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Multicriteria optimization problems of finite horizon stochastic cooperative linear-quadratic difference games

Chenchen PENG & Weihai ZHANG*

College of Electrical Engineering and Automation, Shandong University of Science and Technology, Qingdao 266590, *China*

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Abstract This paper investigates the Pareto optimality of the regular and the indefinite stochastic cooperative linear-quadratic difference games in a finite time horizon. We derive a general form and a linear property of the solution to the linear stochastic difference system by defining several sequences of bounded and linear operators. The performance criteria's convexity can be guaranteed naturally under the weighted matrices' constraints for the regular cooperative game, and the weighting technique can well characterize the Pareto optimality. We also establish a novel convexity criterion for the cost functionals of the indefinite cooperative game, in which we find that the minimization of the performance criteria's weighted sum is equivalent to the Pareto optimal strategies. To derive all the Pareto optimal strategies and solutions, we present a computing algorithm using the weighted difference Riccati equation and the weighted difference Lyapunov equation for the regular and the indefinite cases. We present a practical example in the economy to validate the results.

Keywords multicriteria optimization problems, Pareto optimality, cooperative difference games, discrete-time stochastic LQ theory, the difference Riccati equation, the difference Lyapunov equation

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1 Introduction

A multicriteria optimization problem is referred to as an indefinite stochastic cooperative linear-quadratic (LQ) difference game when all players (more than one) determine to cooperate their strategies with one aim to minimize their quadratic costs, where the weighted matrices are indefinite, and the game's dynamic model is a linear stochastic discrete-time system. The continuous-time indefinite stochastic LQ optimization problem was initiated by Chen et al. [1], in which, the indefinite problem may still be well-posed even when the weighted matrix for the control term is indefinite if the diffusion term of the dynamic system is affected by the control input. Similarly, the discrete-time indefinite stochastic LQ optimal control may be well-posed if the dynamic system relies on the control. For reader's brevity, an example is shown as follows:

$$\min J = \sum_{k=0}^{N-1}(x_k^2 + r_k u_k^2) + x_N^2 \quad \text{s.t.} \quad x_{k+1} = 0,\ x_0 = 0, \tag{1}$$

where $r_k < 0$. It is easy to see that problem (1) is not well-posed. Actually, as $|u_k| \to +\infty$, $J = \sum_{k=0}^{N-1} r_k u_k^2 \to -\infty$. Moreover, consider the following LQ optimization problem:

$$\min J = \sum_{k=0}^{N-1}(x_k^2 + r_k u_k^2) + x_N^2 \quad \text{s.t.} \quad x_{k+1} = u_k,\ x_0 = 0. \tag{2}$$

* Corresponding author (email: w_hzhang@163.com)

A simple calculation shows that $J=\sum_{k=0}^{N-1}(r_k+1)u_k^2$. Thus, optimal control problem (2) is well-posed when $r_k>-1$ (with the optimal control $u_k^*=0$). Now, consider a stochastic case of (2):

$$\min J=\mathbb{E}\left[\sum_{k=0}^{N-1}(x_k^2+r_ku_k^2)+x_N^2\right] \quad \text{s.t.} \quad x_{k+1}=u_k+u_k\omega_k,\ x_0=0, \tag{3}$$

where ω_k denotes a sequence of independent random variables. Substituting the state equation into the cost, we have that $J=\mathbb{E}[\sum_{k=0}^{N-1}(2+r_k)u_k^2]$. Therefore, the problem (3) is sensible when r_k is a deterministic function with $r_k>-2$ (in this situation, the optimal control $u_k^*=0$). Certainly, r_k cannot be negative enough. For example, if $r_k<-2$, the problem (3) would be ill-posed. The aforementioned observation reveals the main difference between the regular case (where the weighted matrix for the state is positive-semidefinite, and the weighted matrix for the control is positive-definite) and the indefinite case in the deterministic and the stochastic LQ optimization problems. For the discrete-time case, Rami et al. [2] studied the finite horizon indefinite stochastic LQ problem where the parameters of difference systems imposed multiplicative and additive non-Gaussian noises for the control and the state terms. Rami et al. [2] introduced a novel difference Riccati equation (DRE) and obtained the optimal state feedback strategy using the matrix minimum principle. In addition, the equivalence between the attainability, the well-posedness of the stochastic LQ control, the solvability of the DRE and the feasibility of the introduced linear matrix inequalities (LMIs) was derived. Zhang et al. [3] accomplished the follow-up investigations on the infinite time horizon.

Generally, by cooperation, the performance criterion of some player in the multiobjective optimization problem will not be changed by himself anymore. This paper is concerned with the cooperative results: no cost can increase without decrement to other performance criteria, yielding Pareto optimality notion. A joint strategy $\hat{u}$ is a Pareto optimal strategy if it has the property that there is not a strategy u such that at least one of the inequalities $J_i(u)\leqslant J_i(\hat{u}),\ i=1,\ldots,\mathcal{N}$ holds strictly. We call the point $(J_1(\hat{u}),J_2(\hat{u}),\ldots,J_{\mathcal{N}}(\hat{u}))\in\mathbb{R}^{\mathcal{N}}$ a Pareto optimal solution and refer to the set comprising all optimal solutions as the Pareto frontier. A Pareto optimal solution can be referred to as an undominated solution because other solutions cannot dominate it. In 1881, Edgeworth [4] discussed the dominated and undominated solutions (see [5, 6] for early developments). In the deterministic cooperative game theory, the Riccati equation method and the maximum principle (MP) approach have been employed to investigate the Pareto optimality (see [7–10] for the differential games, and [11] for the difference games). Engwerda [7] investigated the cooperative LQ differential games with two players over the finite and the infinite time horizons. Engwerda [8] tackled the finite horizon cooperative differential game of nonlinear systems. The action space investigated changes from the convex case [7] to the general case [8]. Later, the results in [8] are generalized to the situation with an infinite time horizon [9, 10]. Furthermore, Peng and Zhang [11] investigated the discrete-time counterpart by [9, 10], in which the necessary conditions were obtained on the basis of the MP of the discrete-time type.

Zhu and Zhang [12] identified the finite horizon stochastic LQ Pareto optimal strategies of time-varying difference systems using the weighting technique for the regular case. The adaptive dynamic programming approach [13] obtained the Pareto optimal strategies of the infinite horizon regular mean-field stochastic cooperative LQ differential game. For more details on the regular stochastic LQ Pareto optimality, see [14,15] and the references therein. However, the weighting technique is a sufficient criterion to acquire Pareto optimal strategies as pointed out by [8,11], which also holds for the indefinite stochastic cooperative LQ game. Although the cooperative game theory has been widely investigated, there still exists an open problem, which is to derive a condition to make the weighting approach a necessary and sufficient condition for finding all the Pareto optimal strategies of the multicriteria optimization problem. This paper investigates the unsolved issue for finite horizon stochastic cooperative LQ difference games in both the regular and the indefinite cases.

The highlights of this paper can be generalized as follows.

(1) Compared with [7–9, 11, 12], this paper considers that the weighted matrices of the performance criteria are indefinite not only for the state term but also for the control term, and that the difference systems have both state- and control-dependent noises.

(2) By defining several sequences of bounded and linear operators, we derive the performance criteria's convexity for the regular stochastic cooperative LQ difference game, which can be guaranteed spontaneously because of the nature of such a problem.

(3) To obtain all the optimal solutions of the indefinite cooperative game, we derive a novel criterion for the convexity of costs using the weighting technique.

(4) All the optimal strategies and solutions are respectively acquired by the solutions to the weighted DRE and the weighted difference Lyapunov equation (DLE) for both the regular and the indefinite cooperative difference games.

The rest of this article is arranged as follows. Section 2 formulates the finite horizon stochastic cooperative LQ difference game and presents some basic results on Pareto optimality. Section 3 firstly studies the regular cooperative LQ game and further considers the indefinite case, where a new criterion for the convexity of the performance criteria is derived and the computational algorithm for Pareto optimality is presented. Section 4 gives an example in the economy to verify the results' correctness. Finally, Section 5 concludes the paper.

2 Problem formulation and preliminaries

Let us consider the following finite horizon stochastic cooperative LQ difference game with $\mathcal{N}$-player:

$$\begin{aligned} &\min J_i(x_0; u_{1,k}, u_{2,k}, \ldots, u_{\mathcal{N},k}) \\ &:= \mathbb{E}\left[\sum_{k=0}^{N-1}\left(x_k^{\mathrm{T}}\mathbb{Q}_i^k x_k + \sum_{j=1}^{\mathcal{N}} u_{j,k}^{\mathrm{T}}\mathbb{R}_{ij}^k u_{j,k}\right) + x_N^{\mathrm{T}}\mathbb{G}_i^N x_N\right],\ i \in \bar{\mathcal{N}} := \{1, \ldots, \mathcal{N}\}, \end{aligned} \tag{4}$$

$$\text{s.t.} \begin{cases} x_{k+1} = \left(\mathbb{A}_1^k x_k + \sum_{i=1}^{\mathcal{N}} \mathbb{B}_{i,1}^k u_{i,k}\right) + \left(\mathbb{A}_2^k x_k + \sum_{i=1}^{\mathcal{N}} \mathbb{B}_{i,2}^k u_{i,k}\right)\omega_k, \\ x_0 \in \mathbb{R}^n,\ k \in \mathbb{N} := \{0, 1, \ldots, N-1\}, \end{cases} \tag{5}$$

where $\mathbb{A}_1^k,\ \mathbb{A}_2^k \in \mathbb{R}^{n\times n}$, $\mathbb{B}_{i,1}^k,\ \mathbb{B}_{i,2}^k \in \mathbb{R}^{n\times m_i}$ are given deterministic matrices. In (5), $\{x_k \in \mathbb{R}^n,\ k \in \bar{\mathbb{N}} := \{0, 1, 2, \ldots, N\}\}$ and $\{u_{i,k} \in \mathbb{R}^{m_i},\ k \in \mathbb{N},\ i \in \bar{\mathcal{N}}\}$ are respectively the state variables and the control inputs. $\{\omega_k,\ k \in \mathbb{N}\}$ defined on a complete probability space $\{\Omega, \mathscr{F}, \mathbb{P}\}$ denotes a sequence of real independent random variables with the properties $\mathbb{E}(\omega_k) = 0$ and $\mathbb{E}(\omega_k\omega_s) = \delta_{sk}$ (δ_{sk} represents the Kronecker function, i.e., $\delta_{sk} = 1$ if $s = k$ while $\delta_{sk} = 0$ if $s \neq k$). Let $\mathscr{F}_k$ denote the σ-algebra developed by $\{\omega_s, s = 0, 1, 2, \ldots, k\}$ and $\mathscr{F}_{-1} = \{\emptyset, \Omega\}$ (Ω is the sample space, $\emptyset$ is the empty set). Clearly, $\mathscr{F}_{k-1} \subset \mathscr{F}_k$. Let $\mathscr{F} = \{\mathscr{F}_k\}_{k=0}^N$. In (4), $\mathbb{Q}_i^k \in \mathbb{S}^n$, $\mathbb{R}_{ij}^k \in \mathbb{S}^{m_j}$ and $\mathbb{G}_i^N \in \mathbb{S}^n$ ($\mathbb{S}^n$ is the set of all $n \times n$ symmetric matrices) are deterministic real matrices with proper dimensions. Let $\mathcal{L}_{\mathscr{F}}^2(\mathbb{N}; \mathbb{R}^m)$ denote the set of all $\mathbb{R}^m$-value processes $\phi = \{\phi_k,\ k \in \mathbb{N}\}$ with ϕ_k being $\mathscr{F}_k$-measurable and $\sum_{k=0}^{N-1}\mathbb{E}|\phi_k|^2 < \infty$. Before proceeding, for player $i \in \bar{\mathcal{N}}$, the strategy space is given by $u_{i,k} \in \mathbb{U}_{ad}^i = \mathcal{L}_{\mathscr{F}}^2(\mathbb{N}; \mathbb{R}^{m_i})$. Furthermore, the joint control u can be denoted by

$$u = (u_{1,k}, \ldots, u_{\mathcal{N},k}) \in \mathbb{U}_{ad}^1 \times \cdots \times \mathbb{U}_{ad}^{\mathcal{N}} = \mathbb{U}_{ad} = \mathcal{L}_{\mathscr{F}}^2(\mathbb{N}; \mathbb{R}^m). \tag{6}$$

Evidently, $\mathbb{U}_{ad}$ is a convex set.

An important approach in tackling the Pareto optimality is the weighting technique. Specially, if both the admissible control space and the cost functionals are convex with respect to strategy u, then the weighting technique is a necessary and sufficient condition for achieving optimal strategies; see [16] for the details.

Lemma 1. For a $\theta \in \Theta := \{\theta = (\theta_1, \theta_2, \ldots, \theta_{\mathcal{N}}) \big| \theta_i > 0 \text{ and } \sum_{i=1}^{\mathcal{N}}\theta_i = 1\}$, if $\hat{u} \in \mathbb{U}_{ad}$ and

$$\hat{u} \in \arg\min_{u\in\mathbb{U}_{ad}}\left\{\sum_{i=1}^{\mathcal{N}}\theta_i J_i(x_0; u)\right\}, \tag{7}$$

then $\hat{u}$ is a Pareto optimal strategy. Furthermore, if both $\mathbb{U}_{ad}^i$ and $J_i(x_0; u)$ are convex with respect to u, $\forall i \in \bar{\mathcal{N}}$, then for an optimal strategy $\hat{u}$, there exists a $\theta \in \Theta$ such that Eq. (7) holds.

Lemma 2 shows that the Pareto optimal solutions for each one of the players can be achieved as the solution to a corresponding optimal control problem with constraints. However, the constrained set of a fixed player relies on the Pareto solutions that denote the deterioration of other players; see [17, 18] for the details.

Lemma 2. $\hat{u} \in \mathbb{U}_{ad}$ is an optimal strategy if and only if (iff) $\forall i \in \bar{\mathcal{N}}$, $J_i(x_0;u)$ is minimized by $\hat{u}$ on the set with constraints defined as

$$\mathbb{U}_{ad}^i := \{u | J_j(x_0;u) \leqslant J_j(x_0;\hat{u}),\ j=1,2,\ldots,\mathcal{N},\ j \neq i\}.$$

3 Pareto optimality

In this section, we investigate the finite horizon stochastic cooperative LQ difference game in both the regular and the indefinite cases. For notational convenience, we rewrite the foregoing LQ problem (4) and (5) in a structure, which coincides with that of the standard discrete-time stochastic LQ control. To do so, let $\mathbb{R}_i^k = \operatorname{diag}\left\{\mathbb{R}_{i1}^k, \mathbb{R}_{i2}^k, \ldots, \mathbb{R}_{i\mathcal{N}}^k\right\}, \mathbb{B}_j^k = [\mathbb{B}_{1,j}^k\ \mathbb{B}_{2,j}^k \ldots \mathbb{B}_{\mathcal{N},j}^k],\ j=1,2, u_k = [u_{1,k}^{\mathrm{T}}\ u_{2,k}^{\mathrm{T}}\ \cdots\ u_{\mathcal{N},k}^{\mathrm{T}}]^{\mathrm{T}}$, and then we have the following multicriteria optimization problem for $i \in \bar{\mathcal{N}}$:

Problem ($\mathbb{P}_N^i$-LQ).

$$\min\ J_i(x_0;u) := \mathbb{E}\left[\sum_{k=0}^{N-1}\left(x_k^{\mathrm{T}}\mathbb{Q}_i^k x_k + u_k^{\mathrm{T}}\mathbb{R}_i^k u_k\right) + x_N^{\mathrm{T}}\mathbb{G}_i^N x_N\right], \tag{8}$$

$$\text{s.t.}\quad x_{k+1} = \left(\mathbb{A}_1^k x_k + \mathbb{B}_1^k u_k\right) + \left(\mathbb{A}_2^k x_k + \mathbb{B}_2^k u_k\right)\omega_k,\ x_0 \in \mathbb{R}^n,\ k \in \mathbb{N}. \tag{9}$$

Given $u. \in \mathbb{U}_{ad}$, $(x., u.)$ is an admissible pair if $x. \in \mathcal{L}_{\mathscr{F}}^2(\bar{\mathbb{N}};\mathbb{R}^n)$ is the solution to (9) corresponding to $u. \in \mathbb{U}_{ad}$.

The lemma presented below will be used throughout the article.

Lemma 3. Let $x_{u_1,k}$ be the solution to (9) corresponding to the strategy u_1 and $\mu \in [0,1]$. For a fixed initial value $x_0 \in \mathbb{R}^n$, we have that

$$x_{\mu u_1 + (1-\mu)u_2,k} = \mu x_{u_1,k} + (1-\mu)x_{u_2,k}. \tag{10}$$

Proof. Let

$$\begin{cases}\Psi(k,\ell) = \left(\mathbb{A}_1^k + \mathbb{A}_2^k\omega_k\right)\left(\mathbb{A}_1^{k-1} + \mathbb{A}_2^{k-1}\omega_{k-1}\right)\cdots\left(\mathbb{A}_1^\ell + \mathbb{A}_2^\ell\omega_\ell\right),\ k \geqslant \ell,\\ \Psi(k,\ell) = I,\ k < \ell.\end{cases}$$

In view of the dynamic system (9) of Problem ($\mathbb{P}_N^i$-LQ), for $k \in \mathbb{N}$, we obtain that

$$x_{k+1} = \Psi(k,0)x_0 + \sum_{\ell=0}^{k}\Psi(k,\ell+1)\left(\mathbb{B}_1^\ell + \mathbb{B}_2^\ell\omega_\ell\right)u_\ell.$$

In the sequel, for any $x_0 \in \mathcal{L}_{\mathscr{F}}^2(0;\mathbb{R}^n)$, $u \in \mathbb{U}_{ad}$, we introduce the following operators:

$$\begin{cases}(\Upsilon x_0) = \{[(\Upsilon x_0)_0, \ldots, (\Upsilon x_0)_{N-1}] | (\Upsilon x_0)_k = \Psi(k-1,0)x_0,\ k \in \mathbb{N}\},\\ \bar{\Upsilon}x_0 = \Psi(N-1,0)x_0,\\ \Gamma u = \{[(\Gamma u)_0, \ldots, (\Gamma u)_{N-1}] \,|(\Gamma u)_k = \sum_{\ell=0}^{k-1}\Psi(k-1,\ell+1)\left(\mathbb{B}_1^\ell + \mathbb{B}_2^\ell\omega_\ell\right)u_\ell,\ (\Gamma u)_0 = 0\},\\ \bar{\Gamma}u = \sum_{\ell=0}^{N-1}\Psi(N-1,\ell+1)\left(\mathbb{B}_1^\ell + \mathbb{B}_2^\ell\omega_\ell\right)u_\ell.\end{cases}$$

As a consequence, we obtain that

$$x_k = (\Upsilon x_0)_k + (\Gamma u)_k,\quad k \in \mathbb{N},\ x_N = \bar{\Upsilon}x_0 + \bar{\Gamma}u. \tag{11}$$

Notice that the operators

$$\begin{cases}\Upsilon : \mathcal{L}_{\mathscr{F}}^2(0;\mathbb{R}^n) \mapsto \mathcal{L}_{\mathscr{F}}^2(\mathbb{N};\mathbb{R}^n),\ \bar{\Upsilon} : \mathcal{L}_{\mathscr{F}}^2(0;\mathbb{R}^n) \mapsto \mathcal{L}_{\mathscr{F}}^2(N;\mathbb{R}^n),\\ \Gamma : \mathbb{U}_{ad} \mapsto \mathcal{L}_{\mathscr{F}}^2(\mathbb{N};\mathbb{R}^n),\ \bar{\Gamma} : \mathbb{U}_{ad} \mapsto \mathcal{L}_{\mathscr{F}}^2(N;\mathbb{R}^n)\end{cases} \tag{12}$$

are all linear and bounded. Therefore, in view of (11), for $k \in \bar{\mathbb{N}}$, we have that

$$\begin{aligned}x_{\mu u_1+(1-\mu)u_2,k} &= (\Upsilon x_0)_k + [\Gamma(\mu u_1 + (1-\mu)u_2)]_k\\ &= (\Upsilon x_0)_k + \mu(\Gamma u_1)_k + (1-\mu)(\Gamma u_2)_k\\ &= \mu x_{u_1,k} + (1-\mu)x_{u_2,k}.\end{aligned}$$

3.1 The regular case

Now, we investigate the regular stochastic cooperative LQ difference game, that is, the weighted matrices in Problem ($\mathbb{P}_N^i$-LQ) satisfy

$$\mathbb{Q}_i^k,\ \mathbb{G}_i^N \geqslant 0,\ \mathbb{R}_i^k > 0,\ i \in \bar{\mathcal{N}},\ k \in \mathbb{N}. \tag{13}$$

Work on the single objective regular stochastic LQ optimization problems can be found in [19] for the backward stochastic differential system with a partial information, in [20,21] for the mean-field stochastic difference systems. Further, the stochastic noncooperative LQ games have been studied in [22] for the Stackelberg differential game with asymmetric information, in [23] for the zero-sum stochastic differential game, and in [24] for a leader-follower differential game. In this subsection, we derive the Pareto optimality under condition (13). To proceed, we define the inner products $\langle x, y\rangle = \mathbb{E}(\sum_{p=0}^k x_p^{\mathrm{T}} y_p)$ for any $x = (x_0, \ldots, x_k)$ and $y = (y_0, \ldots, y_k)$ in Hilbert space.

In the following, we derive the convexity of performance criteria based on Lemma 3 and the Pareto optimality according to Lemma 1 for Problem ($\mathbb{P}_N^i$-LQ).

Theorem 1. Consider Problem ($\mathbb{P}_N^i$-LQ). The performance criteria $J_i(x_0; u)$, $i \in \bar{\mathcal{N}}$ are convex under condition (13). Moreover, for $\theta \in \Theta$, the Pareto optimal strategies are given by

$$\hat{u}(\theta) = \arg\min_{u \in \mathbb{U}_{ad}} \sum_{i=1}^{\mathcal{N}} \theta_i J_i(x_0; u) \quad \text{s.t. (9)}, \tag{14}$$

and the corresponding Pareto optimal solutions are obtained as

$$(J_1(x_0; \hat{u}(\theta)), J_2(x_0; \hat{u}(\theta)), \ldots, J_{\mathcal{N}}(x_0; \hat{u}(\theta))). \tag{15}$$

Proof. It is easy to see that the spaces defined in (12) are all Hilbert spaces. Hence, the corresponding adjoint operators are unique. Moreover, we use the following convention for $i \in \bar{\mathcal{N}}$:

$$(\mathbb{Q}_i x)_k = \mathbb{Q}_i^k x_k,\ k \in \mathbb{N},\ \forall x \in \mathcal{L}_{\mathscr{F}}^2(\mathbb{N}; \mathbb{R}^n), \quad (\mathbb{R}_i u)_k = \mathbb{R}_i^k u_k,\ k \in \mathbb{N},\ \forall u \in \mathbb{U}_{ad}.$$

Based on Lemma 3, the performance criteria $J_i(x_0; u)$, $i \in \bar{\mathcal{N}}$ are expressed as

$$\begin{aligned} J_i(x_0; u) &= \langle \mathbb{Q}_i(\Upsilon x_0 + \Gamma u), \Upsilon x_0 + \Gamma u\rangle_{\mathcal{L}_{\mathscr{F}}^2(\mathbb{N};\mathbb{R}^n)} + \langle \mathbb{R}_i u, u\rangle_{\mathbb{U}_{ad}} \\ &\quad + \left\langle \mathbb{G}_i^N(\bar{\Upsilon} x_0 + \bar{\Gamma} u), \bar{\Upsilon} x_0 + \bar{\Gamma} u\right\rangle_{\mathcal{L}_{\mathscr{F}}^2(N;\mathbb{R}^n)} \\ &= \langle \Phi_1 u, u\rangle_{\mathbb{U}_{ad}} + 2\langle \Phi_2 x_0, u\rangle_{\mathbb{U}_{ad}} + \langle \Phi_3 x_0, x_0\rangle_{\mathcal{L}_{\mathscr{F}}^2(0;\mathbb{R}^n)} \end{aligned} \tag{16}$$

with

$$\Phi_1 = \mathbb{R}_i + \Gamma^* \mathbb{Q}_i \Gamma + \bar{\Gamma}^* \mathbb{G}_i^N \bar{\Gamma},\ \Phi_2 = \Gamma^* \mathbb{Q}_i \Upsilon + \bar{\Gamma}^* \mathbb{G}_i^N \bar{\Upsilon},\ \Phi_3 = \Upsilon^* \mathbb{Q}_i \Upsilon + \bar{\Upsilon}^* \mathbb{R}_i \bar{\Upsilon}.$$

Furthermore, for any $\mu \in [0, 1]$,

$$\begin{aligned} &J_i(x_0; \mu u_1 + (1-\mu) u_2) \\ &= \langle \Phi_1[\mu u_1 + (1-\mu) u_2], \mu u_1 + (1-\mu) u_2\rangle_{\mathbb{U}_{ad}} + 2\langle \Phi_2 x_0, \mu u_1 + (1-\mu) u_2\rangle_{\mathbb{U}_{ad}} + \langle \Phi_3 x_0, x_0\rangle_{\mathcal{L}_{\mathscr{F}}^2(0;\mathbb{R}^n)} \\ &= \mu^2 \langle \Phi_1 u_1, u_1\rangle_{\mathbb{U}_{ad}} + (1-\mu)^2 \langle \Phi_1 u_2, u_2\rangle_{\mathbb{U}_{ad}} + 2\mu(1-\mu)\langle \Phi_1 u_1, u_2\rangle_{\mathbb{U}_{ad}} + 2\mu \langle \Phi_2 x_0, u_1\rangle_{\mathbb{U}_{ad}} \\ &\quad + 2(1-\mu)\langle \Phi_2 x_0, u_2\rangle_{\mathbb{U}_{ad}} + \langle \Phi_3 x_0, x_0\rangle_{\mathcal{L}_{\mathscr{F}}^2(0;\mathbb{R}^n)}. \end{aligned}$$

Therefore, we obtain that

$$\begin{aligned} &\mu J_i(x_0; u_1) + (1-\mu) J_i(x_0; u_2) - J_i(x_0; \mu u_1 + (1-\mu) u_2) \\ &= \mu(1-\mu)\left\{\langle \Phi_1 u_1, u_1\rangle_{\mathbb{U}_{ad}} + \langle \Phi_1 u_2, u_2\rangle_{\mathbb{U}_{ad}} - 2\langle \Phi_1 u_1, u_2\rangle_{\mathbb{U}_{ad}}\right\} \\ &= \mu(1-\mu)\langle \Phi_1(u_1 - u_2), u_1 - u_2\rangle_{\mathbb{U}_{ad}} \\ &= \mu(1-\mu) J_i(0; u_1 - u_2). \end{aligned}$$

Obviously, $J_i(0; u_1 - u_2) \geqslant 0$ under condition (13). Therefore, $J_i(x_0; u)$, $i \in \bar{\mathcal{N}}$ are convex with respect to u. In addition, based on Lemma 1, all the optimal strategies and solutions can be obtained by (14) and (15), respectively.

Remark 1. The computing algorithm for obtaining Pareto optimal strategies and solutions of the regular cooperative game can be covered by that of the indefinite case, which will be shown in Subsection 3.2 and is omitted here.

It is easy to see that the dynamic system (9) of Problem ($\mathbb{P}_N^i$-LQ) is linear. For a fixed initial value $x_0 \in \mathbb{R}^n$, we can acquire the result as follows.

Theorem 2. Suppose $\hat{u}$ is a Pareto optimal strategy for Problem ($\mathbb{P}_N^i$-LQ). Then for a real constant λ, $\lambda\hat{u}$ is an optimal strategy for Problem ($\mathbb{P}_N^i$-LQ) with the initial value λx_0.

Proof. Firstly, consider $\lambda > 0$. Let $x_{(x_0,u),k}$ be the response of (9) corresponding to the initial state x_0 and the strategy u. Then we obtain that $x_{(\lambda x_0,\lambda u),k} = \lambda x_{(x_0,u),k}$ by (11), and $J_i(\lambda x_0; \lambda u) = \lambda^2 J_i(x_0; u)$ by (16), respectively. On the ground of Lemma 2, $\hat{u}$ is an optimal strategy iff $J_i(x_0; u)$ is minimized by $\hat{u}$ on the set

$$\mathbb{U}_{ad}^i(x_0; \hat{u}) := \left\{u | J_j(x_0; u) \leqslant J_j(x_0; \hat{u}),\ j \in \bar{\mathcal{N}},\ j \neq i\right\}. \tag{17}$$

In the following, we prove that $J_i(\lambda x_0; u)$ can be minimized by $\lambda\hat{u}$ on the set

$$\tilde{\mathbb{U}}_{ad}^i := \{u | J_j(\lambda x_0; u) \leqslant J_j(\lambda x_0; \lambda\hat{u}),\ j \in \bar{\mathcal{N}},\ j \neq i\}. \tag{18}$$

Let $u \in \tilde{\mathbb{U}}_{ad}^i$. Based on (18), we have that $J_j(\lambda x_0; u) \leqslant J_j(\lambda x_0; \lambda\hat{u})$, which is equal to $J_j(x_0; \hat{u}) \geqslant J_j(x_0; \frac{1}{\lambda}u)$. Thus, by (17), $\frac{1}{\lambda}u \in \mathbb{U}_{ad}^i(x_0; \hat{u})$, which yields that $J_i(x_0; \frac{1}{\lambda}u) \geqslant J_i(x_0; \hat{u})$, that is, $J_i(\lambda x_0; u) \geqslant J_i(\lambda x_0; \lambda\hat{u})$.

Secondly, when $\lambda = 0$. In this situation, $\lambda = 0$ yields the optimal solution $(0, \ldots, 0)$ with $\hat{u} = 0$. Next, the conclusion will be proved by contradiction. Suppose $(0, \ldots, 0)$ is not an optimal solution. According to the notion of Pareto optimality, then there is a strategy $\bar{u}$ satisfying $J_i(0; \bar{u}) < 0$. In addition, from (16), we have that

$$J_i(x_0; \lambda\bar{u}) = \lambda^2 J_i(0; \bar{u}) + 2\lambda \langle \Phi_2 x_0; \bar{u}\rangle_{\mathbb{U}_{ad}} + \langle \Phi_3 x_0; x_0\rangle_{\mathcal{L}^2_{\mathscr{F}}(0;\mathbb{R}^n)}.$$

However, this indicates that there is a $\tilde{\lambda}$ yielding $J_i(x_0; \tilde{\lambda}\bar{u}) < J_i(x_0; \hat{u})$. This contradicts that $\hat{u}$ is a Pareto optimal strategy.

Finally, the case $\lambda < 0$ can be investigated following the line of $\theta > 0$.

Further, if the initial state $x_0 = 0$, then we can obtain $J_i(0; \lambda u) = \lambda^2 J_i(0; u)$ from (16). Thus, we have the following corollary, for which, the proof is direct.

Corollary 1. Consider Problem ($\mathbb{P}_N^i$-LQ). Suppose $x_0 = 0$ and $\hat{u}$ is optimal for Problem ($\mathbb{P}_N^i$-LQ) resulting in the optimal solution $(J_1(0; \hat{u}), J_2(0; \hat{u}), \ldots, J_{\mathcal{N}}(0; \hat{u}))$. Then $\lambda\hat{u}$ generates the optimal solution $(\lambda^2 J_1(0; \hat{u}), \lambda^2 J_2(0; \hat{u}), \ldots, \lambda^2 J_{\mathcal{N}}(0; \hat{u}))$.

3.2 The indefinite case

For the indefinite LQ control, we need the properties of the Moore-Penrose pseudoinverse [25] of a matrix. For a given matrix $\mathbb{M} \in \mathbb{R}^{m\times n}$, there exists a unique matrix $\mathbb{M}^\dagger \in \mathbb{R}^{n\times m}$ such that

$$(\mathbb{M}^\dagger\mathbb{M})^{\mathrm{T}} = \mathbb{M}^\dagger\mathbb{M}, \quad (\mathbb{M}\mathbb{M}^\dagger)^{\mathrm{T}} = \mathbb{M}\mathbb{M}^\dagger, \mathbb{M}^\dagger\mathbb{M}\mathbb{M}^\dagger = \mathbb{M}^\dagger, \quad \mathbb{M}\mathbb{M}^\dagger\mathbb{M} = \mathbb{M}. \tag{19}$$

Notice that Problem ($\mathbb{P}_N^i$-LQ) may be not well-posed since $\mathbb{Q}_i^k$, $\mathbb{G}_i^N$, $\mathbb{R}_{ij}^k$, $i \in \bar{\mathcal{N}}$ and $k \in \mathbb{N}$ can be indefinite. Thus, the definition of well-posedness is proposed as follows.

Definition 1. Problem ($\mathbb{P}_N^i$-LQ), $i \in \bar{\mathcal{N}}$ are referred to as well-posed for $x_0 \in \mathbb{R}^n$ if

$$-\infty < V_i(x_0) = \inf\nolimits_{u\in\mathbb{U}_{ad}} J_i(x_0; u) < +\infty,\ \forall i \in \bar{\mathcal{N}}.$$

It is attainable with respect to the initial value x_0 if there is a strategy $\hat{u}$ such that the optimal costs $V_i(x_0)$, $i \in \bar{\mathcal{N}}$ can be obtained. In this situation, the strategy $\hat{u}$ is Pareto optimal.

Remark 2. For the regular cooperative game, the performance criteria are naturally convex owing to the constraints of the weighted matrices, which implies that the minimization of the weighted sum of the performance criteria is equivalent to the optimal strategies; see Theorem 1. However, the property for the convexity of costs may be not effective in the indefinite case.

Lemma 1 provides us with a concise way for seeking Pareto optimal strategies. Nevertheless, the weighting technique alone (without the convexity assumption on both the strategy space $\mathbb{U}_{ad}$ and the performance criteria) will be not enough to work out all optimal solutions for the indefinite cooperative game, and the following example is presented to illustrate the viewpoint.

Example 1. Consider the cooperative game with the dynamics

$$x_{1,k+1}=x_{1,k}+u_{1,k}\omega_k,\ x_{1,0}=1,\quad x_{2,k+1}=\frac{\sqrt{2}}{2}(x_{2,k}-u_{2,k})+\frac{\sqrt{2}}{2}(x_{2,k}+u_{2,k})\omega_k,\ x_{2,0}=0$$

together with the following performance criteria:

$$J_1(x_0;u)=\mathbb{E}\left[\sum_{k=0}^{1}\left(u_{1,k}^2+u_{2,k}^2\right)-x_{1,2}^2\right],\quad J_2(x_0;u)=2\mathbb{E}\left[-\sum_{k=0}^{1}\left(u_{1,k}^2+u_{2,k}^2\right)+x_{1,2}^2\right].$$

By construction, it is easy to see that

$$\mathbb{E}\left(x_{1,2}^2\right)=\mathbb{E}\left[\sum_{k=0}^{1}\left(x_{1,k+1}^2-x_{1,k}^2\right)\right]+1=\mathbb{E}\left(\sum_{k=0}^{1}u_{1,k}^2\right)+1,$$
$$\mathbb{E}\left(x_{2,2}^2\right)=\mathbb{E}\left[\sum_{k=0}^{1}\left(x_{2,k+1}^2-x_{2,k}^2\right)\right]=\mathbb{E}\left(\sum_{k=0}^{1}u_{2,k}^2\right).$$

Hence, for all $u:=(u_{1,k},u_{2,k})\in\mathbb{U}_{ad}$, $J_2(x_0;u)=-2\mathbb{E}\left(x_{2,2}^2\right)+2=-2J_1(x_0;u)$, $k\in\{0,1\}$. Clearly, we always have $J_2(x_0;u)=-2J_1(x_0;u)$ no matter what values we select for $u_{i,k}$, $i=1,2$. Therefore, each point on the line $(J_2=-2J_1)$ is a Pareto optimal solution. In addition, consider the minimum of $J_\theta(x_0;u)=\theta J_1(x_0;u)+(1-\theta)J_2(x_0;u)$ subject to the dynamic system. We choose $0<\theta<\frac{2}{3}$, $u_{1,k}=0$ and $u_{2,k}=\chi$. Then, some simple calculations show that $J_\theta(x_0;u)=(3\theta-2)(2\chi^2-1)$. By selecting χ as large as possible, $J_\theta(x_0;u)$ can be made arbitrarily small, i.e., $J_\theta(x_0;u)$ does not have a minimum.

It has been shown in Example 1 that the weighting technique is sufficient, but not necessary in general for obtaining optimal solutions of the indefinite cooperative game. A natural question then is how we can get all optimal solutions. As shown in Lemma 1, if both $J_i(x_0;u)$, $i\in\bar{\mathcal{N}}$ and $\mathbb{U}_{ad}$ are convex, then all Pareto optimal strategies can be achieved by employing the weighting technique, in which, the weighted sum of the cost functionals is minimized. Further, the set of all optimal solutions can be determined as the optimal strategies have been derived. In real life, the admissible control space $\mathbb{U}_{ad}$ can always be chosen as a convex set. Therefore, the key to acquiring all optimal strategies is to determine an appropriate criterion to guarantee the convexity of the performance criteria. To do so, let $\mathbb{X}\subset\mathcal{L}^2_{\mathscr{F}}(\bar{\mathbb{N}};\mathbb{R}^n)\times\mathcal{L}^2_{\mathscr{F}}(\mathbb{N};\mathbb{R}^m)$ be the admissible pairs for (9), i.e.,

$$\mathbb{X}=\left\{(x.,u.)\in\mathcal{L}^2_{\mathscr{F}}(\bar{\mathbb{N}};\mathbb{R}^n)\times\mathcal{L}^2_{\mathscr{F}}(\mathbb{N};\mathbb{R}^m)|u.\in\mathbb{U}_{ad}\text{ and }(x.,u.)\text{ satisfies (9)}\right\}.\tag{20}$$

Let $\mathbb{X}_0\subset\mathcal{L}^2_{\mathscr{F}}(\bar{\mathbb{N}};\mathbb{R}^n)\times\mathcal{L}^2_{\mathscr{F}}(\mathbb{N};\mathbb{R}^m)$ be given as

$$\begin{aligned}\mathbb{X}_0=\big\{&(x.,u.)\in\mathcal{L}^2_{\mathscr{F}}(\bar{\mathbb{N}};\mathbb{R}^n)\times\mathcal{L}^2_{\mathscr{F}}(\mathbb{N};\mathbb{R}^m)\big|u.\in\mathbb{U}_{ad}\text{ and}\\&x_{k+1}=(\mathbb{A}_1^kx_k+\mathbb{B}_1^ku_k)+(\mathbb{A}_2^kx_k+\mathbb{B}_2^ku_k)\omega_k,\ x_0=0\big\}.\end{aligned}\tag{21}$$

Moreover, let $\mathbb{z}.\in\mathcal{L}^2_{\mathscr{F}}(\bar{\mathbb{N}};\mathbb{R}^n)$ be the solution to (9) with $u_k=0$ (a.e. on $\mathbb{N}$, P-a.s.), i.e.,

$$\mathbb{z}_{k+1}=\mathbb{A}_1^k\mathbb{z}_k+\mathbb{A}_2^k\mathbb{z}_k\omega_k,\quad \mathbb{z}_0=x_0.\tag{22}$$

Consequently, based on Lemma 3, $\mathbb{X}_0$ is a linear subspace of $\mathcal{L}^2_{\mathscr{F}}(\bar{\mathbb{N}};\mathbb{R}^n)\times\mathcal{L}^2_{\mathscr{F}}(\mathbb{N};\mathbb{R}^m)$ and

$$\mathbb{X}=\mathbb{X}_0+(\mathbb{z}.,0).\tag{23}$$

Further, $\mathbb{X}$ is an affine subspace of $\mathcal{L}^2_{\mathscr{F}}(\bar{\mathbb{N}};\mathbb{R}^n)\times\mathcal{L}^2_{\mathscr{F}}(\mathbb{N};\mathbb{R}^m)$. As a result of the uniqueness of the solutions to (9) for every given $u.\in\mathbb{U}_{ad}$, there exists a one-to-one correspondence from $\mathbb{U}_{ad}$ to $\mathbb{X}$. In the sequel, we define the following performance criteria $\bar{J}_i(x.;u.)$, $i\in\bar{\mathcal{N}}$:

$$\bar{J}_i(x.;u.)=\mathbb{E}\left[\sum_{k=0}^{N-1}\left(x_k^{\mathrm{T}}\mathbb{Q}_i^kx_k+u_k^{\mathrm{T}}\mathbb{R}_i^ku_k\right)+x_N^{\mathrm{T}}\mathbb{G}_i^Nx_N\right]\tag{24}$$

for $(x.,u.)\in\mathbb{X}_0$. Assume that $(x_1.,u_1.)$, $(x_2.,u_2.)\in\mathbb{X}$ are two given admissible pairs, for $\mu\in[0,1]$, we have (noting Theorem 1)

$$\mu J_i(x_0;u_1)+(1-\mu)J_i(x_0;u_2)-J_i(x_0;\mu u_1+(1-\mu)u_2)=\mu(1-\mu)\bar{J}_i(\bar{x}.;\bar{u}.),\tag{25}$$

where $(\bar{x}.,\bar{u}.)\in\mathbb{X}_0$ is defined by $\bar{x}_k=x_{1k}-x_{2k}$ and $\bar{u}_k=u_{1k}-u_{2k}$. Hence, we have the following result.

Theorem 3. Consider Problem ($\mathbb{P}_N^i$-LQ). $J_i(x_0;u)$, $i\in\bar{\mathcal{N}}$ defined on $\mathbb{X}$ are convex iff $\bar{J}_i(\bar{x}.;\bar{u}.)\geqslant 0$, $i\in\bar{\mathcal{N}}$ for all $(\bar{x}.,\bar{u}.)\in\mathbb{X}_0$.

Proof. " $\Rightarrow$ " Assume that $J_i(x_0;u)$, $i\in\bar{\mathcal{N}}$ are convex on $\mathbb{X}$. Let $(\bar{x}.,\bar{u}.)\in\mathbb{X}_0$ be given. On the ground of (23), we obtain $(x_{1}.,u_{1}.)=(\mathbb{z}.,0)+(\bar{x}.,\bar{u}.)\in\mathbb{X}$ and $(x_{2}.,u_{2}.)=(\mathbb{z}.,0)\in\mathbb{X}$ (as $(x.,u.)\equiv(0,0)\in\mathbb{X}_0$). In addition, $\bar{x}_k=x_{1k}-x_{2k}$ and $\bar{u}_k=u_{1k}-u_{2k}$. Owing to the convexity of $J_i(x_0;u)$, $i\in\bar{\mathcal{N}}$ and (25), we have that $\bar{J}_i(\bar{x}.;\bar{u}.)\geqslant 0$, $i\in\bar{\mathcal{N}}$.

" $\Leftarrow$ " It is easy to see that if $\bar{J}_i(\bar{x}.;\bar{u}.)\geqslant 0$, $i\in\bar{\mathcal{N}}$ for all $(\bar{x}.,\bar{u}.)\in\mathbb{X}_0$, then $J_i(x_0;u)$, $i\in\bar{\mathcal{N}}$ are convex on $\mathbb{X}$ based on (25).

On the basis of Theorem 3, we show the intrinsic relation between the convexity of costs and the well-posedness of the cooperative game.

Corollary 2. The well-posedness of the indefinite cooperative game can ensure the convexity of the cost functionals.

Proof. Suppose that $J_i(x_0;u)$, $i\in\bar{\mathcal{N}}$ are well-posed, and then we have $\bar{J}_i(x.,u.)\geqslant 0$ by Theorem 2 for $u\in\mathbb{U}_{ad}$. If there exists $(\breve{x}.,\breve{u}.)$ such that $\bar{J}_i(\breve{x}.,\breve{u}.)<0$. Then for $\zeta>0$, we obtain $(\zeta\breve{x}.,\zeta\breve{u}.)\in\mathbb{X}_0$ and $\bar{J}_i(\zeta\breve{x}.,\zeta\breve{u}.)=\zeta^2\bar{J}_i(\breve{x}.,\breve{u}.)$, which contradicts the well-posedness assumption of $J_i(x_0;u)$, $i\in\bar{\mathcal{N}}$. Therefore, when the indefinite game is well-posed, we must have that $J_i(x_0;u)$, $i\in\bar{\mathcal{N}}$ are convex.

In the following, based on Lemma 1 and Theorem 3, we derive the Pareto optimality of Problem ($\mathbb{P}_N^i$-LQ) by the weighting technique. First of all, for a fixed $\theta\in\Theta$, consider the minimization of the weighted sum of the performance criteria:

$$
\begin{aligned}
&\text{Problem } (\mathbb{P}_N^\theta\text{-LQ}).\\
&\min\ J_\theta(x_0;u):=\mathbb{E}\left[\sum_{k=0}^{N-1}\left(x_k^{\mathrm{T}}\mathbb{Q}_\theta^k x_k+u_k^{\mathrm{T}}\mathbb{R}_\theta^k u_k\right)+x_N^{\mathrm{T}}\mathbb{G}_\theta^N x_N\right]\\
&\text{s.t}\quad x_{k+1}=\left(\mathbb{A}_1^k x_k+\mathbb{B}_1^k u_k\right)+\left(\mathbb{A}_2^k x_k+\mathbb{B}_2^k u_k\right)\omega_k,\ x_0\in\mathbb{R}^n,\ k\in\mathbb{N},
\end{aligned}
$$

where $\mathbb{Q}_\theta^k=\sum_{i=1}^{\mathcal{N}}\theta_i\mathbb{Q}_i^k$, $\mathbb{R}_\theta^k=\sum_{i=1}^{\mathcal{N}}\theta_i\mathbb{R}_i^k$, $\mathbb{G}_\theta^N=\sum_{i=1}^{\mathcal{N}}\theta_i\mathbb{G}_i^N$. It is worth mentioning that Problem ($\mathbb{P}_N^\theta$-LQ) is an optimization problem with one objective, where the weighted matrices are dependent on the value of θ. It is well known that the feasiblity of Problem ($\mathbb{P}_N^\theta$-LQ) can be well characterized by the solution to the corresponding weighted DRE,

$$
\begin{cases}
\mathbb{F}_\theta^k-\mathbb{H}_\theta^{k^{\mathrm{T}}}\mathbb{W}_\theta^{k^\dagger}\mathbb{H}_\theta^k=0,\\
\mathbb{P}_\theta^N=\mathbb{G}_\theta^N,\\
\mathbb{W}_\theta^k\mathbb{W}_\theta^{k^\dagger}\mathbb{H}_\theta^k=\mathbb{H}_\theta^k,\\
\mathbb{W}_\theta^k\geqslant 0,\ k\in\mathbb{N},
\end{cases}
\tag{26}
$$

where

$$
\begin{cases}
\mathbb{F}_\theta^k=-\mathbb{P}_\theta^k+\mathbb{Q}_\theta^k+\mathbb{A}_1^{k^{\mathrm{T}}}\mathbb{P}_\theta^{k+1}\mathbb{A}_1^k+\mathbb{A}_2^{k^{\mathrm{T}}}\mathbb{P}_\theta^{k+1}\mathbb{A}_2^k,\\
\mathbb{W}_\theta^k=\mathbb{R}_\theta^k+\mathbb{B}_1^{k^{\mathrm{T}}}\mathbb{P}_\theta^{k+1}\mathbb{B}_1^k+\mathbb{B}_2^{k^{\mathrm{T}}}\mathbb{P}_\theta^{k+1}\mathbb{B}_2^k,\\
\mathbb{H}_\theta^k=\mathbb{B}_1^{k^{\mathrm{T}}}\mathbb{P}_\theta^{k+1}\mathbb{A}_1^k+\mathbb{B}_2^{k^{\mathrm{T}}}\mathbb{P}_\theta^{k+1}\mathbb{A}_2^k.
\end{cases}
\tag{27}
$$

Now, what remains to show is how to obtain all Pareto optimal solutions. To this end, we present the following lemma; see Theorem 4.3 of [2] for a detailed proof.

Lemma 4. The statements given below are equivalent:

(1) The weighted DRE (26) is solvable.

(2) Problem ($\mathbb{P}_N^\theta$-LQ) is attainable.

(3) The set $\mathbb{P}:=\left\{\mathbb{P}_\theta^k\in\mathbb{S}^n\,\middle|\,\mathscr{P}_\theta^k\geqslant 0,\ k\in\mathbb{N}\text{ and }\mathbb{P}_\theta^N\leqslant\mathbb{G}_\theta^N\right\}$ is nonempty, in which $\mathscr{P}_\theta^k=\left[\begin{smallmatrix}\mathbb{F}_\theta^k & \mathbb{H}_\theta^{k^{\mathrm{T}}}\\ \mathbb{H}_\theta^k & \mathbb{W}_\theta^k\end{smallmatrix}\right]$.

(4) Problem ($\mathbb{P}_N^\theta$-LQ) is well-posed.

In addition, when each one of the assertions (1)–(4) is satisfied, the optimal strategy of Problem ($\mathbb{P}_N^\theta$-LQ) is obtained as

$$
u_\theta^k=\mathbb{K}_\theta^k x_k=-\mathbb{W}_\theta^{k^\dagger}\mathbb{H}_\theta^k x_k.
\tag{28}
$$

Moreover, the optimal performance criterion is uniquely given by

$$
\min\nolimits_{u\in\mathbb{U}_{ad}}J_\theta(x_0;u)=\mathbb{E}(x_0^{\mathrm{T}}\mathbb{P}_\theta^0 x_0).
\tag{29}
$$

Remark 3. Notice that the convexity of the cost functionals of Problem ($\mathbb{P}_N^i$-LQ) implies that the performance criterion of Problem ($\mathbb{P}_N^\theta$-LQ) is convex, i.e., the convexity of $J_i(x_0;u)$, $i\in\bar{\mathcal{N}}$ yields that $J_\theta(x_0;u)=\sum_{i=1}^{\mathcal{N}}\theta_iJ_i(x_0;u)$ is convex with an arbitrary $\theta\in\Theta$.

Remark 4. Lemma 4 is also applicable for Problem ($\mathbb{P}_N^i$-LQ), $i\in\bar{\mathcal{N}}$, where the corresponding weighting matrices $(\mathbb{Q}_i^k,\mathbb{R}_i^k,\mathbb{G}_i^N)$ replace the weighted weighting matrices $(\mathbb{Q}_\theta^k,\mathbb{R}_\theta^k,\mathbb{G}_\theta^N)$. Moreover, the optimal strategy of Problem ($\mathbb{P}_N^i$-LQ) is presented as

$$u_i^k=\mathbb{K}_i^kx_k=-\mathbb{W}_i^{k^\dagger}\mathbb{H}_i^kx_k \tag{30}$$

with the optimal cost value

$$\min\nolimits_{u\in\mathbb{U}_{ad}}J_i(x_0;u)=\mathbb{E}(x_0^{\mathrm{T}}\mathbb{P}_i^0x_0). \tag{31}$$

It should be noted that the optimal strategy obtained by (30) is the strategy $u_i^k\in\mathbb{R}^m$ that is employed to optimize the cost $J_i(x_0;u)$ of player i by all players, instead of the optimal control $u_{i,k}\in\mathbb{R}^{m_i}$ of player i, which yields a main difference between the multicriteria cooperative game and the single-objective optimization problem. Nevertheless, owing to the fact that the optimal strategy given by (30) relies on the weighting matrices of the cost functionals in the game that may vary among players, it is hard to identify an optimal strategy to satisfy the concept of Pareto optimality. Thus, we employ the weighting technique to identify the Pareto optimal controls.

Theorem 4. Suppose that the corresponding DRE of Problem ($\mathbb{P}_N^i$-LQ) has a solution $\mathbb{P}_i^k$, $k\in\mathbb{N}$ for any $i\in\bar{\mathcal{N}}$. Then the optimal strategy for a fixed $\theta\in\Theta$ can be acquired as

$$\begin{aligned}
u_\theta^{k^*}&=\arg\min\nolimits_{u\in\mathbb{U}_{ad}}\sum_{i=1}^{\mathcal{N}}\theta_iJ_i(x_0;u)\\
&=\left[u_{1\theta,k}^{*\mathrm{T}}\ \cdots\ u_{\mathcal{N}\theta,k}^{*\mathrm{T}}\right]^{\mathrm{T}}\\
&=-\left(\mathbb{R}_\theta^k+\mathbb{B}_1^{k^{\mathrm{T}}}\mathbb{P}_\theta^{k+1}\mathbb{B}_1^k+\mathbb{B}_2^{k^{\mathrm{T}}}\mathbb{P}_\theta^{k+1}\mathbb{B}_2^k\right)^\dagger\left(\mathbb{B}_1^{k^{\mathrm{T}}}\mathbb{P}_\theta^{k+1}\mathbb{A}_1^k+\mathbb{B}_2^{k^{\mathrm{T}}}\mathbb{P}_\theta^{k+1}\mathbb{A}_2^k\right)x_k\\
&=\left[(\mathbb{K}_{1\theta,k}^*x_k)^{\mathrm{T}}\ \cdots\ (\mathbb{K}_{\mathcal{N}\theta,k}^*x_k)^{\mathrm{T}}\right]^{\mathrm{T}},
\end{aligned} \tag{32}$$

where x_k denotes the solution to the stochastic difference system (5).

Proof. Since within the assumption, the corresponding DRE of Problem ($\mathbb{P}_N^i$-LQ) has a solution $\mathbb{P}_i^k$, $k\in\mathbb{N}$, then $J_i(x_0;u)$, $i\in\bar{\mathcal{N}}$ have a minimum by Lemma 4 and Remark 4 and can be convex under the condition presented in Theorem 3. Thus, on the basis of Remark 3 and Lemma 4, $J_\theta(x_0;u)=\sum_{i=1}^{\mathcal{N}}\theta_iJ_i(x_0;u)$ is convex and achieves a minimum $\mathbb{E}(x_0^{\mathrm{T}}\mathbb{P}_\theta^0x_0)$ for any $\theta\in\Theta$. Therefore, in view of Lemma 1, all optimal strategies are given by (32).

Theorem 5. Assume that the corresponding DRE (26) and (27) of Problem ($\mathbb{P}_N^i$-LQ) have a solution $\mathbb{P}_i^k$ for $\theta=\mathscr{E}_i$ (the ith standard unit vector in $\mathbb{R}^{\mathcal{N}}$), $i\in\bar{\mathcal{N}}$. Then for any initial value, there exists a Pareto optimal solution. Additionally, for a fixed $\theta\in\Theta$ and a fixed initial state $x_0\in\mathbb{R}^n$, all Pareto optimal solutions are given as

$$\left\{\left(J_1\left(x_0;\mathbb{K}_{1\theta,k}^*,\mathbb{K}_{2\theta,k}^*,\ldots,\mathbb{K}_{\mathcal{N}\theta,k}^*\right),\ldots,J_{\mathcal{N}}\left(x_0;\mathbb{K}_{1\theta,k}^*,\mathbb{K}_{2\theta,k}^*,\cdots\mathbb{K}_{\mathcal{N}\theta,k}^*\right)\right)\big|\theta\in\Theta\right\}, \tag{33}$$

where

$$J_i(x_0;\mathbb{K}_{1\theta,k}^*,\mathbb{K}_{2\theta,k}^*,\ldots,\mathbb{K}_{\mathcal{N}\theta,k}^*)=\mathbb{E}\left(x_0^{\mathrm{T}}\mathbb{X}_{i\theta}^{0^*}x_0\right). \tag{34}$$

Further, $\mathbb{X}_{i\theta}^{k^*}$, $k\in\mathbb{N}$, $i\in\bar{\mathcal{N}}$ are the solution to the weighted DLE,

$$\begin{cases}
\mathbb{X}_{i\theta}^{k^*}=\mathbb{Q}_i^k+\sum_{j=1}^{\mathcal{N}}\mathbb{K}_{j\theta,k}^{*}{}^{\mathrm{T}}\mathbb{R}_{ij}^k\mathbb{K}_{j\theta,k}^*\left(\mathbb{A}_1^k+\sum_{i=1}^{\mathcal{N}}\mathbb{B}_{i,1}^k\mathbb{K}_{i\theta,k}^*\right)^{\mathrm{T}}\mathbb{X}_{i\theta}^{k+1^*}\left(\mathbb{A}_1^k+\sum_{i=1}^{\mathcal{N}}\mathbb{B}_{i,1}^k\mathbb{K}_{i\theta,k}^*\right)\\
\qquad+\left(\mathbb{A}_2^k+\sum_{i=1}^{\mathcal{N}}\mathbb{B}_{i,2}^k\mathbb{K}_{i\theta,k}^*\right)^{\mathrm{T}}\mathbb{X}_{i\theta}^{k+1^*}\left(\mathbb{A}_2^k+\sum_{i=1}^{\mathcal{N}}\mathbb{B}_{i,2}^k\mathbb{K}_{i\theta,k}^*\right),\\
\mathbb{X}_{i\theta}^{N^*}=\mathbb{G}_i^N.
\end{cases} \tag{35}$$

Proof. Substituting $u_{i\theta,k}^*=\mathbb{K}_{i\theta,k}^*x_k$ into (4) and (5), we obtain the cost functionals and the dynamic model in the closed loop form:

$$J_i\left(x_0;\mathbb{K}_{1\theta,k}^*,\mathbb{K}_{2\theta,k}^*,\ldots,\mathbb{K}_{\mathcal{N}\theta,k}^*\right):=\mathbb{E}\left[\sum_{k=0}^{N-1}x_k^{\mathrm{T}}\left(\mathbb{Q}_i^k+\sum_{j=1}^{\mathcal{N}}\mathbb{K}_{j\theta,k}^{*}{}^{\mathrm{T}}\mathbb{R}_{ij}^k\mathbb{K}_{j\theta,k}^*\right)x_k+x_N^{\mathrm{T}}\mathbb{G}_i^Nx_N\right], \tag{36}$$

and

$$x_{k+1} = \left(\mathbb{A}_1^k + \sum_{i=1}^{\mathcal{N}} \mathbb{B}_{i,1}^k \mathbb{K}_{i\theta,k}^*\right) x_k + \left(\mathbb{A}_2^k + \sum_{i=1}^{\mathcal{N}} \mathbb{B}_{i,2}^k \mathbb{K}_{i\theta,k}^*\right) x_k \omega_k, \quad x_0 \in \mathbb{R}^n. \tag{37}$$

In the following, substituting the equality

$$\begin{aligned}
&\mathbb{E}\left[\sum_{k=0}^{N-1}\left(x_{k+1}^{\mathrm{T}} \mathbb{X}_{i\theta}^{k+1^*} x_{k+1} - x_k^{\mathrm{T}} \mathbb{X}_{i\theta}^{k^*} x_k\right)\right] \\
&= \mathbb{E}\left\{\sum_{k=0}^{N-1} x_k^{\mathrm{T}}\left[\left(\mathbb{A}_1^k + \sum_{i=1}^{\mathcal{N}} \mathbb{B}_{i,1}^k \mathbb{K}_{i\theta,k}^*\right)^{\mathrm{T}} \mathbb{X}_{i\theta}^{k+1^*}\left(\mathbb{A}_1^k + \sum_{i=1}^{\mathcal{N}} \mathbb{B}_{i,1}^k \mathbb{K}_{i\theta,k}^*\right)\right.\right. \\
&\quad \left.\left. + \left(\mathbb{A}_2^k + \sum_{i=1}^{\mathcal{N}} \mathbb{B}_{i,2}^k \mathbb{K}_{i\theta,k}^*\right)^{\mathrm{T}} \mathbb{X}_{i\theta}^{k+1^*}\left(\mathbb{A}_2^k + \sum_{i=1}^{\mathcal{N}} \mathbb{B}_{i,2}^k \mathbb{K}_{i\theta,k}^*\right) - \mathbb{X}_{i\theta}^{k^*}\right] x_k\right\} \\
&= \mathbb{E}\left(x_N^{\mathrm{T}} \mathbb{X}_{i\theta}^{N^*} x_N - x_0^{\mathrm{T}} \mathbb{X}_{i\theta}^{0^*} x_0\right)
\end{aligned}$$

to (36), elementary calculations show that

$$\begin{aligned}
&J_i\left(x_0; \mathbb{K}_{1\theta,k}^*, \mathbb{K}_{2\theta,k}^*, \ldots, \mathbb{K}_{\mathcal{N}\theta,k}^*\right) \\
&= \mathbb{E}\Bigg\{x_0^{\mathrm{T}} \mathbb{X}_{i\theta}^{0^*} x_0 - x_N^{\mathrm{T}} \mathbb{X}_{i\theta}^{N^*} x_N + x_N^{\mathrm{T}} \mathbb{G}_i^N x_N + \sum_{k=0}^{N-1} x_k^{\mathrm{T}}\left[\left(\mathbb{A}_1^k + \sum_{i=1}^{\mathcal{N}} \mathbb{B}_{i,1}^k \mathbb{K}_{i\theta,k}^*\right)^{\mathrm{T}} \mathbb{X}_{i\theta}^{k+1^*}\right. \\
&\quad \times \left(\mathbb{A}_1^k + \sum_{i=1}^{\mathcal{N}} \mathbb{B}_{i,1}^k \mathbb{K}_{i\theta,k}^*\right) - \mathbb{X}_{i\theta}^{k^*} + \mathbb{Q}_i^k + \left(\mathbb{A}_2^k + \sum_{i=1}^{\mathcal{N}} \mathbb{B}_{i,2}^k \mathbb{K}_{i\theta,k}^*\right)^{\mathrm{T}} \mathbb{X}_{i\theta}^{k+1^*} \\
&\quad \times \left.\left(\mathbb{A}_2^k + \sum_{i=1}^{\mathcal{N}} \mathbb{B}_{i,2}^k \mathbb{K}_{i\theta,k}^*\right) + \sum_{j=1}^{\mathcal{N}} {\mathbb{K}_{j\theta,k}^*}^{\mathrm{T}} \mathbb{R}_{ij}^k \mathbb{K}_{j\theta,k}^*\right] x_k\Bigg\}.
\end{aligned} \tag{38}$$

Consequently, the optimal costs (noting (35)) are given as (34).

Remark 5. We can draw the following conclusion on the basis of the results obtained above:

(1) For the regular cooperative game (under condition (13)), Eq. (32) in Theorem 4 reduces to Theorem 1 of [12]. Furthermore, the Pareto optimal strategy is written as

$$u_\theta^{k^*} = -\left(\mathbb{R}_\theta^k + {\mathbb{B}_1^k}^{\mathrm{T}} \mathbb{P}_\theta^{k+1} \mathbb{B}_1^k + {\mathbb{B}_2^k}^{\mathrm{T}} \mathbb{P}_\theta^{k+1} \mathbb{B}_2^k\right)^{-1}\left({\mathbb{B}_1^k}^{\mathrm{T}} \mathbb{P}_\theta^{k+1} \mathbb{A}_1^k + {\mathbb{B}_2^k}^{\mathrm{T}} \mathbb{P}_\theta^{k+1} \mathbb{A}_2^k\right) x_k,$$

and the optimal solutions can be acquired following the line of Theorem 5.

(2) Specifically, in the deterministic situation, the weighted DRE (26) reduces to

$$\begin{cases}
\mathbb{P}_\theta^k = \mathbb{Q}_\theta^k + {\mathbb{A}_1^k}^{\mathrm{T}} \mathbb{P}_\theta^{k+1} \mathbb{A}_1^k - ({\mathbb{B}_1^k}^{\mathrm{T}} \mathbb{P}_\theta^{k+1} \mathbb{A}_1^k)^{\mathrm{T}} (\mathbb{R}_\theta^k + {\mathbb{B}_1^k}^{\mathrm{T}} \mathbb{P}_\theta^{k+1} \mathbb{B}_1^k)^\dagger ({\mathbb{B}_1^k}^{\mathrm{T}} \mathbb{P}_\theta^{k+1} \mathbb{A}_1^k), \\
\mathbb{P}_\theta^N = \mathbb{G}_\theta^N, \\
0 = (\mathbb{R}_\theta^k + {\mathbb{B}_1^k}^{\mathrm{T}} \mathbb{P}_\theta^{k+1} \mathbb{B}_1^k)(\mathbb{R}_\theta^k + {\mathbb{B}_1^k}^{\mathrm{T}} \mathbb{P}_\theta^{k+1} \mathbb{B}_1^k)^\dagger ({\mathbb{B}_1^k}^{\mathrm{T}} \mathbb{P}_\theta^{k+1} \mathbb{A}_1^k) - {\mathbb{B}_1^k}^{\mathrm{T}} \mathbb{P}_\theta^{k+1} \mathbb{A}_1^k, \\
\mathbb{R}_\theta^k + {\mathbb{B}_1^k}^{\mathrm{T}} \mathbb{P}_\theta^{k+1} \mathbb{B}_1^k \geqslant 0, \ k \in \mathbb{N},
\end{cases}$$

and the Pareto optimal strategy is acquired as

$$u_\theta^{k^*} = -(\mathbb{R}_\theta^k + {\mathbb{B}_1^k}^{\mathrm{T}} \mathbb{P}_\theta^{k+1} \mathbb{B}_1^k)^\dagger ({\mathbb{B}_1^k}^{\mathrm{T}} \mathbb{P}_\theta^{k+1} \mathbb{A}_1^k) x_k.$$

Now, we present Algorithm 1 to obtain all Pareto optimal strategies and solutions for both the regular and the indefinite (under the convexity of costs) cooperative games.

Algorithm 1 Pareto optimality algorithm

1: Choose one $\theta \in \Theta$ and work out $\mathbb{P}_\theta^N$ by $\mathbb{P}_\theta^N = \mathbb{G}_\theta^N$ in (26);
2: Calculate $\mathbb{W}_\theta^k$, $\mathbb{W}_\theta^{k\dagger}$ (or $(\mathbb{W}_\theta^k)^{-1}$ for the regular case), $\mathbb{H}_\theta^k$, $\mathbb{P}_\theta^k$ in (26) and (27) from $k = N-1$ to $k = 0$;
3: Calculate the gain matrix $\mathbb{K}_\theta^k$ in (28) based on step 2 from $k = N-1$ to $k = 0$;
4: Determine $\mathbb{K}_{i\theta,k}^*$, $i \in \bar{\mathcal{N}}$ on the ground of step 3 and calculate the Pareto optimal strategy $u_{i\theta,k}^*$ by (32) in Theorem 4;
5: Substitute $\mathbb{K}_{i\theta,k}^*$ into (35) and calculate $\mathbb{X}_{i\theta}^{k^*}$ in Theorem 5 from $k = N$ to $k = 0$;
6: Substitute $\mathbb{X}_{i\theta}^{0^*}$ in step 5 into (34) and then calculate the Pareto optimal solution $\{J_1(x_0; u^*), \ldots, J_{\mathcal{N}}(x_0; u^*)|\theta \in \Theta\}$;
7: Change $\theta \in \Theta$, repeat steps 1–6 and acquire different Pareto optimal strategies and solutions.

Table 1 The coefficients of the finite horizon difference game

Time k	$b_{1,1}^k$	$b_{2,1}^k$	q_1^k	q_2^k	r_{11}^k	r_{12}^k	r_{21}^k	r_{22}^k
0	0.8000	−0.9000	1.0060	0.5020	1.0000	0	−0.1000	2.0900
1	0.8521	−0.9572	0.9500	0.4900	0.9576	0	−0.0800	1.9600
2	0.9800	−0.8556	0.9751	0.4895	0.9751	0	−0.0500	1.8976
3	0.9725	−0.9125	0.9800	0.5100	0.9835	0	0.0100	1.9270
4	1.0000	−0.9913	0.9315	0.4865	0.9637	0	−0.0700	2.0050
5	0.8632	−0.8569	0.9923	0.5060	1.0025	0	0.0600	1.8796
6	0.9600	−0.9625	0.9768	0.5123	0.9968	0	−0.0060	1.9865

4 A numerical example in economy

Example 2. Consider the mutual effect of fiscal policies of two states [16]. Suppose that the competitiveness between the two states can be characterized by

$$s_{k+1} = 0.6s_k + b_{1,1}^k f_{1,k} + b_{2,1}^k f_{2,k} + 0.3s_k\omega_k, \quad s_0 \in \mathbb{R},$$

where s_k represents the price difference of the two states and $f_{i,k}$ denotes the fiscal deficit, which can be dominated by the fiscal authority of state i, $i = 1, 2$. Each one of the fiscal authorities in both states is devoted to minimizing the intertemporal loss performance criteria given as

$$J_i(s_0; f_{1,k}, f_{2,k}) = \mathbb{E}\left[\sum_{k=0}^{N-1}\left(q_i^k s_k^2 + r_{i1}^k f_{1,k}^2 + r_{i2}^k f_{2,k}^2\right) + q_i^N s_N^2\right],\ i = 1, 2.$$

Taking $N = 7$, the coefficients of the equation and the costs are expressed in Table 1. Suppose that the two states decide to cooperate to reduce the initial price difference. Notice that $r_{12}^k = 0$, $k \in \{0, 1, 2, 3, 4, 5, 6\}$ in the game, which is referred to be a singular optimization problem [7]. Furthermore, some of the control weights $r_{21}^k < 0$ indicate that the cost of state 2 can be effectively reduced by the control of state 1 at time k. The weighted sum of the cost functionals can be expressed as

$$\begin{aligned} &J_\theta(s_0; f_{1,k}, f_{2,k}) \\ &= \theta J_1(s_0; f_{1,k}, f_{2,k}) + (1-\theta)J_2(s_0; f_{1,k}, f_{2,k}) \\ &= \mathbb{E}\left\{\sum_{k=0}^{6}\left[((q_1^k - q_2^k)\theta + q_2^k)s_k^2 + ((r_{11}^k - r_{21}^k)\theta + r_{21}^k)f_{1,k}^2 + (1-\theta)r_{22}^k f_{2,k}^2\right] + ((q_1^7 - q_2^7)\theta + q_2^7)s_7^2\right\}. \end{aligned}$$

Let $q_1^7 = 2$ and $q_2^7 = 1$. When $\theta = 0.5$, we solve the weighted DRE of the game stage by stage and derive the gain matrix $\mathbb{K}_\theta^k$, which are exhibited in Table 2. We select several values of θ to show the Pareto optimal solutions in Table 3 and plot the corresponding scatter diagram in Figure 1. When $\theta = 0.5$, $x_0 = 3$, the trajectories of the optimal state and the optimal strategies are shown in Figure 2.

5 Conclusion and future work

This paper has studied the Pareto optimality over a finite horizon for both the regular and the indefinite stochastic cooperative LQ difference games. By defining some sequences of bounded and linear operators, the convexity of costs has been derived to hold naturally as the constraints in the weighted matrices for the regular cooperative game. The Pareto optimality for the indefinite case has been well investigated by

Table 2 The coefficients of the weighted DRE and the gain matrix $\mathbb{K}_\theta^k$

Stage	$\mathbb{W}_\theta^k$	$\mathbb{W}_\theta^{k\dagger}$	$\mathbb{H}_\theta^k$	$\mathbb{P}_\theta^k$	$\mathbb{K}_\theta^k$
6	$\begin{bmatrix} 1.8778 & -1.3860 \\ -1.3860 & 2.3829 \end{bmatrix}$	$\begin{bmatrix} 0.9332 & 0.5428 \\ 0.5428 & 0.7354 \end{bmatrix}$	$\begin{bmatrix} 0.8640 \\ -0.8663 \end{bmatrix}$	0.9836	$\begin{bmatrix} -0.3361 \\ 0.1681 \end{bmatrix}$
5	$\begin{bmatrix} 1.2641 & -0.7275 \\ -0.7275 & 1.6620 \end{bmatrix}$	$\begin{bmatrix} 1.0575 & 0.4629 \\ 0.4629 & 0.8043 \end{bmatrix}$	$\begin{bmatrix} 0.5094 \\ -0.5057 \end{bmatrix}$	0.9502	$\begin{bmatrix} -0.3046 \\ 0.1709 \end{bmatrix}$
4	$\begin{bmatrix} 1.3971 & -0.9419 \\ -0.9419 & 1.9363 \end{bmatrix}$	$\begin{bmatrix} 1.0652 & 0.5182 \\ 0.5182 & 0.7685 \end{bmatrix}$	$\begin{bmatrix} 0.5701 \\ -0.5652 \end{bmatrix}$	0.8788	$\begin{bmatrix} -0.3144 \\ 0.1389 \end{bmatrix}$
3	$\begin{bmatrix} 1.3279 & -0.7799 \\ -0.7799 & 1.6952 \end{bmatrix}$	$\begin{bmatrix} 1.0319 & 0.4747 \\ 0.4747 & 0.8083 \end{bmatrix}$	$\begin{bmatrix} 0.5128 \\ -0.4811 \end{bmatrix}$	0.9163	$\begin{bmatrix} -0.3007 \\ 0.1455 \end{bmatrix}$
2	$\begin{bmatrix} 1.3426 & -0.7683 \\ -0.7683 & 1.6196 \end{bmatrix}$	$\begin{bmatrix} 1.0224 & 0.4850 \\ 0.4850 & 0.8475 \end{bmatrix}$	$\begin{bmatrix} 0.5388 \\ -0.4704 \end{bmatrix}$	0.9062	$\begin{bmatrix} -0.3227 \\ 0.1374 \end{bmatrix}$
1	$\begin{bmatrix} 1.0968 & -0.7391 \\ -0.7391 & 1.8103 \end{bmatrix}$	$\begin{bmatrix} 1.2579 & 0.5136 \\ 0.5136 & 0.7621 \end{bmatrix}$	$\begin{bmatrix} 0.4633 \\ -0.5204 \end{bmatrix}$	0.8990	$\begin{bmatrix} -0.3155 \\ 0.1587 \end{bmatrix}$
0	$\begin{bmatrix} 1.0254 & -0.6473 \\ -0.6473 & 1.7732 \end{bmatrix}$	$\begin{bmatrix} 1.2673 & 0.4626 \\ 0.4626 & 0.7328 \end{bmatrix}$	$\begin{bmatrix} 0.4315 \\ -0.4855 \end{bmatrix}$	0.9437	$\begin{bmatrix} -0.3223 \\ 0.1561 \end{bmatrix}$

Table 3 Values of θ and the corresponding Pareto optimal solutions

	θ								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$J_1(s_0; f_{1,k}, f_{2,k})$	14.2773	12.8205	12.2252	11.8025	11.5204	11.1826	10.8059	10.3684	9.8629
$J_2(s_0; f_{1,k}, f_{2,k})$	4.2030	4.6896	5.0382	5.3501	5.6575	6.0514	6.6190	7.5452	9.2752

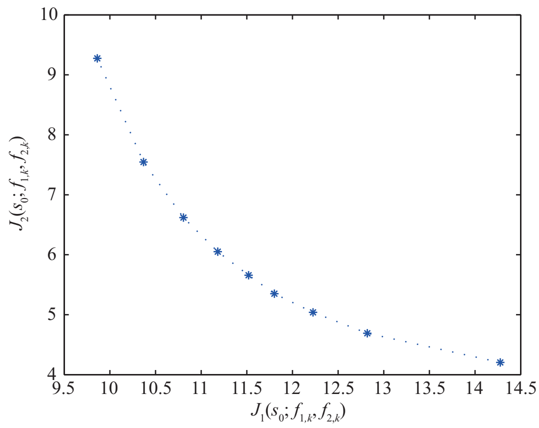

Figure 1 (Color online) The scatter diagram of the finite horizon Pareto solutions.

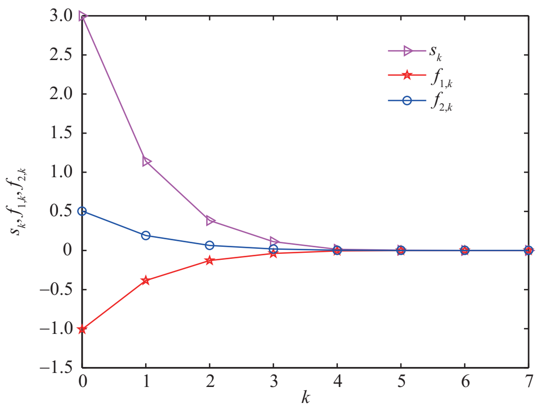

Figure 2 (Color online) The optimal state and control trajectories of the game.

the weighting technique, in which a new criterion for the convexity of the costs has been established. All the optimal strategies and solutions have been identified by an algorithm based on the solutions to the weighted DRE and the weighted DLE. For further investigations, we may study the finite horizon Pareto optimality based on the stochastic MP of the discrete-time type [26] and the infinite horizon Pareto optimality.

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