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# An online quantum state filter with sparse disturbance and Gaussian noise

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Dear editor,

Quantum communication and quantum computation, whose essence is using quantum mechanics to control the quantum state, are the cutting-edge technologies for information security and efficient computing. Quantum state estimation (QSE) is considered as a fundamental problem in high-precision quantum state feedback control, and the estimation accuracy significantly affects subsequent applications [1]. As proposed by Zhang et al. [2], this problem can be equivalently regarded as a convex optimization problem with the physical constraints of positive semidefinite and unit-trace Hermitian, and the disturbance of quantum system can be divided into two cases as measurement noise and system disturbance. Ignoring any of them, the estimated state will deviate from the true quantum state [3], which inspired us to do further research.

In this study, we propose an online quantum state filter (OQSF) to solve the problem of online quantum state estimation with both sparse disturbance and Gaussian noise. To the best of our knowledge, this is the first time that two types of disturbances are considered simultaneously in the online quantum state estimation problem.

*Problem statement.* Considering an $n$-qubit quantum system, we can design the measurement operator $\mathcal{A}_k$ and the measurement record sequence $b_k$ at the $k$-th sampling [4] according to the output $y$, where $b_k$ is contaminated by Gaussian measurement noise $e$ and sparse state disturbance $S$. To estimate the quantum state $\hat{\rho}_k$ at the current sampling $k$ is the aim of OQSF. First, the quadratic ellipsoidal norm is defined as $\|x\|_G^2 = x^\dagger G x$, where $x \in \mathbb{C}^{m\times 1}$ is the target column vector and $G \in \mathbb{C}^{m\times m}$ is a symmetric positive matrix, which can be freely selected as a weight matrix. With prior knowledge that $\rho_k$ is complex and satisfies the constraints of positive semidefinite and unit-trace Hermitian, $e$ obeys Gaussian distribution and $S$ is sparse. Therefore, the OQSF problem at the $k$-th ($k = 1, 2, \dots, N$) sampling can be converted into the following convex optimization problem:

$$\begin{aligned} \min_{\hat{\rho},\hat{S},\hat{e}} \quad & \|\mathrm{vec}(\hat{\rho},\hat{\rho}_{k-1})\|_{\omega I_1} + I_C(\hat{\rho}) + \theta\|\hat{S}\|_1 + \|\hat{e}\|_{\gamma I_2}, \\ \text{s.t.} \quad & \mathcal{A}_k\,\mathrm{vec}(\hat{\rho}+\hat{S}) + \hat{e} = b_k, \end{aligned} \tag{1}$$

where $\omega{>}0$, $\theta{>}0$ and $\gamma{>}0$ are the regularization parameters, $\|\mathrm{vec}(\hat{\rho},\hat{\rho}_{k-1})\|_{\omega I_1}$ and $\|\hat{e}\|_{\gamma I_2}$ are the quadratic ellipsoidal norms. $\omega I_1$ and $\gamma I_2$ are the weight matrices of the ellipsoidal norms. $I_1$ and $I_2$ represent the identity matrices with dimension $\min(k,l)$. Convex set $C := \rho \in \mathbb{C}^{d\times d} | \hat{\rho}^\dagger = \hat{\rho}, \hat{\rho} \succeq 0, \mathrm{tr}(\rho) = 1$ represents the quantum state constraint; when $\hat{\rho} \in C$, the indicator function $I_C = 0$, otherwise $I_C = \infty$.

*Online quantum state filter.* The online alternating direction multiplier (OADM) method is introuduced to derive the online quantum state filter. The OADM framework decomposes the problem into two optimization subproblems, then minimizes the augmented Lagrangian function corresponding to the original variables, and finally updates Lagrangian multipliers through dual gradient ascent to solve the global optimization problem. Although Eq. (1) contains three optimization variables, it can still be preliminarily decomposed by OADM. The augmented Lagrangian function of (1) is

$$\begin{aligned} L_\alpha := {} & \|\mathrm{vec}(\hat{\rho},\hat{\rho}_{k-1})\|_{\omega I_1} + \tfrac{\alpha}{2}\|\mathcal{A}_k\,\mathrm{vec}(\hat{\rho}+\hat{S}) + \hat{e} - b_k\| \\ & - \langle \lambda, \mathcal{A}_k\,\mathrm{vec}(\hat{\rho}+\hat{S}) + \hat{e} - b_k\rangle + \theta\|\hat{S}\|_1 + \|\hat{e}\|_{\gamma I_2}, \end{aligned} \tag{2}$$

where $\alpha > 0$ is a positive penalty parameter; $\lambda$ is a Lagrange multiplier and a real vector. During the optimization process, we ensure that $\rho$ belongs to the quantum constraint set $C$ at every moment. Thus, $\mathcal{A}_k\,\mathrm{vec}(\rho)$ is real and so is $\mathcal{A}_k\,\mathrm{vec}(\rho+S) - b_k$.

At this point, by OADM, we decompose the problem (1) into

$$\left\{ \begin{aligned} & (\hat{\rho}_k, \hat{S}_k) = \operatorname*{argmin}_{\hat{\rho},\hat{S}} \Big\{ \theta\|\hat{S}\|_1 + I_C(\hat{\rho}) + \|\mathrm{vec}(\hat{\rho}-\hat{\rho}_{k-1})\|_{\omega I_1}^2 \\ & \qquad + \frac{\alpha}{2}\left\| \mathcal{A}_k\,\mathrm{vec}(\hat{\rho}+\hat{S}) + \hat{e}_{k-1} - b_k - \frac{\lambda_{k-1}}{\alpha}\right\|_2^2 \Big\}, && \text{(3a)} \\ & \hat{e_k} = \operatorname*{argmin}_{\hat{e}} \Big\{ \frac{\alpha}{2}\left\| \mathcal{A}_k\,\mathrm{vec}(\hat{\rho}+\hat{S}) + \hat{e} - b_k - \frac{\lambda_{k-1}}{\alpha}\right\|_2^2 \\ & \qquad + \|e\|_{\gamma I_2}^2 \Big\}, && \text{(3b)} \\ & \lambda_k = \lambda_{k-1} - \alpha(\mathcal{A}_k\,\mathrm{vec}(\hat{\rho}+\hat{S}) + \hat{e}_k - \hat{b}_k). && \text{(3c)} \end{aligned} \right.$$

**Subproblem 1.** For the subproblem (3a) with dual variables $\hat{\rho}$ and $\hat{S}$, firstly ignore the indicator function $I_C(\hat{\rho})$

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and then replace the constant $(b_k + \lambda_{k-1}/\alpha - \hat{e}_{k-1})$ with $u$ in (3a). The unconstrained form of (3a) can be obtained:

$$(\tilde{\rho_k}, \hat{S}_k) = \underset{\hat{\rho},\hat{S}}{\operatorname{argmin}}\left\{\theta\|\hat{S}\|_1 + \|\operatorname{vec}(\hat{\rho} - \hat{\rho}_{k-1})\|^2_{\omega I_1} + \frac{\alpha}{2}\left\|\mathcal{A}_k \operatorname{vec}(\hat{\rho} + \hat{S}) - u\right\|_2^2\right\}, \tag{4}$$

where $\tilde{\rho}$ denotes the unconstrained density matrix.

All terms about $\hat{\rho}$ in (4) are differentiable. By using the first-order optimal condition and partitioned matrix inversion lemma, we can get the solution as

$$\operatorname{vec}(\tilde{\rho}_k) = \operatorname{vec}(\hat{\rho}_{k-1}) + K(z - \mathcal{A}_k \operatorname{vec}(\hat{S})), \tag{5}$$

where $K = \mathcal{A}_k^\dagger(\frac{2\omega}{\alpha} + \mathcal{A}_k^\dagger \mathcal{A}_k)^{-1}$, $z = u - \mathcal{A}_k \operatorname{vec}(\hat{\rho}_{k-1})$.

Substituting (5) into (4), we will get the optimization problem only with $S$:

$$\min_{\hat{S}}\{\|z - \mathcal{A}_k \operatorname{vec}(\hat{S})\|^2_R + \theta\|\hat{S}\|_1\}, \tag{6}$$

where $R = \alpha(I_3 - \mathcal{A}_k K)^\dagger(I_3 - \mathcal{A}_k) + \omega K^\dagger K$.

In problem (6), the ellipsoidal norm $\|\cdot\|_R$ is continuously differentiable, but the $l_1$ norm $\|\cdot\|_1$ is not differentiable. Let $f(\hat{S}) = \|z - \mathcal{A}_k \operatorname{vec}(\hat{S})\|^2_R$, $g(\hat{S}) = \theta\|\hat{S}\|_1$. It is obviously the problem that can be solved by iterative soft threshold algorithm (ISTA). We can get the calculation of sparse disturbance $S$ at the $k$-th sampling:

$$\hat{S}_k = M_{\beta_{k-1}\theta}(d_k), \tag{7}$$

where $M_{\beta_{k-1}\theta}(d_k)$ is the soft threshold operator, $M_{\beta_{k-1}\theta}(d_k) = \operatorname{sgn}(d_k) \times \max(d_k - \omega_{k-1}, 0)$. $d_k$ is the gradient of smoothing term $f(\hat{S})$ at the $k$-th sampling.

According to (7), calculate the estimated value of sparse disturbance $\hat{S}_k$ at time $k$ and substitute it into (5) to obtain the unconstrained estimated value density matrix $\tilde{\rho}_k$:

$$\operatorname{vec}(\tilde{\rho}_k) = \operatorname{vec}(\hat{\rho}_{k-1}) + K(z - \mathcal{A}_k \operatorname{vec}(\hat{S}_k)). \tag{8}$$

Now, considering the indicator function $I_C(\hat{\rho})$ previously ignored, the estimated value $\hat{\rho}_k$ that satisfies the physical constraints can be obtained by solving the following semidefinite programming (SDP) problem:

$$\hat{\rho}_k = \arg\min_{\hat{\rho}} \|\operatorname{vec}(\hat{\rho} - \tilde{\rho}_k)\|_F, \quad \text{s.t.} \quad \hat{\rho} \succeq 0,\ \operatorname{tr}(\hat{\rho}) = 1,\ \hat{\rho}^\dagger = \hat{\rho}, \tag{9}$$

where $\|\cdot\|_F$ represents Frobenius norm.

Note that the SDP problem (9) has an optimal solution [5], which can be obtained by solving Karush-Kuhn-Tucker (KKT) conditions. So far, we have obtained the estimation of $\hat{\rho}_k$ and $\hat{S}_k$.

**Subproblem 2.** Eq. (3b) is an unconstrained quadratic problem. The update formula of $e_k$ can be obtained directly through the first-order optimal condition as

$$\hat{e}_k = \frac{\alpha}{2\gamma + \alpha}\left(b_k + \frac{\lambda_{k-1}}{\alpha} - \mathcal{A}_k \operatorname{vec}(\hat{\rho})_k + \hat{S}_k\right). \tag{10}$$

*Experiments.* For an $n$-qubit quantum system, the initial state density matrix is chosen as $\rho_1^n = \underbrace{\rho_1 \otimes \cdots \otimes \rho_1}_{n}$ where $\rho_1 = [0.5, (1-i)/(\sqrt{8}); (1+i)/(\sqrt{8}), 0.5]$. Gaussian noise is generated by the MATLAB command randn$(n, 1)$ at each sampling, which is amplified by a selected constant multiple to obtain a desired signal-to-noise ratio (SNR), and SNR = 40 dB. The sparse disturbance matrix $S \in \mathbb{R}^{d\times d}$, whose value meets the Gaussian distribution $N(0, \|\rho\|_F/20)$, has $d^2/10$ nonzero entries, and the positions of nonzero elements are randomly distributed in the true density matrix. For the performance of an estimated $\rho_k$, we adopt the normalized distance $D(\hat{\rho}_k, \rho_k)$ between the estimated $\hat{\rho}_k$ and the true density matrix $\rho_k$, which is defined as $D(\hat{\rho}_k, \rho_k) = \|\hat{\rho}_k - \rho_k\|^2_F / \|\rho_k\|^2_F$.

To verify the superior performance, we compare the proposed OQSF with the existing algorithms ALR-MEG, OPG-ADMM and QSE-OADM, in terms of the normalized distance $D(\hat{\rho}_k, \rho_k)$ between the estimate $\hat{\rho}_k$ and the true density matrix $\rho_k$. In order to reflect the accuracy difference between the four algorithms more clearly, we set the ordinate representing the estimation accuracy to logarithmic form.

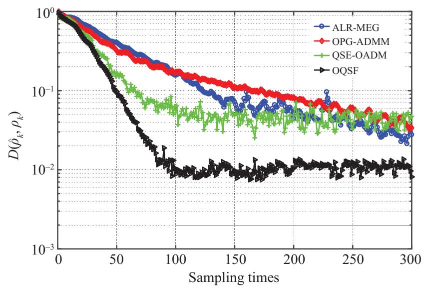


**Figure 1** (Color online) Normalized distance changes with sampling times for 4 qubits.

Figure 1 depicts the normalized distance $D(\hat{\rho}_k, \rho_k)$ of four algorithms with respect to sampling times for a 4-qubit system. We can see that at the 100th sampling, the performance $D(\hat{\rho}_k, \rho_k)$ of ALR-MEG, OPG-ADMM, QSE-OADM and OQSF is 0.16, 0.17, $5.3 \times 10^{-2}$ and $6.2 \times 10^{-3}$, respectively, which shows that under the same sampling, the estimation accuracy of OQSF based on the normalized distance is about an order of magnitude higher than that of QSE-OADM, which has the second highest accuracy. Furthermore, we calculate the running time required by the four algorithms to reach the performance indicators: 0.063 s in ALR-MEG, 0.3116 s in OPG-ADMM, 0.073 s in QSE-OADM and 0.043 s in OQSF. Our proposed OQSF can obtain the state estimation under the target accuracy with the least running time.

*Conclusion.* An OQSF was proposed in this study, which online estimated the density matrix in the time-varying quantum system with Gaussian measurement noise and sparse state disturbance. The experimental results showed the superiority of OQSF as an online state estimation algorithm for multi-qubit quantum systems, which is ready to realize the high-precision quantum estimated state feedback control in actual quantum device systems.

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**References**

1 Gao Q, Dong D, Petersen I R, et al. Design of a discrete-time fault-tolerant quantum filter and fault detector. IEEE Trans Cybern, 2021, 51: 889–899
2 Zhang J, Cong S, Ling Q, et al. Quantum state filter with disturbance and noise. IEEE Trans Automat Contr, 2020, 65: 2856–2866
3 Wu C, Qi B, Chen C, et al. Robust learning control design for quantum unitary transformations. IEEE Trans Cybern, 2017, 47: 4405–4417
4 Zhang K, Cong S, Li K. An efficient online estimation algorithm with measurement noise for time-varying quantum states. Signal Process, 2021, 180: 107887
5 Gonçalves D S, Gomes-Ruggiero M A, Lavor C. A projected gradient method for optimization over density matrices. Optim Methods Software, 2016, 31: 328–341