Common quantitative characteristics of music melodies —
pursuing the constrained entropy maximization casually in composition

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CONTENS

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PART 1

Introduction —— Fundamental questions

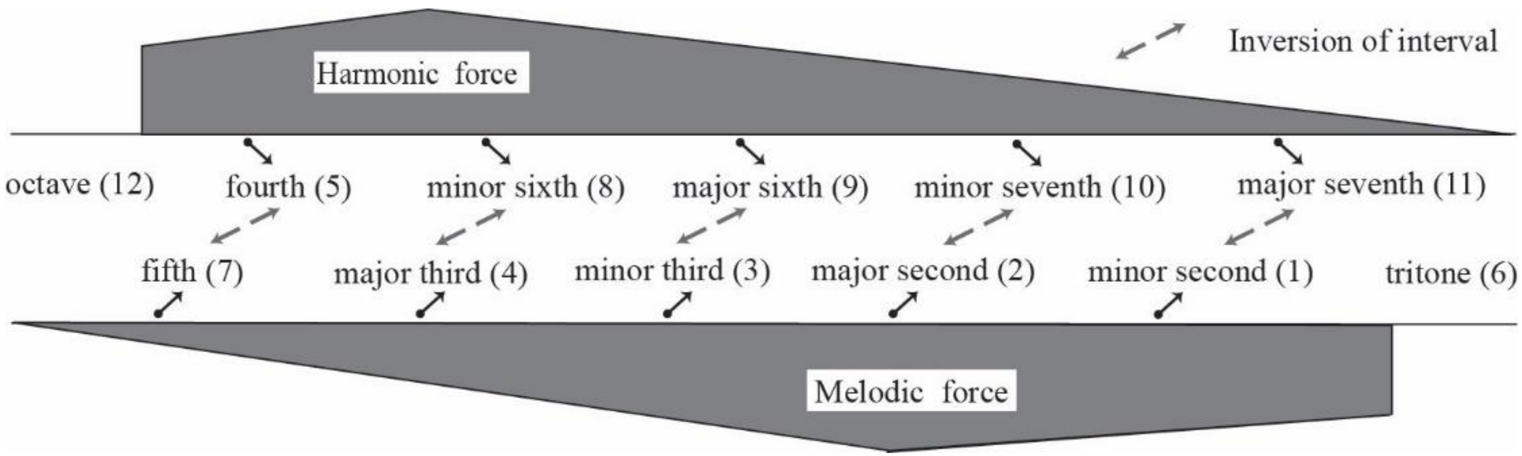
Fundamental questions on music

- ☐ Why general public without music training likes beautiful melody?
- ☐ What in common music composers have been pursuing?
- ☐ If there is any scientific mechanism behind?

PART 1 Introduction —— Related musical terms

- **Melody**: pitch and rhythm
- **Tonal music**: composed around a tonal center (tonic)
- **Melodic interval or melody variation**: the difference of two adjacent notes in a melody measured by number of semitones
- **Semitone** (a half step or a half tone): the smallest music interval, e.g. from C to C♯

Introduction —— Quantifying melodic intervals or melody variations



Driving force of melody movement

harmonic force and melodic force of intervals in Hindemith's theory

PART 1

Introduction —— Efforts made to quantify melodic intervals or melody variations

- The *General Pitch Interval Representation*
- Relations to consonance and predicting music emotion
- …

Few efforts made to connect the composition theory and the quantitative characteristics of melodic intervals

CONTENS

PART 2

Three characteristics of tonal music melody

Three mathematical characteristics of melody variations:

I. if the notes of a music piece are considered as a stochastic sequence, the probability distribution of a particular melodic interval is stationary. That is, the probability of a particular melodic interval tends to be constant as the note sequence moves on.

II. The curve of melody movement is smooth and its smoothness tends to be a small constant “smoothness attractor”.

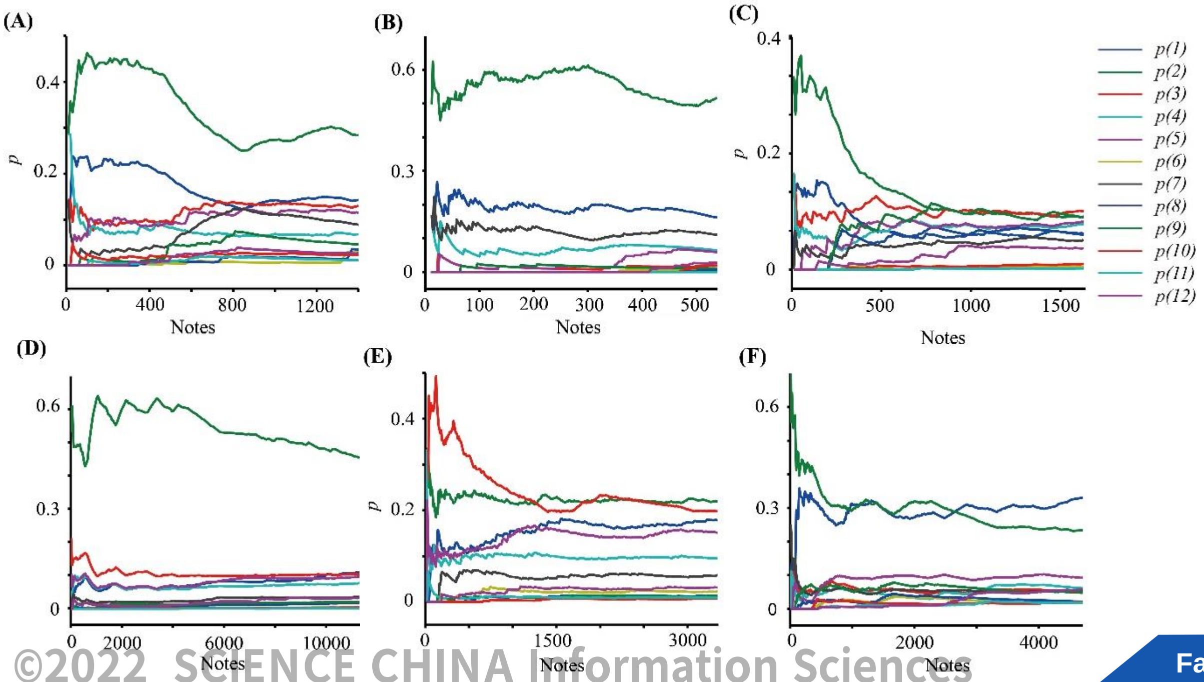
III. The melodic intervals are diversified so that the interval entropy is maximized subject to the above two.

Characteristic I: *Stationary distribution of melody variations*

The distribution of the melody variation in semitone unit is steady due to frequent repetition in melody and highly consistency of music structure

$$\lim_{N\to\infty} p(i,N) = p(i)$$

$p(i,N)$: frequencies of melodic intervals i with N notes.



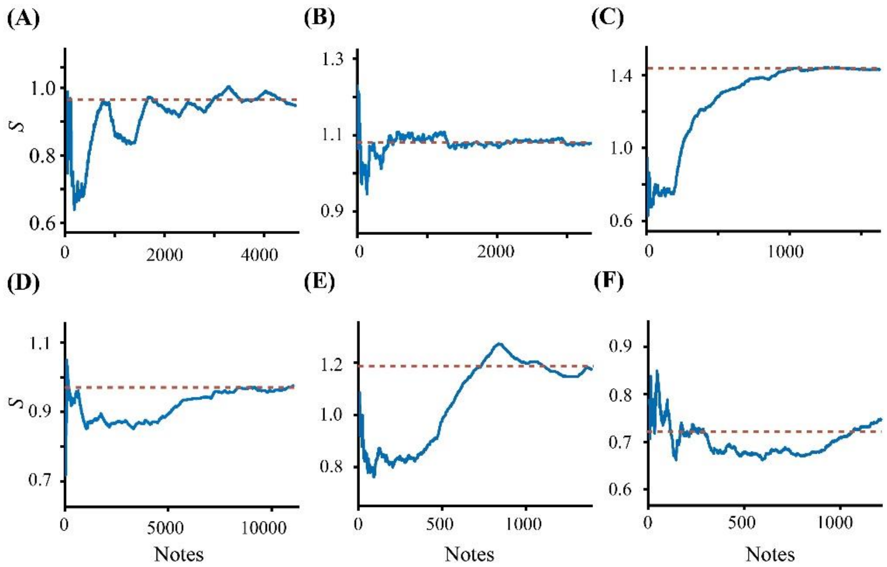
Frequencies of melody variations (semitones) with the melody developing. (A) Webber, Memory, Cats. (B) Vivaldi, the Four Seasons, Spring. (C) Schubert, the Linden Tree. (D) Mendelssohn, Wedding March. (E) Beethoven, Symphony No. 5. (F) Mozart, K. 265.

PART 2

Characteristic II: *Smoothness of melody curves*

Melody variation has clear and similar envelop and the smoothness factor approximates to a small constant: melody smooth attractor

$$S = \sum_i p(i) \log i = E(\log i) \rightarrow m$$

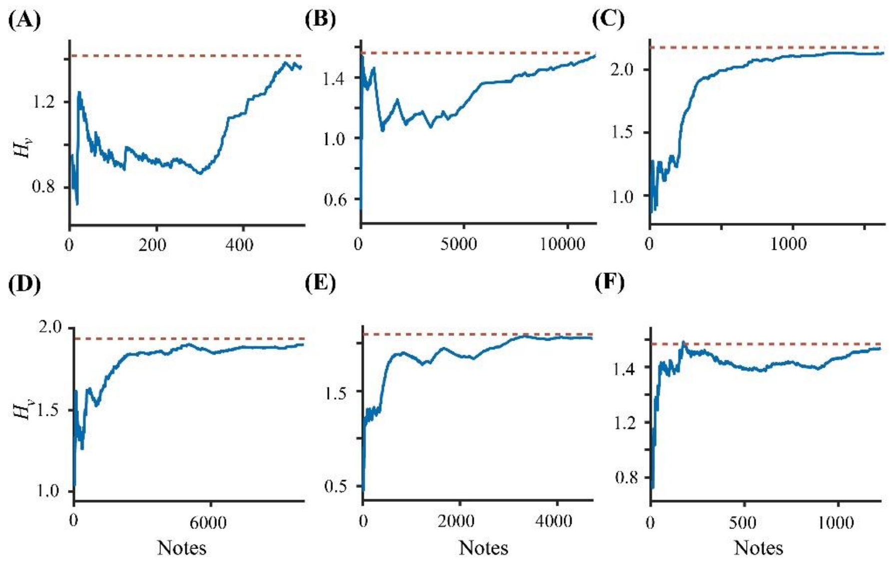


The smoothness of melody S tends to a small "smoothness attractor" with the melody developing. (A) Mozart, K. 265. (B) Beethoven, Symphony No. 5. (C) Schubert, The Linden Tree. (D) Mendelssohn, Wedding March. (E) Sibelius, Finlandia. (F) Bach, BWV 1060, II.

Characteristic III: *Entropy maximization of melody variations*

The functions of various melody variations are different and melody must be fully developed with diversity so that the variation entropy should be maximized

$$H_v = -\sum_i p(i) \log p(i)$$



The melody variation entropy H_v achieves its maximum with the melody developing. (A) Vivaldi, The Four Seasons, Spring. (B) Mendelssohn, Wedding March. (C) Shubert, The Linden Tree. (D) Beethoven, Symphony No 5. (E) Mozart, K. 265. (F) Jay Chou, Blue and White Porcelain**.**

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PART 3

Mathematical model of music melodies

Based on the three characteristics above, the following constrained entropy maximization problem can be formulated as necessity for composers, where $p(i)$ is the probability density function of melodic interval for semitone i :

$$\max H_v = -\sum_i p(i) \ln p(i)$$

Maximization interval entropy

subject to

$$\begin{cases} \sum_i p(i) = 1 \\ \sum_i p(i) \ln i = m \end{cases}$$

Melodic smoothness attractor m is a small constant.

PART 3

Solving for probability density function of melodic intervals (1)

By using the Lagrange multiplier method, we obtain

$$L = -\sum_i [p(i)\, ln\, p(i) + \lambda_0 p(i) + \lambda_1 p(i)\, ln\, i] + \lambda_0 + \lambda_1 m;\ \lambda_0, \lambda_1 > 0$$

Let

$$J = \sum_i p(i)\, ln\, p(i) + \lambda_0 p(i) + \lambda_1\, p(i) ln\, i = \sum_i L_i$$

By calculus of variations, we have

$$\delta J = \sum_i \left[\frac{\partial L_i}{\partial p(i)} \delta p(i) \right]$$

where $\delta p(i)$ denotes the first-order variation of $p(i)$.

PART 3

Solving for probability density function of melodic intervals (2)

Functional extremum condition：

$$\frac{\partial L_i}{\partial p(i)}\delta p(i) = 0$$

Solving the equation above and then using the approximate integration method, we obtain the CCDFs $T(i)$ (complementary commutative distribution function) of melodic intervals of tonal music observe the Power Law:

$$\boldsymbol{T(i) = P(I \geq i) = 1 - F(i) = ci^{-D}}$$

where $c > 0, D > 0, i = \{I_a, I_a + 1, \dots, I_b\}$.

The Power Law of the tonal music melody is the result of the composition theory taught in music schools for more than three hundred years.

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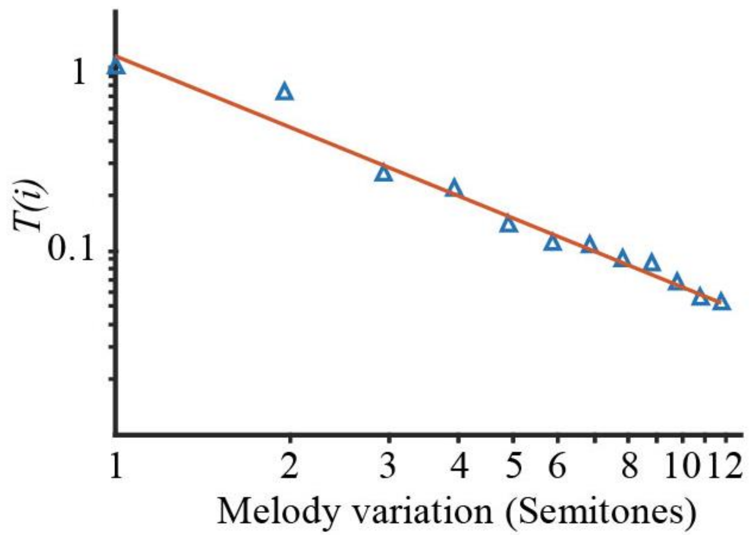
PART 4 Power Law of melodic intervals of tonal music

Melodic intervals of tonal music observing the Power Law is derived

$$T(i) = P(X \geq i) = ci^{-D}$$

where $\log T(i)$ versus $\log i$ is an affine function or a straight line geometrically, and is consistent with the observation results on any tonal music without exception!

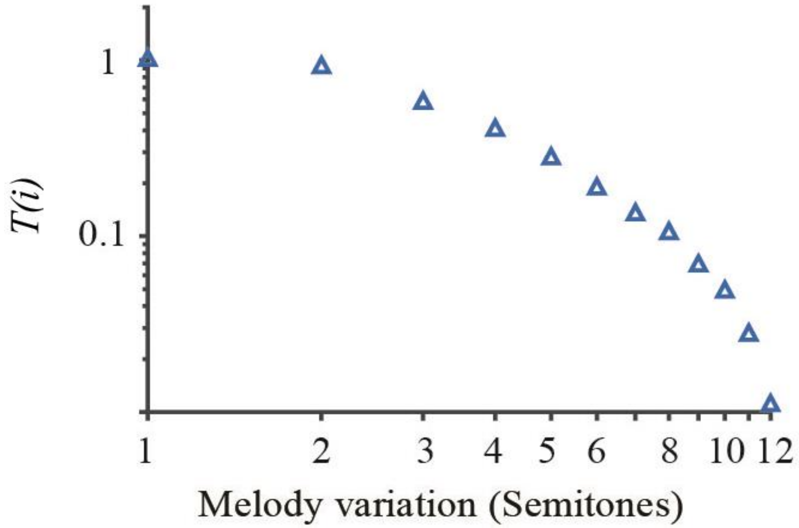


Mozart, Concerto for flute and harp in C Major.

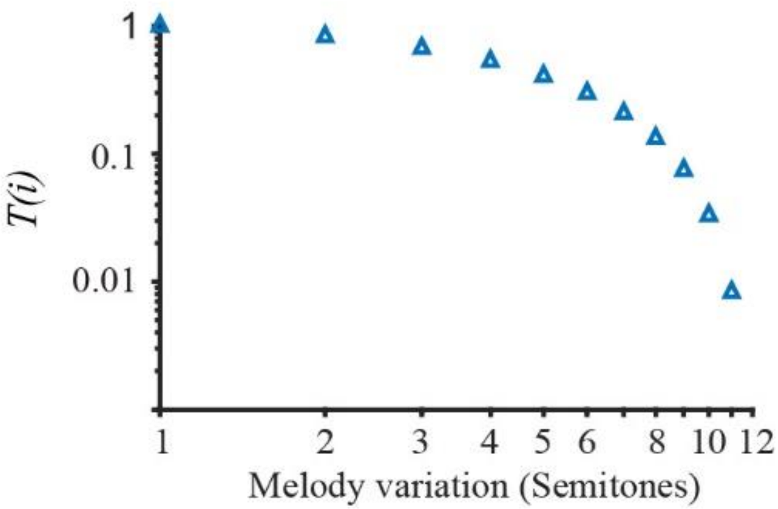
PART 4

The atonal music that does not observe Power Law

In atonal music including the twelve-tone system or the serialism advocated by Schoenberg, etc., the CCDFs of “melodic” intervals DO NOT observe the Power Law.



Messiaen, Turangalila Symphony, Mov. II.



Simulation based on Schoenberg’s twelve-tone system

PART 5

Conclusions

- Melody of tonal music has three mathematical characteristics: the stationary distribution of melody variations; the smooth curve with a small smoothness attractor; the entropy maximization of melody variations with constraints.
- The Power Law on melodic intervals of tonal music can be derived based on the model formulated with the above three characteristics.
- Th finding can help us comprehend music with quantification, can serve as a basis for investigating if there is any physiological reason why the public without music training like the music with melodies, and provides a necessity for music composition aided by artificial intelligence.

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Thank You!