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Practical tracking of MIMO uncertain stochastic systems driven by colored noises via active disturbance rejection control

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Abstract This study addresses the practical tracking problem for a class of multi-input multi-output (MIMO) uncertain stochastic systems driven by colored noises via the active disturbance rejection control approach. The extended state observer is designed to estimate in real time the unmeasurable states and the stochastic total disturbance of each subsystem, including unknown coupling system dynamics between subsystems, colored noises, and uncertainty caused by partly unknown control parameters. Active disturbance rejection controllers based on the timely estimation of the extended state observer and compensation are then designed, guaranteeing the mean square practical convergence of the tracking errors. Some numerical simulations are performed to validate the effectiveness of the proposed control strategy.

Keywords uncertain stochastic systems, active disturbance rejection control, extended state observer, tracking, colored noise

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1 Introduction

Coping with disturbances and uncertainties is a distinct paradigm in control theory since the emergence of the modern control theory in the later years of the 1950s, as cited in the study of [1], who stated that the control operation "must not be influenced by internal and external disturbances" [1, p.228]. Many control approaches have been developed since the 1970s to deal with disturbances and uncertainties. Among many others, stochastic control [2–4] and robust control [5–7] are two important disturbance attenuation methods. The former is often made available for attenuating stochastic noises with known statistical characteristics, while the latter can be applied in attenuating more general disturbances assumed to be energy bounded. However, note that most of the robust control approaches are in view of the worst-case scenario that makes the control design comparatively conservative.

Inspired by the powerful proportional-integral-derivative (PID) error-driven thought rather than the model-based thought, an almost model-free anti-disturbance control technology known as active disturbance rejection control (ADRC), was proposed by Han [8] in the later 1980s. Compared with disturbance attenuation methods like stochastic control and robust control, ADRC is an active anti-disturbance control, albeit not a passive one, which addresses disturbance rejection for systems with disturbances and uncertainties. This innovative ideology arises from the estimation/cancelation strategy of ADRC by using an extended state observer (ESO), which is its key part. In the ADRC framework, internal unmodeled system dynamics and unknown external disturbances that affect system performance are lumped together to be a time signal, called total disturbance. Using the measurement output of a plant, the ESO is designed for the real-time estimation of the unmeasurable states and the total disturbance. Once the total

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disturbance is estimated in real time, the ADRC controller is designed based on the timely estimation of the ESO and compensation to acquire the desired performance of closed-loop systems. This estimation/compensation characteristic of ADRC makes it capable of eliminating the total disturbance in real time. The control energy can then be substantially saved in engineering applications [9].

On the one hand, the past two decades have seen the effectiveness and the practicality of ADRC in many engineering applications, such as the energy storage system [10], permanent magnet synchronous motor [11], gasoline engines [12], power plant [13], and raceway photobioreactor [14], among others. On the other hand, the past decade has observed many studies on the theoretical foundation of ADRC for stabilization, tracking, performance analysis of uncertain systems, and so on (see, for example, [15–25]). However, the disturbances considered in these studies all had no random characteristic. Disturbances in the form of stochastic noises are inescapable in both nature and man-made systems and often have negative effects on the system performance. Therefore, an interesting and important problem naturally arises. What kinds of stochastic noises can be estimated and rejected by ADRC? This topic is quite different from those discussed in most of the available studies on the disturbance attenuation for stochastic systems driven by white noise like [2, 3, 7, 26–28], where the white noise is the generalized derivative of the Brownian motion (see, for example, [29, p.51, Theorem 3.14]), and the plant is modeled by Itô-type stochastic systems. The Brownian motion is unbounded in the sense of almost everywhere or moment; thus, the estimation and cancelation strategy of ADRC leaves out the widely considered white noise in control theory. In view of the abovementioned consideration, ADRC has been applied in the output-feedback stabilization of single-input single-output and multi-input multi-output (MIMO) uncertain stochastic systems driven by bounded stochastic noises in [30, 31], where the noises are regarded as part of the stochastic total disturbances to be estimated and canceled by ADRC.

As mentioned, stochastic disturbances in the control theory are often modeled by white noise that is a stationary stochastic process with zero mean and constant spectral density. Nevertheless, white noise does not always describe well the stochastic disturbances in practice because its δ-function correlation is an idealization of the correlations of real processes which often have finite, or even long, correlation time [32]. A more practical characterization could be given by an exponentially correlated process, known as the colored noise or Ornstein-Uhlenbeck process [32,33]. We develop herein ADRC for a class of MIMO uncertain stochastic systems with the estimation and cancelation of realistic colored noises.

The main contributions and the novelty of this study are summed up as follows: (a) disturbances in the form of colored noises are first coped with in terms of estimation and disturbance rejection by ADRC; (b) the stochastic uncertainties are in a large scale, including unknown coupling system dynamics between subsystems, colored noises, and uncertainty caused by partly unknown control parameters; and (c) the designed ADRC controllers of a very simple structure guarantee a satisfactory practical tracking performance and an approximate decoupling of MIMO uncertain stochastic systems driven by colored noises.

We proceed as follows: Section 2 formulates the problem, Section 3 presents the main framework of the ADRC controller design and the main result, Section 4 provides the proof of the main result, Section 5 discusses the numerical simulations, and Section 6 concludes the paper.

The following notations are used all throughout the paper. $\mathbb{R}^n$: n-dimensional Euclidean space; $\mathrm{E}X$: mathematical expectation of a random variable X; X^{T}: transpose for vector or matrix X; $|X|$: absolute value of a scalar X; $\|X\|$: 2-norm (or Euclidean norm) of a vector X or induced 2-norm of a matrix X; I_n: n-dimensional identity matrix; $\lambda_{\min}(X)$ and $\lambda_{\max}(X)$: minimum and maximum eigenvalues of a positive definite matrix X, respectively.

2 Problem formation

The plant considered in this paper is the partial exact feedback linearizable MIMO system [34] with unknown coupling system dynamics and colored noises as follows:

$$\begin{cases} \mathrm{d}x_i(t) = A_{n_i}x_i(t)\mathrm{d}t + B_{n_i}\left[f_i(t,x(t)) + \displaystyle\sum_{j=1}^{m} p_{ij}w_j(t) + \sum_{j=1}^{m} q_{ij}u_j(t)\right]\mathrm{d}t, \\ y_i(t) = C_{n_i}x_i(t),\ i = 1,2,\ldots,m, \end{cases} \tag{1}$$

where $x = (x_1^{\mathrm{T}}, \ldots, x_m^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^n$ with $x_i \in \mathbb{R}^{n_i}$ and $n = n_1 + \cdots + n_m$, $u = (u_1, \ldots, u_m)^{\mathrm{T}} \in \mathbb{R}^m$, and $y = (y_1, \ldots, y_m)^{\mathrm{T}} \in \mathbb{R}^m$ are the state, control input, and output of the system, respectively; the functions $f_i(\cdot) : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ satisfying the following Assumption 1 are unknown; the constants q_{ij} $(i, j = 1, 2, \ldots, m)$ are the control coefficients which are not completely known with nominal values q_{ij}^* adequately close to q_{ij}. The system matrices in (1) are specified as

$$A_{n_i} = \begin{pmatrix} 0 & I_{n_i-1} \\ 0 & 0 \end{pmatrix}_{n_i \times n_i}, \quad B_{n_i} = (0, \ldots, 0, 1)^{\mathrm{T}}_{n_i \times 1}, \quad C_{n_i} = (1, 0, \ldots, 0)_{1 \times n_i}. \tag{2}$$

p_{ij} $(i, j = 1, 2, \ldots, m)$ are parameters that could be unknown, and $w_j(t)$ $(j = 1, 2, \ldots, m)$ are colored noises whose mathematical descriptions are specified in (3).

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a complete filtered probability space with a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \geqslant 0}$ on which m mutually independent one-dimensional standard Wiener processes $W_j(t)$ $(j = 1, 2, \ldots, m)$ are defined. Mathematically, the colored noises $w_j(t)$ $(j = 1, 2, \ldots, m)$ are the solutions to the Itô-type stochastic differential equations (see, for example, [32, p.426], [35, p.101]):

$$\mathrm{d}w_j(t) = -\alpha_j w_j(t)\mathrm{d}t + \alpha_j \sqrt{2\beta_j}\mathrm{d}W_j(t), \tag{3}$$

where $\alpha_j > 0$ and $\beta_j > 0$ are constants representing the correlation time and the noise intensity, respectively, and the initial values $w_j(0) \in L^2(\Omega; \mathbb{R})$ are independent of $W_j(t)$. From the point of physic meaning, the parameters α_j represent the bandwidth of the noise, β_j are their spectral height, and the correlation functions of the processes $w_j(t)$ are the more realistic exponential functions but not the δ-ones (see, for example, [32]). It should be noted that α_j, β_j could be unknown parameters throughout this paper.

It follows from Itô isometry that

$$\sup_{t \geqslant 0} \mathrm{E}|w_j(t)|^2 \leqslant \mathrm{E}|w_j(0)|^2 + \alpha_j \beta_j \triangleq \gamma_j. \tag{4}$$

The boundedness of the colored noises in the second moment is a key reason why the colored noises can be estimated and rejected by the ADRC approach.

Let $v_i(t)$ $(i = 1, 2, \ldots, m)$ be given reference signals which are supposed to be (n_i+1)-order continuously differentiable. The control objective in this paper is to design ADRC controllers such that for any initial states, the output $y_i(t)$ of each subsystem tracks $v_i(t)$ in practically mean square sense, and at the same time $x_{ij}(t)$ tracks $v_i^{(j-1)}(t)$ in practically mean square sense for all $j = 2, \ldots, n_i$.

3 ADRC controller design and the main result

The unknown system dynamics and external disturbances affecting performance of each subsystem are regarded as the stochastic total disturbance or extended state from the "time scale" to be estimated by ESO, no matter what the mathematical expressions of the system dynamics and external disturbances are. For each $1 \leqslant i \leqslant m$, the stochastic total disturbance (extended state) of i-subsystem is as follows:

$$x_{i(n_i+1)}(t) \triangleq f_i(t, x(t)) + \sum_{j=1}^{m} p_{ij} w_j(t) + \sum_{j=1}^{m} (q_{ij} - q_{ij}^*) u_j(t), \tag{5}$$

which contains unknown coupling system dynamics between subsystems, colored noises, and uncertainty caused by the deviation of control parameters from their nominal values. So it can be seen that the stochastic total disturbance is with stochastic uncertainty in large scale.

In order to estimate in real time the unmeasurable states and stochastic total disturbance of each subsystem, a set of ESOs are designed by using the inputs and outputs of system (1) as follows:

$$\begin{cases} \mathrm{d}\hat{x}_{i1}(t) = [\hat{x}_{i2}(t) + a_{i1} r(y_i(t) - \hat{x}_{i1}(t))]\mathrm{d}t, \\ \mathrm{d}\hat{x}_{i2}(t) = [\hat{x}_{i3}(t) + a_{i2} r^2(y_i(t) - \hat{x}_{i1}(t))]\mathrm{d}t, \\ \qquad \vdots \\ \mathrm{d}\hat{x}_{in_i}(t) = \left[\hat{x}_{i(n_i+1)}(t) + a_{in_i} r^{n_i}(y_i(t) - \hat{x}_{i1}(t)) + \displaystyle\sum_{j=1}^{m} q^*_{ij} u_j(t)\right]\mathrm{d}t, \\ \mathrm{d}\hat{x}_{i(n_i+1)}(t) = a_{i(n_i+1)} r^{n_i+1}(y_i(t) - \hat{x}_{i1}(t))\mathrm{d}t, \ 1 \leqslant i \leqslant m, \end{cases} \tag{6}$$

where $\hat{x}_{ij}(t)$ $(j = 1, 2, \ldots, n_i)$ are the estimates of the states $x_{ij}(t)$, $\hat{x}_{i(n_i+1)}(t)$ is the estimate of the stochastic total disturbance $x_{i(n_i+1)}(t)$ of i-subsystem, $r > 0$ is the gain parameter to be tuned, and a_{ij} $(j = 1, 2, \ldots, n_i + 1)$ are designed parameters such that the following matrices are Hurwitz:

$$E_i = \begin{pmatrix} -a_{i1} & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -a_{in_i} & 0 & 0 & \ddots & 1 \\ -a_{i(n_i+1)} & 0 & 0 & \cdots & 0 \end{pmatrix}_{(n_i+1)\times(n_i+1)}. \tag{7}$$

Compared with the traditional state observer design, an augmented state variable $\hat{x}_{i(n_i+1)}(t)$ is added in each ESO for real-time estimation of the stochastic total disturbance $x_{i(n_i+1)}(t)$ of i-subsystem.

For $1 \leqslant i \leqslant m$, set

$$(v_{i1}(t), v_{i2}(t), \ldots, v_{i(n_i+2)}(t)) = \left(v_i(t), \dot{v}_i(t), \ldots, v_i^{(n_i+1)}(t)\right). \tag{8}$$

After ESOs (6) are designed, the ESO-based controllers are designed as follows:

$$u_i(t) = \sum_{l=1}^{m} \hat{q}^*_{il} \left\{ \sum_{j=1}^{n_l} k_{lj}(\hat{x}_{lj}(t) - v_{lj}(t)) - \hat{x}_{l(n_l+1)}(t) + v_{l(n_l+1)}(t) \right\}, \quad 1 \leqslant i \leqslant m, \tag{9}$$

where $\hat{q}^*_{il}$ are defined in (14) and the feedback gain parameters k_{lj} $(l = 1, 2, \ldots, m, j = 1, 2, \ldots, n_l)$ are chosen such that the following matrices are Hurwitz:

$$F_i = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ddots & 1 \\ k_{i1} & k_{i2} & \cdots & k_{i(n_i-1)} & k_{in_i} \end{pmatrix}_{n_i \times n_i}. \tag{10}$$

The $-\hat{x}_{l(n_l+1)}(t)$ is a compensation term designed for real-time cancelation of the stochastic total disturbance.

In order to obtain the mean square practical convergence of the resulting closed-loop system composed of (1), (6) and (9), the following Assumptions are required.

The following Assumption 1 is about the unknown functions $f_i(\cdot)$ $(i = 1, 2, \ldots, m)$.

Assumption 1. The unknown functions $f_i : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ $(i = 1, \ldots, m)$ are continuously differentiable with respect to their arguments. There exist known constants $D_{ij} > 0$ $(j = 1, 2, 3)$ such that for all $t \geqslant 0$, $x \in \mathbb{R}^n$, it holds that

$$\left| \frac{\partial f_i(t, x)}{\partial t} \right| \leqslant D_{i1} + D_{i2}\|x\|, \quad \left\| \frac{\partial f_i(t, x)}{\partial x} \right\| \leqslant D_{i3}, \quad i = 1, 2, \ldots, m. \tag{11}$$

Remark 1. Since the coupling system dynamic of each i-subsystem is regarded as the part of the stochastic total disturbance to be estimated by ESO, it is reasonable to assume by (11) that its partial derivatives (or "variations") with respect to t and x are linear growth and bounded, respectively.

The following Assumption 2 is about the reference signals and their derivatives.

Assumption 2. Suppose that there exist known positive constants N_i $(i = 1, 2, \dots, m)$ such that

$$\sup_{t\geqslant 0} |v_{ij}(t)| \leqslant N_i, \quad \forall j = 1, 2, \dots, n_i + 2. \tag{12}$$

Remark 2. In Assumptions 1 and 2, the upper bounds D_{ij} $(i = 1, 2, \dots, m, j = 1, 2, 3)$ and N_i $(i = 1, 2, \dots, m)$ are assumed to be known, which is theoretically for the purpose of guaranteeing that the lower bound r^* of the tuning gain parameter r specified in the following (31) is known. It should be noted that these bounds can be relaxed to be unknown in practical applications by some experience in tuning gain parameter.

Let matrices U_i and Q_i be the unique positive definite matrix solutions of the Lyapunov equations:

$$U_iE_i + E_i^{\mathrm{T}}U_i = -I_{n_i+1} \ \text{ and } \ Q_iF_i + F_i^{\mathrm{T}}Q_i = -I_{n_i}, \tag{13}$$

respectively.

The following Assumption 3 is mainly about the deviation level of nominal values q^*_{ij} from the unknown control coefficients q_{ij}, which should be not so large as assumed in (15).

Assumption 3. The matrix with the nominal values q^*_{ij} $(i, j = 1, 2, \dots, m)$ as entries is invertible whose inverse matrix is given by

$$\begin{pmatrix} q^*_{11} & q^*_{12} & \cdots & q^*_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ q^*_{m1} & q^*_{m2} & \cdots & q^*_{mm} \end{pmatrix}^{-1} = \begin{pmatrix} \hat{q}^*_{11} & \hat{q}^*_{12} & \cdots & \hat{q}^*_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{q}^*_{m1} & \hat{q}^*_{m2} & \cdots & \hat{q}^*_{mm} \end{pmatrix}, \tag{14}$$

and the deviations of nominal values q^*_{ij} from the unknown control coefficients q_{ij} satisfy

$$\xi \triangleq 1 - \sum_{i,j,l=1}^{m} 2\lambda_{\max}(U_i)|(q_{ij} - q^*_{ij})\hat{q}^*_{jl}a_{l(n_l+1)}| > 0, \tag{15}$$

where ξ is known and U_i are specified in (13).

The main result on mean square practical convergence of the resulting closed-loop system composed of (1), (6) and (9) is summarized in the succeeding Theorem 1, which includes the mean square practical convergence of the estimation errors of unmeasurable states and stochastic total disturbance of each subsystem and mean square practical convergence of the tracking errors.

Theorem 1. Suppose that Assumptions 1–3 hold. Then, the closed-loop of system (1) under ESOs (6) based controllers (9) has the unique global solutions and the mean square practical convergence in the sense that there are a known constant $r^* > 0$ (specified in (31)) and a constant $t_r \triangleq 2r^{\varrho} + 1$ with any $r \geqslant r^*$ and any given $\varrho > 0$, such that for any initial values $x(0) \in \mathbb{R}^n, \hat{x}(0) \in \mathbb{R}^{n+m}$, and for all $t \geqslant t_r$, it holds that

$$\mathrm{E}|x_{ij}(t) - \hat{x}_{ij}(t)|^2 \leqslant \frac{\Upsilon}{r^{2n_i+3-2j}}, \quad 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n_i + 1, \tag{16}$$

$$\mathrm{E}|x_{ij}(t) - v_{ij}(t)|^2 \leqslant \frac{\Upsilon}{r}, \quad 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n_i, \tag{17}$$

where $\Upsilon > 0$ is a constant independent of the gain constant r and specified in (50).

Remark 3. It should be pointed out that the mean square practical convergence of the output tracking errors is included as

$$\mathrm{E}|y_i(t) - v_i(t)|^2 \leqslant \frac{\Upsilon}{r}, \quad 1 \leqslant i \leqslant m, \tag{18}$$

and the stabilization problem at the origin in practically mean square sense is a special case by letting $v_i(t) \equiv 0$ for all $t \geqslant 0$ and $i = 1, 2, \dots, m$. By adjusting the gain parameter r, the estimation errors (16) and tracking errors (17) could be arbitrarily small for sufficiently large $t \geqslant t_r$, where the convergence practicality is embodied by the fact that the estimation and tracking accuracy and t_r are dependent on the tuning gain parameter r.

Remark 4. The stochastic total disturbance $x_{i(n_i+1)}(t)$ of each i-subsystem is estimated in real time by ESOs (6) and is approximatively canceled by the compensation term of the ADRC controllers (9). In what follows, it can be seen that the ϖ_i-subsystem of the equivalent closed-loop system (25) is approximately decoupled, which reveals a natural decoupling function of ADRC.

4 Proof of the main result

Proof of Theorem 1. For $1 \leqslant i \leqslant m$, set

$$\begin{aligned}
&\eta_{ij}(t) = r^{n_i+1-j}[x_{ij}(t) - \hat{x}_{ij}(t)], \quad 1 \leqslant j \leqslant n_i + 1,\\
&\varpi_{ij}(t) = x_{ij}(t) - v_{ij}(t), \quad 1 \leqslant j \leqslant n_i,\\
&\eta_i = (\eta_{i1}, \ldots, \eta_{i(n_i+1)})^{\mathrm{T}}, \quad \varpi_i = (\varpi_{i1}, \ldots, \varpi_{in_i})^{\mathrm{T}},\\
&\Theta_i = \left(x_{i2}, \ldots, x_{i(n_i-1)}, \quad \sum_{j=1}^{n_i} k_{ij}(\hat{x}_{ij} - v_{ij}) + x_{i(n_i+1)} - \hat{x}_{i(n_i+1)} + v_{i(n_i+1)} \right)^{\mathrm{T}},\\
&\Theta = (\Theta_1^{\mathrm{T}}, \ldots, \Theta_m^{\mathrm{T}})^{\mathrm{T}}.
\end{aligned} \tag{19}$$

We choose $r \geqslant 1$. Find the derivative with respect to the time variable t along the closed-loop of system (1) under ESOs (6) based controllers (9). It is obtained that

$$\frac{\mathrm{d}f_i(t, x(t))}{\mathrm{d}t} = \frac{\partial f_i(t, x(t))}{\partial t} + \left(\frac{\partial f_i(t, x(t))}{\partial x}\right)^{\mathrm{T}} \Theta(t) \triangleq \Delta_{i1}(t). \tag{20}$$

By Assumptions 1 and 2, it can be easily concluded that there exist known positive constants $\delta_{i1}, \delta_{i2}, \delta_{i3}$ independent of r such that

$$|\Delta_{i1}(t)| \leqslant \delta_{i1} + \delta_{i2}\|\eta(t)\| + \delta_{i3}\|\varpi(t)\|, \quad \forall t \geqslant 0. \tag{21}$$

In addition,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=1}^{m}(q_{ij} - q_{ij}^*)u_j(t)\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\sum_{j,l=1}^{m}(q_{ij} - q_{ij}^*)\hat{q}_{jl}^*\left\{\sum_{s=1}^{n_l} k_{ls}(\hat{x}_{ls}(t) - v_{ls}(t)) - \hat{x}_{l(n_l+1)}(t) + v_{l(n_l+1)}(t)\right\}\\
&= \sum_{j,l=1}^{m}(q_{ij} - q_{ij}^*)\hat{q}_{jl}^*\left\{\sum_{s=1}^{n_l-1} k_{ls}[\hat{x}_{l(s+1)}(t) + a_{ls}r^s(y_l(t) - \hat{x}_{l1}(t)) - v_{l(s+1)}(t)]\right\}\\
&\quad + \sum_{j,l=1}^{m}(q_{ij} - q_{ij}^*)\hat{q}_{jl}^*\left\{k_{ln_l}\left(\hat{x}_{l(n_l+1)}(t) + a_{ln_l}r^{n_l}(y_l(t) - \hat{x}_{l1}(t)) + \sum_{s=1}^{m} q_{ls}^* u_s(t) - v_{l(n_l+1)}(t)\right)\right\}\\
&\quad + \sum_{j,l=1}^{m}(q_{ij} - q_{ij}^*)\hat{q}_{jl}^*\left\{-a_{l(n_l+1)}r^{n_l+1}(y_l(t) - \hat{x}_{l1}(t)) + \dot{v}_{l(n_l+1)}(t)\right\} \triangleq \Delta_{i2}(t).
\end{aligned} \tag{22}$$

By Assumption 2, a direct computation shows that there exist known positive constants $\delta_{i4}, \delta_{i5}, \delta_{i6}$ independent of r such that

$$|\Delta_{i2}(t)| \leqslant \delta_{i4} + \delta_{i5}\|\eta(t)\| + \delta_{i6}\|\varpi(t)\| + r\Lambda_i\|\eta(t)\|, \quad \forall t \geqslant 0, \tag{23}$$

where

$$\Lambda_i = \sum_{j,l=1}^{m} |(q_{ij} - q_{ij}^*)\hat{q}_{jl}^* a_{l(n_l+1)}|. \tag{24}$$

After making some direct deductions, it can be obtained further that the closed-loop system composed of system (1), ESOs (6), controllers (9) and the reference signals is equivalent to the one as follows:

$$\begin{cases} \mathrm{d}\varpi_i(t) = A_{n_i}\varpi_i(t)\mathrm{d}t + B_{n_i}\left[\sum\limits_{j=1}^{n_i} k_{ij}\varpi_{ij}(t) - \sum\limits_{j=1}^{n_i}\frac{k_{ij}}{r^{n_i+1-j}}\eta_{ij}(t) + \eta_{i(n_i+1)}(t)\right]\mathrm{d}t, \\ \mathrm{d}\eta_i(t) = rA_{n_i+1}\eta_i(t)\mathrm{d}t - r\begin{pmatrix} a_{i1}\eta_{i1}(t) \\ \cdots \\ a_{i(n_i+1)}\eta_{i1}(t)\end{pmatrix}\mathrm{d}t + B_{n_i+1}\left(\Delta_{i1}(t)+\Delta_{i2}(t) - \sum\limits_{j=1}^{m} p_{ij}\alpha_j w_j(t)\right)\mathrm{d}t \\ \qquad +B_{n_i+1}\sum\limits_{j=1}^{m} p_{ij}\alpha_j\sqrt{2\beta_j}\mathrm{d}W_j(t), \quad 1 \leqslant i \leqslant m, \end{cases} \tag{25}$$

where the ϖ_i-subsystem and η_i-subsystem represent the dynamic model of the tracking error and the estimation error, respectively.

For $1 \leqslant i \leqslant m$, we define the Lyapunov functions $V_{i1} : \mathbb{R}^{n_i} \to \mathbb{R}$ by $V_{i1}(\varpi_i) = \varpi_i^{\mathrm{T}} Q_i \varpi_i$ for $\varpi_i \in \mathbb{R}^{n_i}$ and $V_{i2} : \mathbb{R}^{n_i+1} \to \mathbb{R}$ by $V_{i2}(\eta_i) = \eta_i^{\mathrm{T}} U_i \eta_i$ for $\eta_i \in \mathbb{R}^{n_i+1}$. Define the positive definite function V : $\mathbb{R}^{2n_1+\cdots+2n_m+m} \to \mathbb{R}$ by

$$V(\varpi,\eta) = V_1(\varpi_1,\ldots,\varpi_m) + V_2(\eta_1,\ldots,\eta_m) = \sum_{i=1}^{m}[V_{i1}(\varpi_i) + V_{i2}(\eta_i)]. \tag{26}$$

Next, the proof is proceeded in three steps as follows.

Step 1. The existence of the unique global solutions $\varpi(t)$, $\eta(t)$ to system (25) and their mean square practical boundedness are proved.

By (3), $w_j(t)$ $(j = 1,2,\ldots,m)$ can be regarded as augmented state variables of (25). It follows directly from the existence-and-unique theorem for Itô-type stochastic systems (see, for example, [35, p.58,Theorem 3.6]) that there exist unique global solutions $\varpi(t)$, $\eta(t)$, $w_j(t)$ $(j = 1,2,\ldots,m)$ to the equivalent closed-loop system (25).

Apply Itô's formula to $V(\varpi(t),\eta(t))$ with respect to t along system (25) to obtain that

$$\begin{aligned} &\mathrm{d}V(\varpi(t),\eta(t)) \\ &= \sum_{i=1}^{m}\Bigg\{\sum_{j=1}^{n_i-1}\frac{\partial V_{i1}(\varpi_i(t))}{\partial \varpi_{ij}}\varpi_{i(j+1)}(t)\mathrm{d}t + \frac{\partial V_{i1}(\varpi_i(t))}{\partial \varpi_{in_i}}\Bigg\{\sum_{j=1}^{n_i}k_{ij}\varpi_{ij}(t) - \sum_{j=1}^{n_i}\frac{k_{ij}}{r^{n_i+1-j}}\eta_{ij}(t) \\ &\quad +\eta_{i(n_i+1)}(t)\Bigg\}\mathrm{d}t + r\Bigg\{\sum_{j=1}^{n_i}\frac{\partial V_{i2}(\eta_i(t))}{\partial \eta_{ij}}[\eta_{i(j+1)}(t) - a_{ij}\eta_{i1}(t)] - \frac{\partial V_{i2}(\eta_i(t))}{\partial \eta_{i(n_i+1)}}a_{i(n_i+1)}\eta_{i1}(t)\Bigg\}\mathrm{d}t \\ &\quad +\frac{\partial V_{i2}(\eta_i(t))}{\partial \eta_{i(n_i+1)}}\Bigg\{\Delta_{i1}(t)+\Delta_{i2}(t) - \sum_{j=1}^{m}p_{ij}\alpha_j w_j(t)\Bigg\}\mathrm{d}t + \frac{\partial^2 V_{i2}(\eta_i(t))}{\partial \eta^2_{i(n_i+1)}}\sum_{j=1}^{m}p_{ij}^2\alpha_j^2\beta_j\mathrm{d}t \\ &\quad +\frac{\partial V_{i2}(\eta_i(t))}{\partial \eta_{i(n_i+1)}}\sum_{j=1}^{m}p_{ij}\alpha_j\sqrt{2\beta_j}\mathrm{d}W_j(t)\Bigg\}. \end{aligned} \tag{27}$$

It follows from (21), (23), (25), (27) and Young's inequality that

$$\begin{aligned} &\mathrm{d}V(\varpi(t),\eta(t)) \\ &\leqslant \sum_{i=1}^{m}\Bigg\{ -\|\varpi_i(t)\|^2 + \frac{1}{r}\|\varpi_i(t)\|^2 + \frac{1}{r}\lambda^2_{\max}(Q_i)\left(\sum_{j=1}^{n_i}k_{ij}\right)^2\|\eta_i(t)\|^2 + \mu_i\|\varpi_i(t)\|^2 \\ &\quad +\frac{\lambda^2_{\max}(Q_i)}{4\mu_i}\|\eta_i(t)\|^2 - r\|\eta_i(t)\|^2 + 2\lambda_{\max}(U_i)\|\eta_i(t)\|\cdot\Bigg\{\delta_{i1} + \delta_{i2}\|\eta(t)\| + \delta_{i3}\|\varpi(t)\| \\ &\quad +\delta_{i4} + \delta_{i5}\|\eta(t)\| + \delta_{i6}\|\varpi(t)\| + r\Lambda_i\|\eta(t)\| + \sum_{j=1}^{m}|p_{ij}\alpha_j w_j(t)|\Bigg\} + 2\lambda_{\max}(U_i)\sum_{j=1}^{m}p_{ij}^2\alpha_j^2\beta_j\Bigg\}\mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{m}\frac{\partial V_{i2}(\eta_i(t))}{\partial\eta_{i(n_i+1)}}\sum_{j=1}^{m}p_{ij}\alpha_j\sqrt{2\beta_j}\mathrm{d}W_j(t)\\
&\leqslant\sum_{i=1}^{m}\Bigg\{-\|\varpi_i(t)\|^2+\frac{1}{r}\|\varpi_i(t)\|^2+\frac{1}{r}\lambda_{\max}^2(Q_i)\left(\sum_{j=1}^{n_i}k_{ij}\right)^2\|\eta_i(t)\|^2+\mu_i\|\varpi_i(t)\|^2\\
&+\frac{\lambda_{\max}^2(Q_i)}{4\mu_i}\|\eta_i(t)\|^2-r\|\eta_i(t)\|^2+\|\eta_i(t)\|^2+\lambda_{\max}^2(U_i)\delta_{i1}^2+2\lambda_{\max}(U_i)\delta_{i2}\|\eta(t)\|^2\\
&+\mu_i\|\varpi(t)\|^2+\frac{\lambda_{\max}^2(U_i)\delta_{i3}^2}{\mu_i}\|\eta_i(t)\|^2+\|\eta_i(t)\|^2+\lambda_{\max}^2(U_i)\delta_{i4}^2+2\lambda_{\max}(U_i)\delta_{i5}\|\eta(t)\|^2\\
&+\mu_i\|\varpi(t)\|^2+\frac{\lambda_{\max}^2(U_i)\delta_{i6}^2}{\mu_i}\|\eta_i(t)\|^2+2r\lambda_{\max}(U_i)\Lambda_i\|\eta(t)\|^2+\|\eta_i(t)\|^2+\lambda_{\max}^2(U_i)\left(\sum_{j=1}^{m}|p_{ij}\alpha_jw_j(t)|\right)^2\\
&+2\lambda_{\max}(U_i)\sum_{j=1}^{m}p_{ij}^2\alpha_j^2\beta_j\Bigg\}\mathrm{d}t+\sum_{i=1}^{m}\frac{\partial V_{i2}(\eta_i(t))}{\partial\eta_{i(n_i+1)}}\sum_{j=1}^{m}p_{ij}\alpha_j\sqrt{2\beta_j}\mathrm{d}W_j(t),
\end{aligned}\tag{28}
$$

where μ_i $(i=1,2,\ldots,m)$ are some positive constants to be specified in (29).

Choose $\mu_i>0$ $(i=1,2,\ldots,m)$ and sufficiently large $r_1>0$ to guarantee that

$$
\begin{aligned}
&\theta_0\triangleq 1-\frac{1}{r_1}-\left(\max_{1\leqslant i\leqslant m}\mu_i+2\sum_{i=1}^{m}\mu_i\right)>0,\\
&\frac{\xi r_1}{2}-\Bigg\{\frac{1}{r_1}\max_{1\leqslant i\leqslant m}\left(\lambda_{\max}^2(Q_i)\left(\sum_{j=1}^{n_i}k_{ij}\right)^2\right)+\max_{1\leqslant i\leqslant m}\left(\frac{\lambda_{\max}^2(Q_i)}{4\mu_i}\right)+3\\
&\quad+\sum_{i=1}^{m}2\lambda_{\max}(U_i)(\delta_{i2}+\delta_{i5})+\max_{1\leqslant i\leqslant m}\frac{\lambda_{\max}^2(U_i)\delta_{i3}^2}{\mu_i}+\max_{1\leqslant i\leqslant m}\left(\frac{\lambda_{\max}^2(U_i)\delta_{i6}^2}{\mu_i}\right)\Bigg\}>0,
\end{aligned}\tag{29}
$$

where ξ is specified in (15). Set

$$
\begin{aligned}
&\theta_1=\sum_{i=1}^{m}\lambda_{\max}^2(U_i)\delta_{i1}^2+\sum_{i=1}^{m}\lambda_{\max}^2(U_i)\delta_{i4}^2+\sum_{i=1}^{m}2\lambda_{\max}(U_i)\sum_{j=1}^{m}p_{ij}^2\alpha_j^2\beta_j,\\
&\theta_2=\frac{\theta_0}{\max_{1\leqslant i\leqslant m}\lambda_{\max}(Q_i)}.
\end{aligned}\tag{30}
$$

Let

$$
r\geqslant r^*\triangleq\max\left\{1,r_1,\frac{2\theta_0\max_{1\leqslant i\leqslant m}(\lambda_{\max}(U_i))}{\xi\max_{1\leqslant i\leqslant m}(\lambda_{\max}(Q_i))}\right\}.\tag{31}
$$

It follows that

$$
\begin{aligned}
&\mathrm{d}V(\varpi(t),\eta(t))\leqslant-\theta_0\|\varpi(t)\|^2\mathrm{d}t-\frac{\xi r}{2}\|\eta(t)\|^2\mathrm{d}t+\sum_{i=1}^{m}\lambda_{\max}^2(U_i)\left(\sum_{j=1}^{m}|p_{ij}\alpha_jw_j(t)|\right)^2\mathrm{d}t\\
&\quad+\theta_1\mathrm{d}t+\sum_{i=1}^{m}\frac{\partial V_{i2}(\eta_i(t))}{\partial\eta_{i(n_i+1)}}\sum_{j=1}^{m}p_{ij}\alpha_j\sqrt{2\beta_j}\mathrm{d}W_j(t)\\
&\leqslant-\theta_2V(\varpi(t),\eta(t))\mathrm{d}t+\sum_{i=1}^{m}\lambda_{\max}^2(U_i)\left(\sum_{j=1}^{m}|p_{ij}\alpha_jw_j(t)|\right)^2\mathrm{d}t+\theta_1\mathrm{d}t\\
&\quad+\sum_{i=1}^{m}\frac{\partial V_{i2}(\eta_i(t))}{\partial\eta_{i(n_i+1)}}\sum_{j=1}^{m}\alpha_j\sqrt{2\beta_j}\mathrm{d}W_j(t),
\end{aligned}\tag{32}
$$

and then

$$
V(\varpi(t),\eta(t))\leqslant\mathrm{e}^{-\theta_2t}V(\varpi(0),\eta(0))+m\sum_{i=1}^{m}\lambda_{\max}^2(U_i)\max_{1\leqslant j\leqslant m}|p_{ij}\alpha_j|^2\int_0^t\mathrm{e}^{-\theta_2(t-s)}\sum_{j=1}^{m}|w_j(s)|^2\mathrm{d}s
$$

$$+\int_0^t \mathrm{e}^{-\theta_2(t-s)}\theta_1 \mathrm{d}s + \int_0^t \mathrm{e}^{-\theta_2(t-s)} \sum_{i=1}^m \frac{\partial V_{i2}(\eta_i(s))}{\partial \eta_{i(n_i+1)}} \sum_{j=1}^m \alpha_j \sqrt{2\beta_j} \mathrm{d}W_j(s). \tag{33}$$

A direct computation shows that

$$\begin{aligned} V(\varpi(0),\eta(0)) &= \sum_{i=1}^m [V_{i1}(\varpi_i(0)) + V_{i2}(\eta_i(0))] \leqslant \sum_{i=1}^m [\lambda_{\max}(Q_i)\|\varpi_i(0)\|^2 + \lambda_{\max}(U_i)\|\eta_i(0)\|^2] \\ &\leqslant \sum_{i=1}^m \left[\lambda_{\max}(Q_i)\sum_{j=1}^{n_i} |x_{ij}(0) - v_{ij}(0)|^2 + \lambda_{\max}(U_i)\sum_{j=1}^{n_i+1} r^{2n_i+2-2j}|x_{ij}(0) - \hat{x}_{ij}(0)|^2\right]. \end{aligned} \tag{34}$$

Choose a $\varrho > 0$ and define

$$\begin{aligned} \Upsilon_1 &= \sup_{r\in[r^*,\infty)} \mathrm{e}^{-\theta_2 r^\varrho} \sum_{i=1}^m \left[\lambda_{\max}(Q_i)\sum_{j=1}^{n_i} |x_{ij}(0) - v_{ij}(0)|^2 \right. \\ &\quad \left. +\lambda_{\max}(U_i)\sum_{j=1}^{n_i+1} r^{2n_i+2-2j}|x_{ij}(0) - \hat{x}_{ij}(0)|^2\right]. \end{aligned} \tag{35}$$

Hence for any $r \geqslant r^*$ and all $t \geqslant r^\varrho$,

$$\mathrm{e}^{-\theta_2 t} V(\varpi(0),\eta(0)) \leqslant \mathrm{e}^{-\theta_2 r^\varrho} V(\varpi(0),\eta(0)) \leqslant \Upsilon_1. \tag{36}$$

As mentioned above, $w_j(t)$ $(j = 1,2,\ldots,m)$ can be considered as augmented state variables of system (25). Thus, it can be seen that the drift and the diffusion terms of system (25) satisfy linear growth condition. So, it can be easily concluded (see, for example, [35, p.51, Lemma 3.2]) that $\mathrm{E}\int_0^t (\mathrm{e}^{-\theta_2(t-s)} \sum_{i=1}^m \frac{\partial V_{i2}(\eta_i(s))}{\partial \eta_{i(n_i+1)}} \sum_{j=1}^m \alpha_j \sqrt{2\beta_j})^2 \mathrm{d}s < \infty$ for any $t \geqslant 0$. That is,

$$\int_0^t \mathrm{e}^{-\theta_2(t-s)} \sum_{i=1}^m \frac{\partial V_{i2}(\eta_i(s))}{\partial \eta_{i(n_i+1)}} \sum_{j=1}^m \alpha_j \sqrt{2\beta_j} \mathrm{d}W_j(s)$$

is a martingale for any $t \geqslant 0$, not just a local martingale. Then, by taking mathematical expectation on both sides of (33), it is obtained that for any $r \geqslant r^*$,

$$\mathrm{E}V(\varpi(t),\eta(t)) \leqslant \Upsilon_1 + \frac{m}{\theta_2}\sum_{i=1}^m \lambda_{\max}^2(U_i) \max_{1\leqslant j\leqslant m} |p_{ij}\alpha_j|^2 \sum_{j=1}^m \gamma_j + \frac{\theta_1}{\theta_2} \triangleq \Upsilon_2, \quad \forall t \geqslant r^\varrho. \tag{37}$$

This completes the proof of Step 1.

Step 2. The existence of the unique global solutions $x(t)$, $\hat{x}(t)$ to the closed-loop system composed of (1), (6) and (9) and the mean square practical convergence of ESOs (6) are proved.

The existence of the unique global solutions $x(t)$, $\hat{x}(t)$ to the closed-loop system composed of (1), (6) and (9) follows directly from the one of $\varpi(t)$, $\eta(t)$ to the equivalent system (25) concluded in Step 1.

Similar to (28), by (29) and Young's inequality, we apply Itô's formula again to $V_2(\eta(t))$ with respect to t along η_i-subsystem of (25) to obtain that

$$\begin{aligned} \mathrm{d}V_2(\eta(t)) &\leqslant \sum_{i=1}^m \Bigg\{ - r\|\eta_i(t)\|^2 + 2\lambda_{\max}(U_i)\|\eta_i(t)\| \cdot \Bigg[\delta_{i1} + \delta_{i2}\|\eta(t)\| + \delta_{i3}\|\varpi(t)\| + \delta_{i4} \\ &\quad +\delta_{i5}\|\eta(t)\| + \delta_{i6}\|\varpi(t)\| + r\Lambda_i\|\eta(t)\| + \sum_{j=1}^m |p_{ij}\alpha_j w_j(t)|\Bigg] + 2\lambda_{\max}(U_i)\sum_{j=1}^m p_{ij}^2\alpha_j^2\beta_j \Bigg\}\mathrm{d}t \\ &\quad + \sum_{i=1}^m \frac{\partial V_{i2}(\eta_i(t))}{\partial \eta_{i(n_i+1)}} \sum_{j=1}^m \alpha_j \sqrt{2\beta_j} \mathrm{d}W_j(t) \\ &\leqslant \sum_{i=1}^m \Bigg\{ - r\|\eta_i(t)\|^2 + \|\eta_i(t)\|^2 + \lambda_{\max}^2(U_i)\delta_{i1}^2 + 2\lambda_{\max}(U_i)\delta_{i2}\|\eta(t)\|^2 + \mu_i\|\varpi(t)\|^2 \end{aligned}$$

$$
\begin{aligned}
&+\frac{\lambda_{\max}^2(U_i)\delta_{i3}^2}{\mu_i}\|\eta_i(t)\|^2+\|\eta_i(t)\|^2+\lambda_{\max}^2(U_i)\delta_{i4}^2+2\lambda_{\max}(U_i)\delta_{i5}\|\eta(t)\|^2+\mu_i\|\varpi(t)\|^2\\
&+\frac{\lambda_{\max}^2(U_i)\delta_{i6}^2}{\mu_i}\|\eta_i(t)\|^2+2r\lambda_{\max}(U_i)\Lambda_i\|\eta(t)\|^2+\|\eta_i(t)\|^2+\lambda_{\max}^2(U_i)\left(\sum_{j=1}^m|p_{ij}\alpha_jw_j(t)|\right)^2\\
&\left.+2\lambda_{\max}(U_i)\sum_{j=1}^m p_{ij}^2\alpha_j^2\beta_j\right\}\mathrm{d}t+\sum_{i=1}^m\frac{\partial V_{i2}(\eta_i(t))}{\partial\eta_{i(n_i+1)}}\sum_{j=1}^m\alpha_j\sqrt{2\beta_j}\mathrm{d}W_j(t)\\
\leqslant&-\frac{\xi r}{2\max_{1\leqslant i\leqslant m}\lambda_{\max}(U_i)}V_2(\eta(t))\mathrm{d}t+m\sum_{i=1}^m\lambda_{\max}^2(U_i)\max_{1\leqslant j\leqslant m}|p_{ij}\alpha_j|^2\sum_{j=1}^m|w_j(t)|^2\mathrm{d}t+\theta_1\mathrm{d}t\\
&+2\sum_{i=1}^m\mu_i\|\varpi(t)\|^2\mathrm{d}t+\sum_{i=1}^m\frac{\partial V_{i2}(\eta_i(t))}{\partial\eta_{i(n_i+1)}}\sum_{j=1}^m\alpha_j\sqrt{2\beta_j}\mathrm{d}W_j(t),
\end{aligned}\tag{38}
$$

where θ_1 is specified in (30).

Set

$$
\theta_3=\frac{\xi}{2\max_{1\leqslant i\leqslant m}\lambda_{\max}(U_i)}.\tag{39}
$$

Thus, it follows from (38) that

$$
\begin{aligned}
V_2(\eta(t))\leqslant&\ \mathrm{e}^{-\theta_3r(t-r^\varrho)}V_2(\eta(r^\varrho))+m\sum_{i=1}^m\lambda_{\max}^2(U_i)\max_{1\leqslant j\leqslant m}|p_{ij}\alpha_j|^2\int_{r^\varrho}^t\mathrm{e}^{-\theta_3r(t-s)}\sum_{j=1}^m|w_j(s)|^2\mathrm{d}s\\
&+\int_{r^\varrho}^t\mathrm{e}^{-\theta_3r(t-s)}\theta_1\mathrm{d}s+\int_{r^\varrho}^t\mathrm{e}^{-\theta_3r(t-s)}2\sum_{i=1}^m\mu_i\|\varpi(s)\|^2\mathrm{d}s\\
&+\int_{r^\varrho}^t\mathrm{e}^{-\theta_3r(t-s)}\sum_{i=1}^m\frac{\partial V_{i2}(\eta_i(s))}{\partial\eta_{i(n_i+1)}}\sum_{j=1}^m\alpha_j\sqrt{2\beta_j}\mathrm{d}W_j(s),\quad\forall t\geqslant r^\varrho.
\end{aligned}\tag{40}
$$

Similar to the deduction in Step 1, it is easy to conclude that

$$
\int_{r^\varrho}^t\mathrm{e}^{-\theta_3r(t-s)}\sum_{i=1}^m\frac{\partial V_{i2}(\eta_i(s))}{\partial\eta_{i(n_i+1)}}\sum_{j=1}^m\alpha_j\sqrt{2\beta_j}\mathrm{d}W_j(s)
$$

is a martingale for any $t\geqslant r^\varrho$. Therefore, by taking mathematical expectation on both sides of (40), it follows from (37) that

$$
\begin{aligned}
\mathrm{E}V_2(\eta(t))\leqslant&\ \mathrm{e}^{-\theta_3r}\mathrm{E}V_2(\eta(r^\varrho))+\frac{m}{r\theta_3}\sum_{i=1}^m\lambda_{\max}^2(U_i)\max_{1\leqslant j\leqslant m}|p_{ij}\alpha_j|^2\sum_{j=1}^m\gamma_j\\
&+\frac{\theta_1}{r\theta_3}+\frac{2\sum_{i=1}^m\mu_i\Upsilon_2}{r\theta_3\min_{1\leqslant i\leqslant m}\lambda_{\min}(Q_i)},\quad\forall t\geqslant r^\varrho+1.
\end{aligned}\tag{41}
$$

Let

$$
\Upsilon_3=\sup_{r\in[r^*,\infty)}r\mathrm{e}^{-\theta_3r}\Upsilon_2.\tag{42}
$$

Therefore, by (37), for any $r\geqslant r^*$ and all $t\geqslant r^\varrho+1$, we have

$$
\begin{aligned}
\mathrm{E}V_2(\eta(t))\leqslant&\ \frac{\Upsilon_3}{r}+\frac{m}{r\theta_3}\sum_{i=1}^m\lambda_{\max}^2(U_i)\max_{1\leqslant j\leqslant m}|p_{ij}\alpha_j|^2\sum_{j=1}^m\gamma_j\\
&+\frac{\theta_1}{r\theta_3}+\frac{2\sum_{i=1}^m\mu_i\Upsilon_2}{r\theta_3\min_{1\leqslant i\leqslant m}\lambda_{\min}(Q_i)}\triangleq\frac{\Upsilon_4}{r},
\end{aligned}\tag{43}
$$

and thus

$$
\mathrm{E}\,\|\eta_i(t)\|^2\leqslant\mathrm{E}\,\|\eta(t)\|^2\leqslant\frac{\mathrm{E}V_2(\eta(t))}{\min_{1\leqslant i\leqslant m}\{\lambda_{\min}(U_i)\}}\leqslant\frac{\Upsilon_4}{r\min_{1\leqslant i\leqslant m}\{\lambda_{\min}(U_i)\}}.\tag{44}
$$

For all $1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n_i + 1$, it follows from (44) that

$$\mathrm{E}[x_{ij}(t) - \hat{x}_{ij}(t)]^2 = \frac{1}{r^{2n_i+2-2j}}\mathrm{E}\left|\eta_{ij}(t)\right|^2 \leqslant \frac{\Upsilon_4}{r^{2n_i+3-2j}\min_{1\leqslant i\leqslant m}\{\lambda_{\min}(U_i)\}} \leqslant \frac{\Upsilon}{r^{2n_i+3-2j}}, \tag{45}$$

where $\Upsilon \geqslant \frac{\Upsilon_4}{\min_{1\leqslant i\leqslant m}\{\lambda_{\min}(U_i)\}}$ is specified in (50). This completes the proof of Step 2.

Step 3. The mean square practical convergence of the tracking errors is proved.

Similar to (28), by (29) and Young's inequality, find the derivative of $V_1(\varpi(t))$ with respect to t along ϖ_i-subsystem of (25) to obtain that

$$\begin{aligned}
&\mathrm{d}V_1(\varpi(t))\\
&\leqslant \sum_{i=1}^{m}\left\{-\|\varpi_i(t)\|^2 + \frac{1}{r}\|\varpi_i(t)\|^2 + \frac{\lambda_{\max}^2(Q_i)}{r}\left(\sum_{j=1}^{n_i}k_{ij}\right)^2\|\eta_i(t)\|^2 + \mu_i\|\varpi_i(t)\|^2 + \frac{\lambda_{\max}^2(Q_i)}{4\mu_i}\|\eta_i(t)\|^2\right\}\mathrm{d}t\\
&\leqslant -\frac{\theta_0}{\max_{1\leqslant i\leqslant m}\lambda_{\max}(Q_i)}V_1(\varpi(t))\mathrm{d}t + \max_{1\leqslant i\leqslant m}\left(\frac{1}{r}\lambda_{\max}^2(Q_i)\left(\sum_{j=1}^{n_i}k_{ij}\right)^2 + \frac{\lambda_{\max}^2(Q_i)}{4\mu_i}\right)\|\eta(t)\|^2\mathrm{d}t.
\end{aligned} \tag{46}$$

Then, it follows from (37) and (44) that for all $t \geqslant t_r \triangleq 2r^{\varrho} + 1$, we have

$$\begin{aligned}
\mathrm{E}V_1(\varpi(t)) \ &\leqslant \mathrm{e}^{-\frac{\theta_0}{\max_{1\leqslant i\leqslant m}\lambda_{\max}(Q_i)}(t-r^{\varrho}-1)}\mathrm{E}V_1(\varpi(r^{\varrho}+1))\\
&\quad + \max_{1\leqslant i\leqslant m}\left(\frac{1}{r}\lambda_{\max}^2(Q_i)\left(\sum_{j=1}^{n_i}k_{ij}\right)^2 + \frac{\lambda_{\max}^2(Q_i)}{4\mu_i}\right)\int_{r^{\varrho}+1}^{t}\mathrm{e}^{-\frac{\theta_0}{\max_{1\leqslant i\leqslant m}\lambda_{\max}(Q_i)}(t-s)}\mathrm{E}\|\eta(s)\|^2\mathrm{d}s\\
&\leqslant \mathrm{e}^{-\frac{\theta_0}{\max_{1\leqslant i\leqslant m}\lambda_{\max}(Q_i)}(t-r^{\varrho}-1)}\Upsilon_2 + \frac{\theta_4}{r} = \frac{\Upsilon_5}{r},
\end{aligned} \tag{47}$$

where

$$\theta_4 \triangleq \max_{1\leqslant i\leqslant m}\left(\frac{1}{r^*}\lambda_{\max}^2(Q_i)\left(\sum_{j=1}^{n_i}k_{ij}\right)^2 + \frac{\lambda_{\max}^2(Q_i)}{4\mu_i}\right)\frac{\Upsilon_4\max_{1\leqslant i\leqslant m}\lambda_{\max}(Q_i)}{\theta_0\min_{1\leqslant i\leqslant m}\lambda_{\min}(U_i)} \tag{48}$$

and

$$\Upsilon_5 \triangleq \sup_{r\in[r^*,\infty)} r\mathrm{e}^{-\frac{\theta_0}{\max_{1\leqslant i\leqslant m}\lambda_{\max}(Q_i)}r^{\varrho}}\Upsilon_2 + \theta_4. \tag{49}$$

Set

$$\Upsilon = \max\left\{\frac{\Upsilon_4}{\min_{1\leqslant i\leqslant m}\{\lambda_{\min}(U_i)\}}, \frac{\Upsilon_5}{\min_{1\leqslant i\leqslant m}\lambda_{\min}(Q_i)}\right\}. \tag{50}$$

Therefore, for any $r \geqslant r^*$, there exists an r-dependent constant $t_r \triangleq 2r^{\varrho} + 1$, such that for all $t \geqslant t_r$ and $1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n_i + 1$, we have

$$\mathrm{E}|x_{ij}(t) - v_{ij}(t)|^2 \leqslant \mathrm{E}\|\varpi(t)\|^2 \leqslant \frac{\Upsilon_5}{r\min_{1\leqslant i\leqslant m}\lambda_{\min}(Q_i)} \leqslant \frac{\Upsilon}{r}. \tag{51}$$

This completes the proof of Step 3 and also Theorem 1.

5 Numerical simulations

In this section, some numerical simulations are performed to illustrate the effectiveness of the proposed ADRC approach. Consider the following MIMO uncertain stochastic system driven by colored noises with unmodeled dynamics as follows:

$$\begin{cases} \mathrm{d}x_{11}(t) = x_{12}(t)\mathrm{d}t, \\ \mathrm{d}x_{12}(t) = \left[f_1(t,x(t)) + \sum\limits_{j=1}^{2} p_{1j}w_j(t) + u_1(t) + u_2(t)\right]\mathrm{d}t, \\ \mathrm{d}x_{21}(t) = x_{22}(t)\mathrm{d}t, \\ \mathrm{d}x_{22}(t) = \left[f_2(t,x(t)) + \sum\limits_{j=1}^{2} p_{2j}w_j(t) + u_1(t) - u_2(t)\right]\mathrm{d}t, \\ y_1(t) = x_{11}(t),\ y_2(t) = x_{21}(t). \end{cases} \tag{52}$$

It can be seen that system (52) is a special case of system (1), where $m = 2, n_1 = n_2 = 2, n = n_1 + n_2 = 4, q_{11} = q_{12} = q_{21} = 1, q_{22} = -1,\ f_i(\cdot)\ (i = 1, 2)$ are unknown functions satisfying Assumption 1, and $p_{ij}\ (i, j = 1, 2)$ are uncertain parameters. The reference signals are specified as $v_1(t) = \sin(2t + 3)$ and $v_2(t) = 2\cos(t + 2)$. The stochastic total disturbance (extended state) of each subsystem is defined as

$$x_{13}(t) = f_1(t,x(t)) + \sum_{j=1}^{2} p_{1j}w_j(t),\ x_{23}(t) = f_2(t,x(t)) + \sum_{j=1}^{2} p_{2j}w_j(t). \tag{53}$$

Motivated from (6), we can design ESOs (54) for system (52) as follows:

$$\begin{cases} \mathrm{d}\hat{x}_{11}(t) = [\hat{x}_{12}(t) + 3\cdot 100\cdot(y_1(t) - \hat{x}_{11}(t))]\mathrm{d}t, \\ \mathrm{d}\hat{x}_{12}(t) = [\hat{x}_{13}(t) + 3\cdot 100^2\cdot(y_1(t) - \hat{x}_{11}(t)) + u_1(t) + u_2(t)]\mathrm{d}t, \\ \mathrm{d}\hat{x}_{13}(t) = 100^3\cdot(y_1(t) - \hat{x}_{11}(t))\mathrm{d}t, \\ \mathrm{d}\hat{x}_{21}(t) = [\hat{x}_{22}(t) + 3\cdot 100\cdot(y_2(t) - \hat{x}_{21}(t))]\mathrm{d}t, \\ \mathrm{d}\hat{x}_{22}(t) = [\hat{x}_{23}(t) + 3\cdot 100^2\cdot(y_2(t) - \hat{x}_{21}(t)) + u_1(t) - u_2(t)]\mathrm{d}t, \\ \mathrm{d}\hat{x}_{23}(t) = 100^3\cdot(y_2(t) - \hat{x}_{21}(t))\mathrm{d}t, \end{cases} \tag{54}$$

where the corresponding matrices in (7) are

$$E_1 = E_2 = \begin{pmatrix} -3 & 1 & 0 \\ -3 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \tag{55}$$

with eigenvalues identical to -1 and then are Hurwitz, and the gain is chosen as $r = 100$.

The ADRC controllers are designed as

$$\begin{aligned} u_1(t) = {} & \frac{1}{2}[-2(\hat{x}_{11}(t) - \sin(2t+3)) - 3(\hat{x}_{12}(t) - 2\cos(2t+3)) - \hat{x}_{13}(t) - 4\sin(2t+3)] \\ & + \frac{1}{2}[-3(\hat{x}_{21}(t) - 2\cos(t+2)) - 4(\hat{x}_{22}(t) + 2\sin(t+2)) - \hat{x}_{23}(t) - 2\cos(t+2)], \\ u_2(t) = {} & \frac{1}{2}[-2(\hat{x}_{11}(t) - \sin(2t+3)) - 3(\hat{x}_{12}(t) - 2\cos(2t+3)) - \hat{x}_{13}(t) - 4\sin(2t+3)] \\ & - \frac{1}{2}[-3(\hat{x}_{21}(t) - 2\cos(t+2)) - 4(\hat{x}_{22}(t) + 2\sin(t+2)) - \hat{x}_{23}(t) - 2\cos(t+2)], \end{aligned} \tag{56}$$

where in this case the $\hat{q}^*_{il}\ (i, l = 1, 2)$ in (9) is specified as $\hat{q}^*_{11} = \hat{q}^*_{12} = \hat{q}^*_{21} = \frac{1}{2}, \hat{q}^*_{22} = -\frac{1}{2}$, and the matrices in (10)

$$F_1 = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix} \tag{57}$$

are Hurwitz. In Figures 1 and 2, the initial values are chosen as

$$w_1(0) = 0, \quad w_2(0) = 0 \tag{58}$$

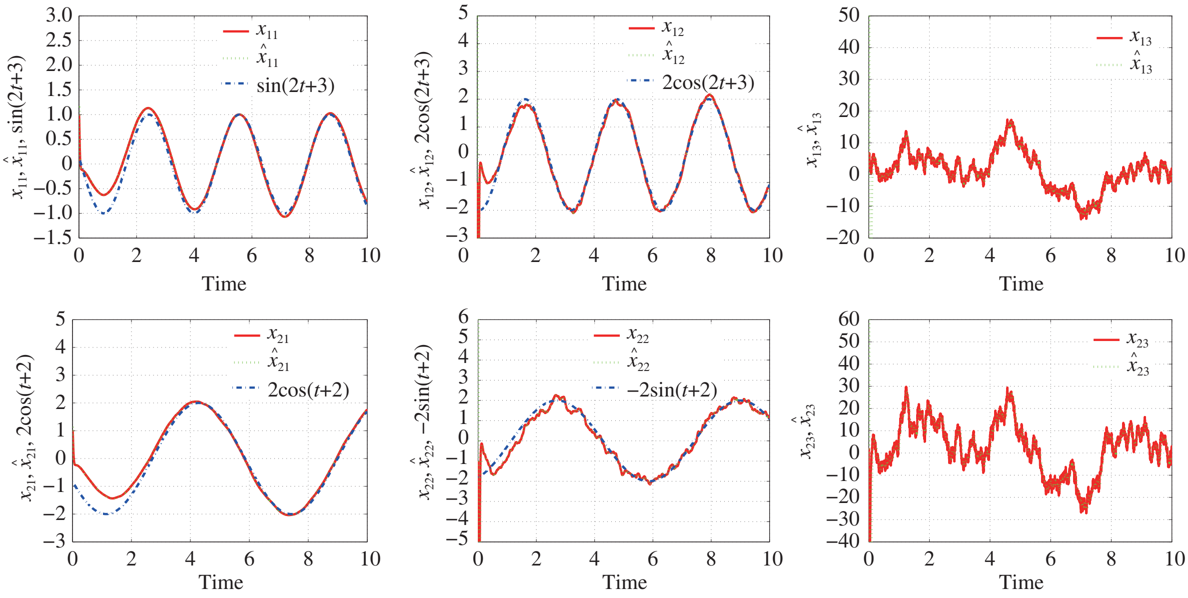

Figure 1 (Color online) The estimation and tracking effects with uncertain parameters given in (60) and (61).

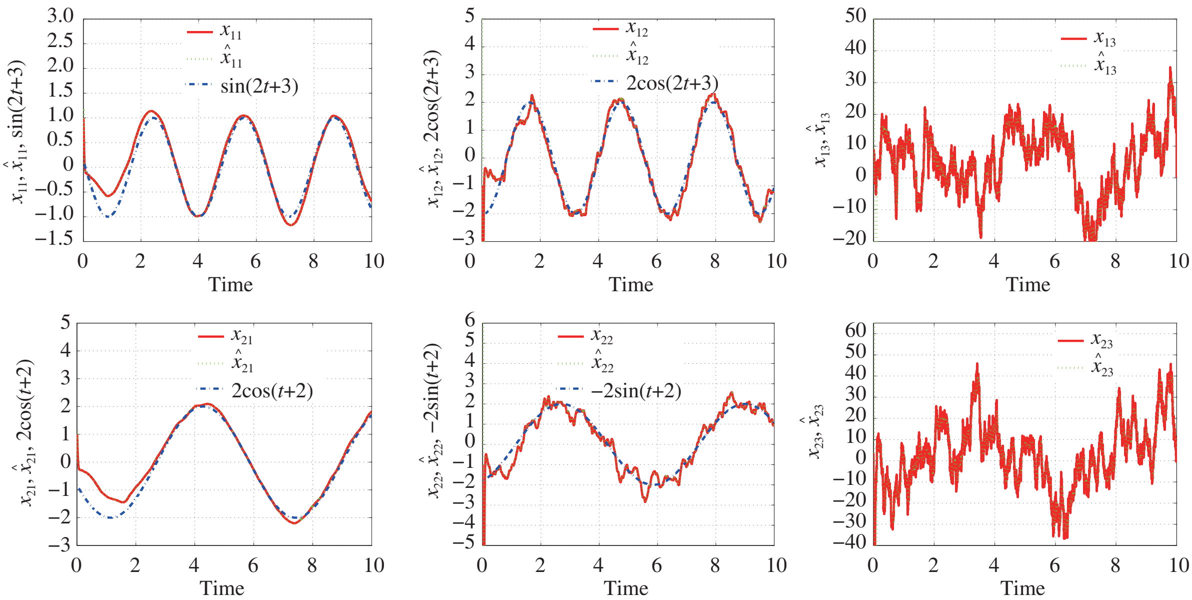

Figure 2 (Color online) The estimation and tracking effects with uncertain parameters given in (62) and (63).

and

$$\begin{aligned} &x_{11}(0) = x_{21}(0) = 1, \quad x_{12}(0) = x_{22}(0) = -1, \\ &\hat{x}_{11}(0) = \hat{x}_{12}(0) = \hat{x}_{21}(0) = \hat{x}_{22}(0) = 0. \end{aligned} \tag{59}$$

In Figure 1, the unknown system functions and uncertain parameters about colored noises are specified as

$$\begin{aligned} &f_1(t, x(t)) = 2\mathrm{e}^{-t} + x_{11}(t) + 2x_{21}(t) + \cos(x_{12}(t) + x_{22}(t)), \\ &f_2(t, x(t)) = \mathrm{e}^{-t} + x_{11}(t) + 2x_{21}(t) + 3x_{22}(t) + \sin(x_{12}(t) + x_{21}(t)), \end{aligned} \tag{60}$$

and

$$\alpha_1 = \beta_1 = 2, \quad \alpha_2 = \beta_2 = 3, \quad p_{11} = p_{12} = 1, \quad p_{21} = p_{22} = 2, \tag{61}$$

respectively.

It can be observed from Figure 1 that for the first subsystem of (52) the output $y_1(t)$ tracks the reference signal $v_1(t) = \sin(2t+3)$, the state $x_{12}(t)$ tracks the derivative of the reference signal $\dot{v}_1(t) = 2\cos(2t+3)$, and the state of ESO $\hat{x}_{1j}(t)$ $(j = 1, 2, 3)$ estimates $x_{1j}(t)$ $(j = 1, 2, 3)$ very effectively. It can be also seen from Figure 1 that for the second subsystem of (52) the tracking effects of the output $y_2(t)$ to the reference signal $v_2(t) = 2\cos(t + 2)$, the state $x_{22}(t)$ to the derivative of the reference signal $\dot{v}_2(t) = -2\sin(t + 2)$ and the estimation effects of the state of ESO $\hat{x}_{2j}(t)$ $(j = 1, 2, 3)$ to $x_{2j}(t)$ $(j = 1, 2, 3)$ are all very satisfactory. In Figure 2, the unknown system functions and uncertain parameters about colored noises are specified as

$$\begin{aligned} &f_1(t, x(t)) = 3\mathrm{e}^{-t} + 2x_{11}(t) + 3x_{21}(t) + 2\cos(x_{12}(t) + x_{22}(t)),\\ &f_2(t, x(t)) = 2\mathrm{e}^{-t} + 2x_{11}(t) + 3x_{21}(t) + 4x_{22}(t) + 3\sin(x_{12}(t) + x_{21}(t)) \end{aligned} \tag{62}$$

and

$$\alpha_1 = 2, \quad \alpha_2 = 3, \quad \beta_1 = \beta_2 = 4, \quad p_{11} = 2, p_{12} = 3, p_{21} = p_{22} = 4, \tag{63}$$

respectively. Although the relevant parameters concerning the intensity of the stochastic total disturbance (53) are increased from (60) and (61) to (62) and (63), in Figure 2 the corresponding estimation and tracking effects maintain very effective. This partly reflects the robust performance of the ADRC controllers (56).

6 Concluding remarks

The ADRC approach was applied herein to the practical tracking problem for a class of MIMO uncertain stochastic systems driven by colored noises. A set of extended state observers were designed for a real-time estimation of the stochastic total disturbance of each subsystem, including unknown coupling system dynamics, colored noises, and deviation uncertainty of control parameters from their nominal values. ADRC controllers based on the estimation and compensation by using extended state observers were then designed. The mean square practical convergence of the closed-loop, including the mean square practical convergence of the output tracking errors, was obtained with a rigorous theoretical analysis. Finally, some numerical simulations were performed to demonstrate the practicality of the proposed active anti-disturbance control design strategy.

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