CCDF is 12. (B) Turkish March in B^b Major, Op. 113, No. 4 by Beethoven. The largest melody variation I_b is 24, the lower and the upper bound of melody variations of the CCDF, I_a and I_T, are 2 and 10 respectively. (C) Piano Sonata No. 16 in C Major, K. 545, Mvt. 1 by Mozart. I_a is 1, I_b is 12 and I_T is 7. (D) Blue and White Porcelain, a piece of modern pop music with an accompaniment by Jay Chou from China with I_b is 29 and I_a, I_T are the same as (A).

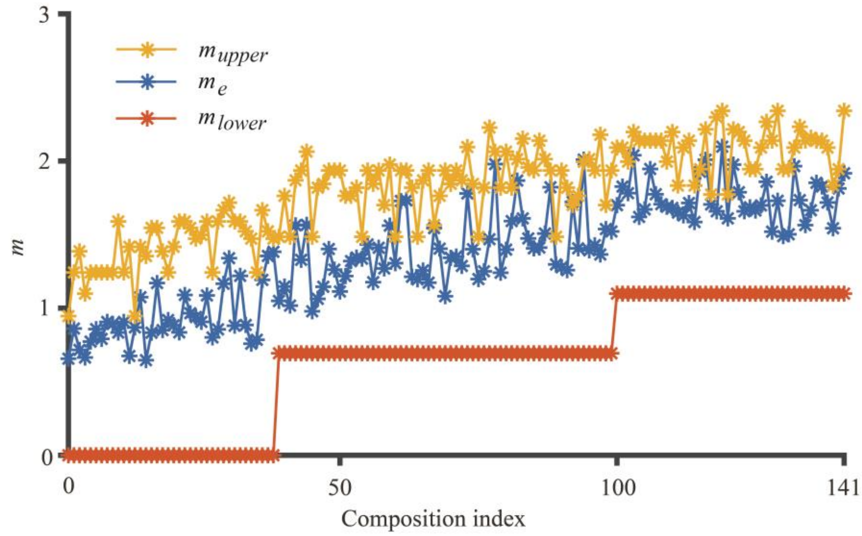

Figure SMII 8. Comparison of the smoothness attractor m between the theoretical range (m_{lower}, m_{upper}) and the actual value m_e. The 141 compositions of the horizontal ordinate are sorted by the value m_{lower} in order to have a good view.

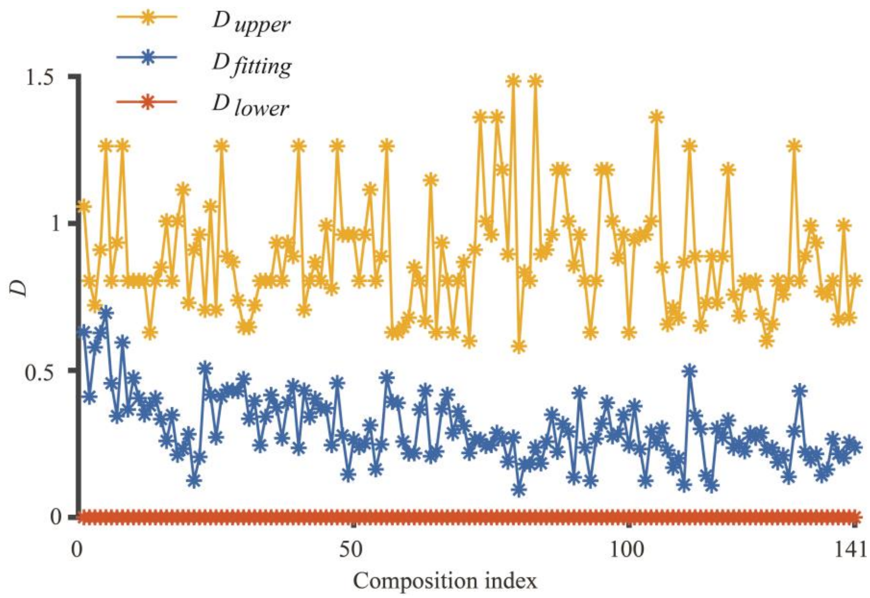

Figure SMII 9. Comparison of the exponent $\boldsymbol{D}$ between the theoretical range $(\boldsymbol{D_{lower}}, \boldsymbol{D_{upper}})$ and the actual value $\boldsymbol{D_{fitting}}$. $D_{lower} = 0$ for all compositions.

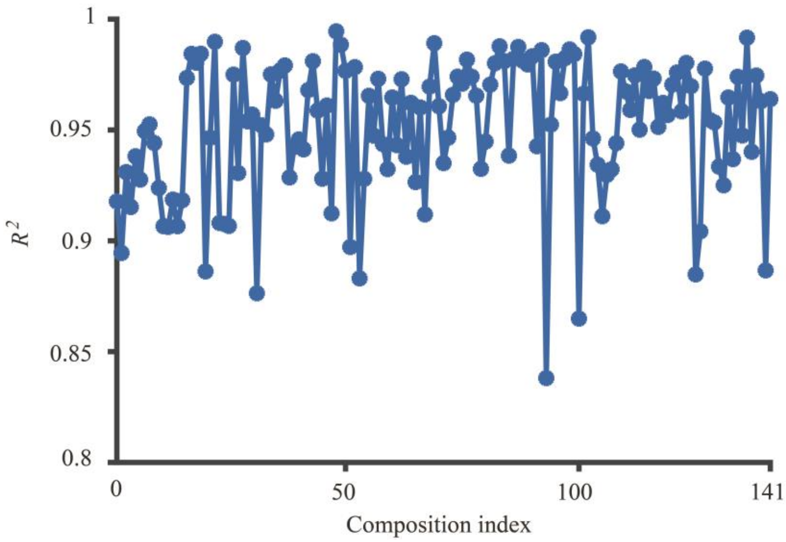

Figure SMII 10. The coefficient of determination R-squared (goodness of fit) for CCDF fitting the Power Law in log-log coordinate system. R^2 of all pieces are larger than 0.83 and most of them (132 pieces) are larger than 0.9.

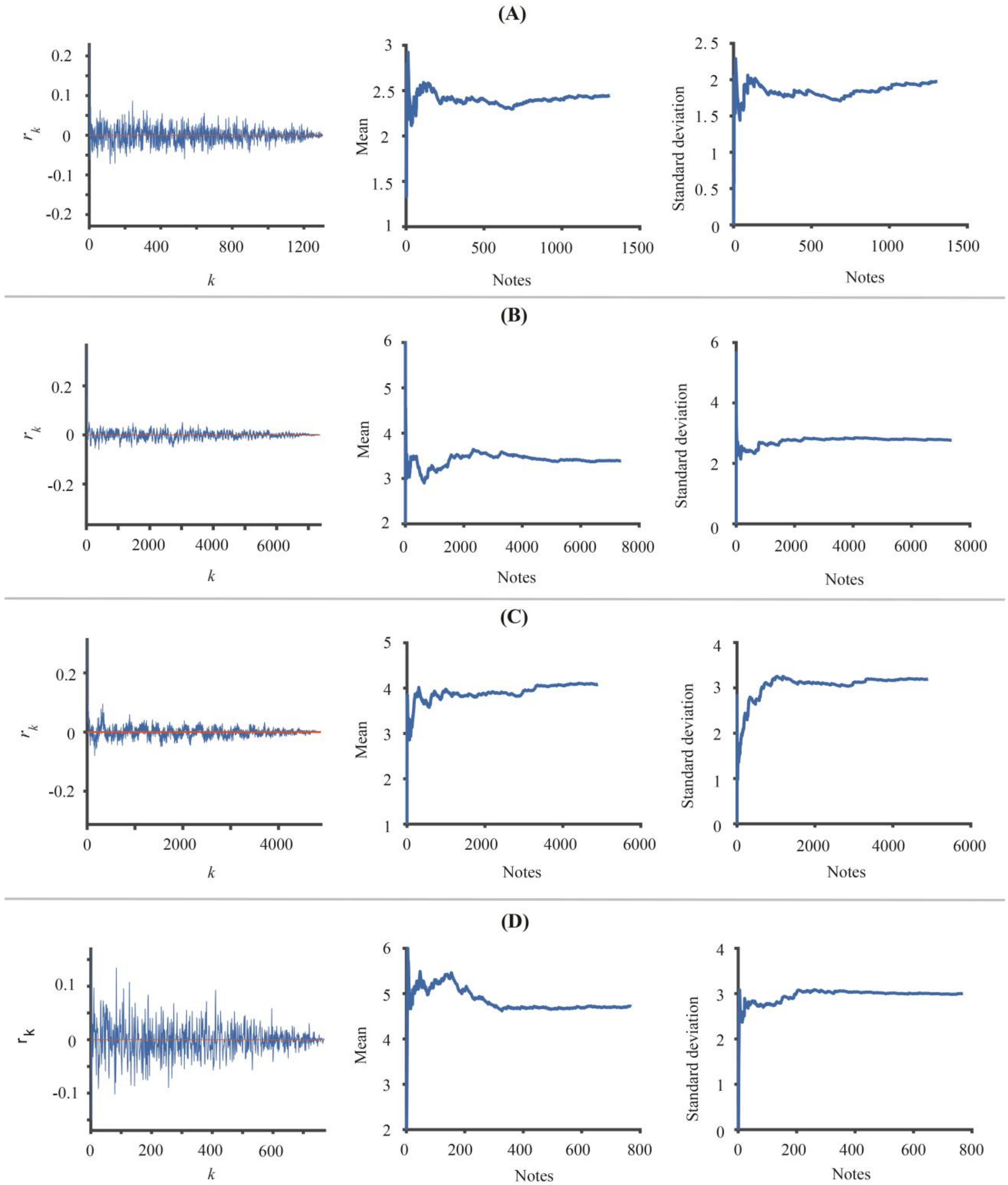

Figure SMII 11. The AFCs, means, and standard deviations with the latter half of note sequences of four music pieces. The AFCs approach 0, meanwhile the means and standard deviations are stationary. (A) The Art of Fugue, Counterpoint, BWV 1080 by J. S. Bach. (B) Fantastic Symphony, Op. 14, Mvt. 5 Songe d'une nuit du sabbat by Berlioz. (C) Symphony No. 2 in D Major, Op. 73, Mvt. 1 by Brahms. (D) Thriller by Michael Jackson.

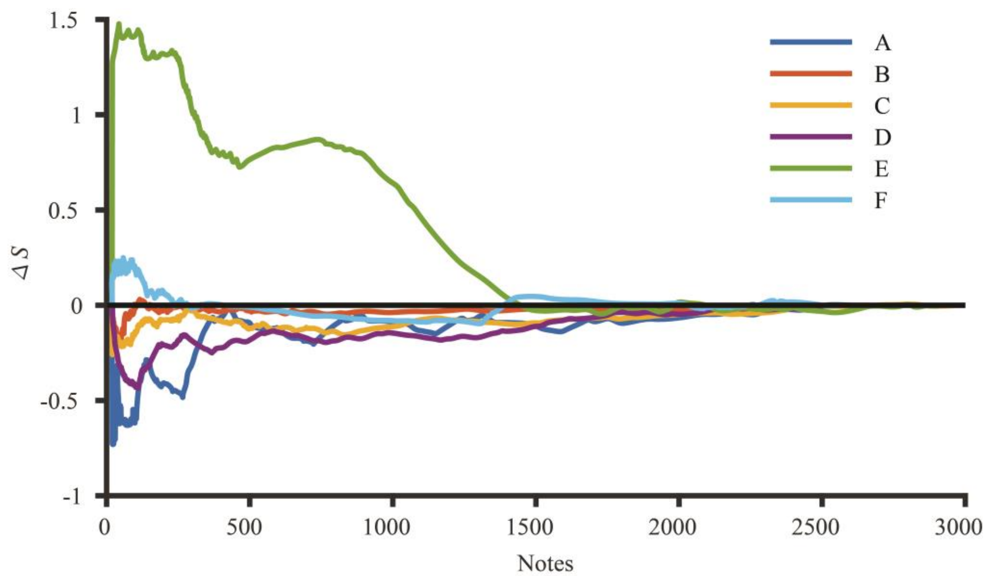

Figure SMII 12. The difference ΔS between the smoothness of melody curves S and the smoothness attractor m_e with the melody developing. ΔS approaches to 0 with the note sequence moving on, which illustrates the smoothness of melody curves S tending to smoothness attractor m_e. (A) Sonata D Minor, K. 141, by D. Scarlatti. (B) The Art of Fugue, Counterpoint, BWV 1080, by J. S. Bach. (C) Messiah HWV 56, Part 1: Every valley shall be exalted (tenor), by Handel. (D) Piano Sonata No. 16 in C Major, K. 545, Mvt. 1, by Mozart. (E) La campanella Grandes etudes de Paganini, S. 141 No. 3, by Liszt. (F) Der Erlkonig, Op. 1 D. 328, by Schubert.

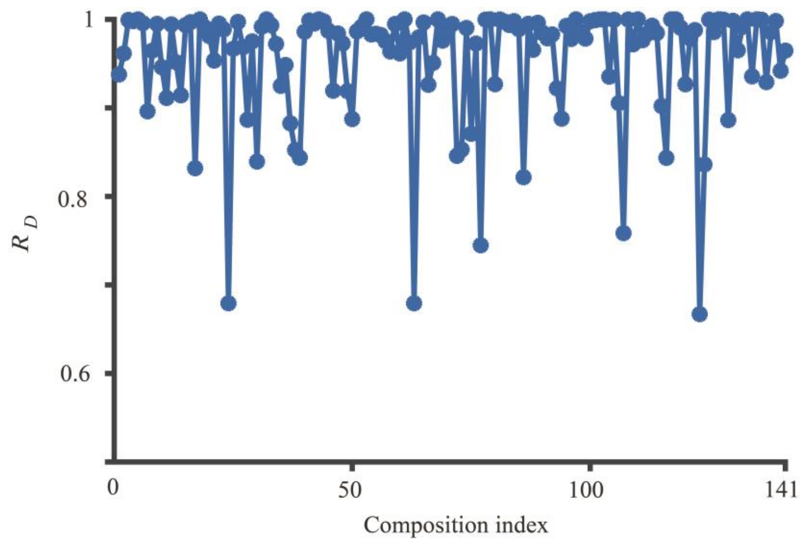

Figure SMII 13. The ratios of the total difference to the total positive difference of the arithmetic average filtering H_v^f. All R_D of 141 music pieces are above 0.67 and 85.1% of them are larger than 0.9 such that the melody variations entropy H_v reaches the maximum with the melody developing.

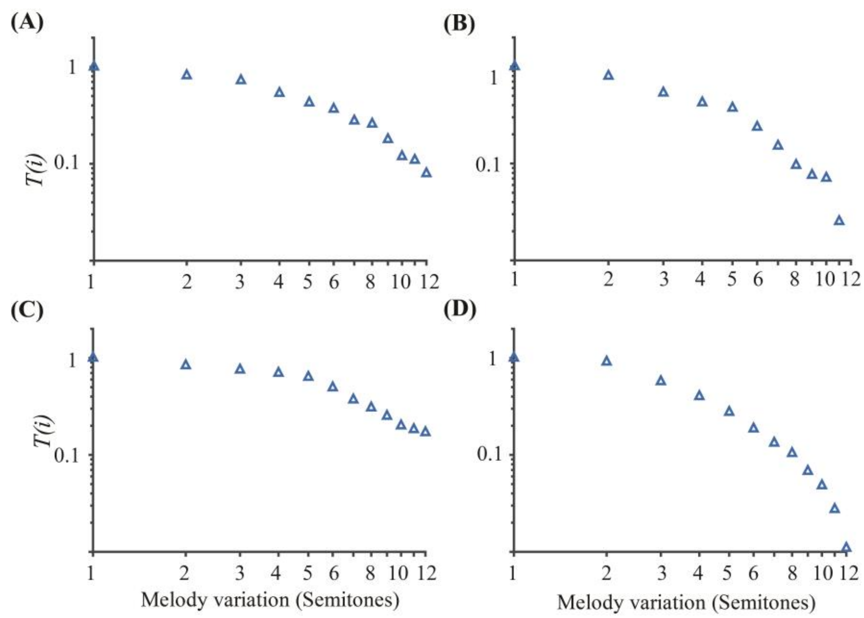

Figure SMII 14. The CCDFs of the variations (in semitones) of the atonal music DO NOT observe the Power Law. (A) Schoenberg, Madonna (vocal) from Pierrot Lunaire. (B) Berg, Prolog 1 from Lulu. (C) Webern, the second song from Der siebente Ring. (D) Messiaen, Turangalila Symphony, Mov. II.

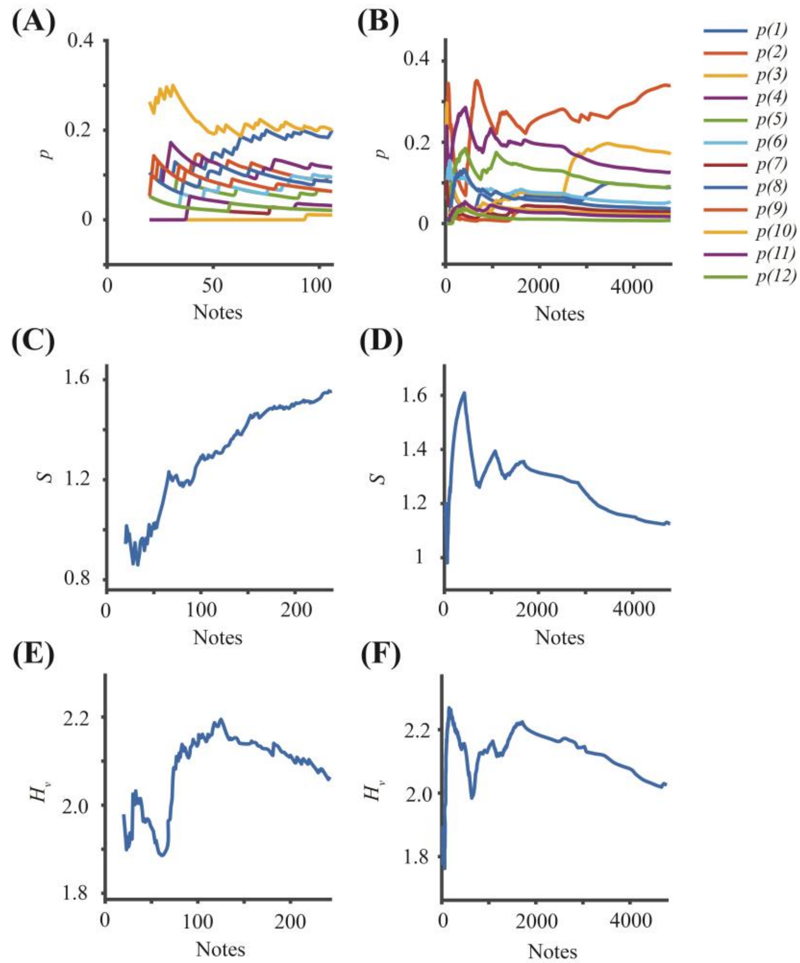

Figure SMII 15. The three mathematical characteristics of some atonal music that do not observe the Power Law. (A) and (B) show the probability $p(i)$ of Madonna (vocal channel) from Pierrot Lunaire by Schoenberg and the second movement from Turangalila Symphony by Messiaen beginning with the first 20 notes. With the melody developing, the frequencies are not stationary. (C) and (D) are the smoothness of melody curves S of the second song from Der siebente Ring by Webern and the second movement from Turangalila Symphony by Messiaen, beginning with the first 20 notes. They do not tend to be any constant. (E) and (F) show that the melody variations entropies decrease around the note of the 120th and 1800th. These two compositions are the second song from Der siebente Ring by Webern and the second movement from Turangalila Symphony by Messiaen, respectively.

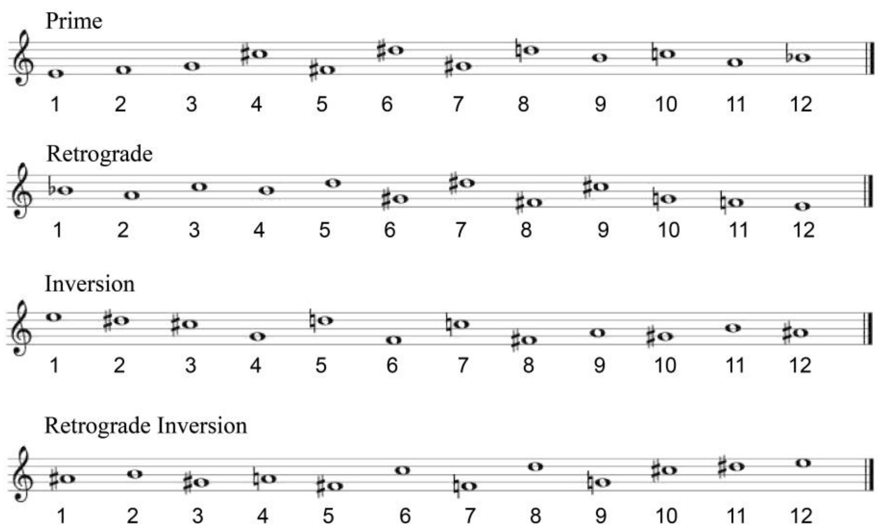

Figure SMII 16. The four basic forms (prime, retrograde, inversion, and retrograde inversion) of the twelve-tone row from Suite, Op. 25 (1923) by Schoenberg [5].

Table SMII 2. An overview of 141 tonal compositions of 14 periods and genres from pre-Baroque to modern pop music.

Group index	Period	Genre	Composition	Composer	Music index
1	Before 17th	Before Baroque	(1). Conductus: Ave virgo virginum (2). Bon jour, bon mois, bon an et bonne estraine (3). Chanson balladée (4). Quant en moy/Amour et biaute/Amara valde (5). Miserere (6). Missa Pange lingua (7). Si dolce el tormento (8). Goe yee, my canzonets (9). Es taget vor dem Walde (10). Missa Ecce ancilla domini (11). Quam pulchra es (12). Thus Sings My Dearest Jewel (13). Voce mea ad Dominum (14). Voglio Di Vita Uscir	Perotin Dufay Machaut Machaut William Byrd Josquin Monteverdi Thomas Morley Elslein Ockeghem John Dunstable Thomas Weelkes Benedetto Croce Monteverdi	1 2 3 4 5 6 7 8 9 10 11 12 13 14
2	1600s - 1750s	Baroque Instrumental Music	(1). La Follia (2). The Four Seasons, Concerto NO. 4 in F Minor, Winter, Op. 8 No. 4, RV. 297 (3). Concerto for 2 Cellos in G Minor, RV. 531 (4). Bassoon Concerto in A Minor, RV. 497 (5). La Follia, Sonata in D Minor, Op. 1 No. 12, RV. 63 (6). Sonata D Minor, K. 141 (7). Keyboard Sonata in A Minor, K. 532 (8). Keyboard Sonata in D Minor, K. 138	F. Couperin Vivaldi Vivaldi Vivaldi Vivaldi D.Scarlatti D.Scarlatti D.Scarlatti	15 16 17 18 19 20 21 22
3	17th	Early Baroque Vocal Music	(1). Bellerophon Ouverture (2). L'Orfeo Vieni Imeneo (3). Verleih uns Frieden (4). O primavera	J. B. Lully Monteverdi Heinrich Schutz Monteverdi	23 24 25 26
4	1600s - 1750s	Baroque	(1). Concerto for oboe and violin in c minor, BWV 1060, II (2). "Peasant Cantata" Mer hahn en neue Oberkeet N0. 8 Aria (soprano): Unser trefflicher, in B Minor, BWV 212 (3). The Art of Fugue, Counterpoint, BWV 1080 (4). Violin Concerto in E Major, BWV 1042 (5). Prelude and Fugue in C$^{\#}$ Major, BWV 848 (6). Minuet in G major, BWV Anh. 115 (7). Toccata and Fugue in D Minor, BWV 565 (8). Violin Concerto in A Minor, BWV 1041 (9). Messiah HWV 56, Part 3: The trumpet shall sound (10). Messiah HWV 56, Part 1: Every valley shall be exalted (tenor) (11). Suite No. 4 in D Minor, HWV 437: III. Sarabande (12). Duetto “Son nata a lagrimar”, HWV 17 (13). Zadok the Priest (Coronation Anthem No. 1, HWV 258) (14). Passacaglia, HWV 432 (15). Les Boreades, Act 4 Entree de Polymnie (16). Las indias galantes Les Sauvage (17). La poule	J. S. Bach J. S. Bach J. S. Bach J. S. Bach J. S. Bach J. S. Bach J. S. Bach J. S. Bach Handel Handel Handel Handel Handel Handel Rameau Rameau Rameau	27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43
5	1730s - 1750s	Early Classicism	(1). Che farò senza Euridice From Orfeo ed Euridice (2). La serva padrona Se tu m'ami	C. W. Gluck G. B. Pergoles	44 45
6	1730s - 1815s	Classicism	(1). Piano Sonata No. 15 in D Major, Op. 28 "Pastoral" (2). Symphony No. 5 in c minor, Op. 67 (3). Violin Sonata No. 5 in F Major, Spring Mvt. 1 Allegro, Op. 24 (4). Symphony No. 9 in D Minor, Mvt. 2 Part 1, Op. 125 (5). Hallelujah From Christ On The Mount Of Olives, Op. 85 (6). "Moonlight Sonata" Piano Sonata No. 14 in C sharp Minor "Quasi una fantasia", Op. 27, No. 2 (7). Symphony No. 6 in F Major, Op. 68 (8). Serenade in D Major for Violin, Viola and Cello, Op. 8 (9). Turkish March in B^{b} Major, Op. 113, No. 4 (10). Overture From Egmont, Op. 84 (11). Deutsche Nationalhymne (Das Lied der deutschen) (12). The tears of Caledonia (Hob XXXIa:201) (13). L Concerto for flute and harp in C Major, K. 299, I (14). Symphony No. 40 in G Minor, K. 550	Beethoven Beethoven Beethoven Beethoven Beethoven Beethoven Beethoven Beethoven Beethoven Beethoven Haydn Haydn Mozart Mozart	46 47 48 49 50 51 52 53 54 55 56 57 58 59

			(15). 12 Variations on “Ah vous dirai-je Maman”, K. 265 (16). Die Zauberflote (The Magic Flute), K. 620, Chor der Priester: O, Isis und Osiris (17). Piano Sonata No. 16 in C Major, K. 545, Mvt. 1 (18). Die Zauberflote (The Magic Flute), K. 620, Queen of the Night Aria (19). Requiem in D Minor, K. 626 Dies irae (20). Dona Nobis Pacem, K. 257, No. 7 (21). Piano Concerto No. 21 in C Major K. 467, Mvt. 2	Mozart Mozart Mozart Mozart Mozart Mozart Mozart	60 61 62 63 64 65 66
7	1820s - 1900s	Romanticism	(1). L'Enfance du Christ, Op. 25, Part II: Ladieu Des Bergers (2). Fantastic Symphony, Op. 14, Mvt. 1 Reveries Passions (3). Fantastic Symphony, Op. 14, Mvt. 4 Marche au supplice (4). Fantastic Symphony, Op. 14, Mvt. 5 Songe d'une nuit du sabbat (5). Symphony No. 2 in D Major, Op. 73, Mvt. 1 (6). Symphony No. 3 in F Major, Op. 90, Mvt. 1 (7). Symphony No. 3 in F Major, Op. 90, Mvt. 3 (8). Symphony No. 4 in E Minor, Op. 98, Mvt. 1 (9). Symphony No. 4 in E Minor, Op. 98, Mvt. 4 (10). Symphony No. 4 in E Minor, Op. 98, Mvt. 2 (11). Mazurka in g minor, Op. 67, No. 2 (12). 12 Etudes In G^b Major, Black Keys, Op. 10 No. 5 (13). Consolation No. 3 in D^b Major, S. 172 (14). La campanella Grandes etudes de Paganini, S. 141 No. 3 (15). Bndiction de Dieu dans la solitude, S. 173 (16). Hungarian Rhapsody in $C^\#$ Minor, S. 244 No. 2 (17). Liebestraum No. 3 In A^b Major, S. 541 No. 3, Notturno III O lieb (18). Transcendental Etudes, S. 139 No. 2, Molto Vivace (19). Transcendental Etudes, S. 139 No. 5, Feux Follets (20). Wedding March, Op. 61 (21). Elijah, Op. 70, MWV A25, So ihr mich von ganzem Herzen suchet (22). Symphony No. 2 in B^b Major, Op. 52, Hymn of Praise (23). Lieder ohne Worte, Op. 67 No. 2, Lost Illusions (24). Piano Sonata in G Minor, Op. 105 (25). Ruh Du bist die Ruh', D. 776 , Op. 59 No. 3 (26). Impromptu No. 1 in F Minor, Op. 142, D. 935 (27). Der Erlkonig, Op. 1 D. 328 (28). Quintet in A Major, Op. 114, The Trout (29). Allegretto in C Minor, D. 915 (30). Symphony No. 8 in B Minor, D. 759 (31). Die Forelle, D. 550 (32). Impromptu in A^b Major, Op. 90 No. 4, D. 899 (33). String Trio in B^b Major, D. 471 Allegro (34). Symphony No. 9 in C Major, D. 944, The Great Mvt. 1 (35). String Quartet Death and the Maiden in D Minor, D. 810 (36). 5 Lieder, Op. 106 No. 1, Stndchen (37). Kinderszenen, Op. 15 (38). Frauenliebe und Leben, Op. 42 No. 2, Er der Herrlichste von allen	Berlioz Berlioz Berlioz Berlioz Brahms Brahms Brahms Brahms Brahms Brahms Chopin Chopin Liszt Liszt Liszt Liszt Liszt Liszt Liszt Mendelssohn Mendelssohn Mendelssohn Mendelssohn Mendelssohn Schubert Schubert Schubert Schubert Schubert Schubert Schubert Schubert Schubert Schubert Schubert Schubert Schumann Schumann	67 68 69 70 71 72 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99 100 101 102 103 104
8	19th	19th Century Opera	(1). Der Freischutz Jagerchor (2). Faust Avant de quitter ces lieux (3). William Tell Overture (4). Ride of the Valkyries	Weber Gounod Rossini Wagner	105 106 107 108
9	1850s -1900s	19th Century Nationalism	(1). Swan Lake Complete Ballet, Op. 20 (2). Norwegian Dance, Op. 35 No. 3 (3). Fire Salmer: I Himmelen, Op. 74 No. 4 (4). Peer Gynt Suite, Op. 46 No. 1, Morning Mood (5). Piano Concerto in A Minor, Op. 16 Mvt. 1 (6). Norwegian Dance, Op. 35 No. 1 (7). Norwegian Dance, Op. 35 No. 2	Tchaikovsky Grieg Grieg Grieg Grieg Grieg Grieg	109 110 111 112 113 114 115
10	1880s - 1910s	Late Romanticism	Symphony No. 6 in A Minor, Mvt. 1	Mahler	116
11	1880s - 1930s	Impressionism	(1). Reverie (2). Mirrors, M. 43, I. Noctuelles (3). Mirrors, M. 43, IV. Alborada del gracioso (4). Gaspard de la nuit, M. 55, I. Ondine	Debussy Ravel Ravel Ravel	117 118 119 120

12	20th	20th Century Nationalism	(1). Andante Festivo (2). Valse Triste (3). Violin Concerto (4). Appalachian Spring (5). Rhapsody in Blue, Piano solo (6). An American in Paris, Piano solo (7). Piano Concerto in F Major, I. Allegro for 2 pianos (8). Piano Concerto in F Major, II. Adagio Andante con moto for 2 pianos (9). Piano Concerto in F Major, III. Allegro agitato for 2 pianos	Sibelius Sibelius Sibelius Copland Gershwin Gershwin Gershwin Gershwin Gershwin	121 122 123 124 125 126 127 128 129
13	1900s - 1950s	Neoclassicism	(1). Trumpet Sonata, Mvt. 2 (2). Scaramouche, II Modéré	Hindemith Milhaud	130 131
14	1950s - now	Pop Music	(1). Nikita (2). Yesterday (3). Memory Cats (4). As Long As You Love Me (5). Thriller (6). Wine (7). Roppongi Pure School (8). Forever love (9). Blue and White Porcelain (10). Coral Sea	Elton John The Beatles Lloyd Webber Backstreet Boys Michael Jackson Japan and Korea Japan and Korea Japan and Korea Jay Chou (China) Jay Chou (China)	132 133 134 135 136 137 138 139 140 141

3 Legends for Datasets (separate file)

Dataset S1.xlsx: An overview of tonal music. This table includes three worksheets. The first worksheep named "MusicInf" provides all the information of the music pieces with the index, file name, composer, and title. The second worksheet "Genre&Period" gives the group index, genres, periods, and the typical musicians, and the last worksheet "MidiPitch" provides all the pitches and channels of 141 compositions.

Dataset S2.xlsx: The probability P of all 141 music pieces. The worksheet "P_Figure" provides the figures of P, from the first 20 notes to the entire notes and the figures of mean, standard deviation, and SAF of the latter half of the musical note sequence.

Dataset S3.xlsx: The smoothness of melody curves S of music pieces. The first worksheet "S_Figure" depicts the figures of ΔS from the first 20 notes to N notes. The second worksheet "m_e" presents the mean value and the standard deviation sum of S of the last half of notes.

Dataset S4.xlsx: The variation entropy H_v of all music. The worksheet "Hv_Figure" shows the figures of H_v from the first 20 notes to the entire notes.

Dataset S5.xlsx: The CCDFs of all 141 music pieces in Dataset S1.xlsx. The first worksheet "Figure" provides the CCDF figures in the log-log coordinate system. The second worksheet "Fitting" is the fitting results by *polyfit* function in the log-log coordinate system. The third worksheet "T(i)" lists the CCDF data.

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