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Consensus of switched multi-agent systems with binary-valued communications

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Abstract This paper studies the consensus of switched multi-agent systems (MAS) with binary-valued communications. Different from the existing studies on switched MAS considering precise observations, each agent studied in this research only receives binary-valued information with stochastic noises from its neighbors' states. Further, unlike the existing studies on MAS with binary-valued information in a fixed topology, in this paper, we consider the jointly connected undirected graphs, each of which switches with non-zero probability. The consensus algorithm comprises of two stages: first, the connected agents employ a recursive projection algorithm to estimate their neighbors' states based on the binary-valued communications; second, the control law of the connected agents is developed based on the estimations to upgrade their states. It is proved that both the speed of the estimation convergence to the real states and the consensus speed of the states can achieve $O(1/t)$ when the iteration step is given a proper value. Furthermore, the results indicate that the larger the value of the lowest probability that a graph emerges with, the more easily the consensus could be achieved. Finally, a simulation is presented to demonstrate the theoretical analysis.

Keywords binary-valued system, switched multi-agent system, recursive projection algorithm, consensus, jointly connected undirected topologies

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1 Introduction

In the past decade, multi-agent systems (MAS) have been extensively studied in many studies given their relevant applications, such as coordination behavior of blocks [1], sensor networks [2, 3], and underwater vehicles [4]. Researchers have studied MAS from different perspectives, such as the type of measurements (precise or set-valued) and type of agents' topologies (fixed graphs or switched).

For fixed-graph topologies with precise measurements, several researchers have presented different theoretical results. For example, Ref. [5] designed an average-control law to analyze the consensus of agents for obtaining precise information of the neighbors' states. Ref. [6] presented the necessary and sufficient conditions of mean square average consensus with measurement noises. Ref. [7] proposed a distributed approach for solving convex feasibility problem, and all the states of the agents can attain a common asymptotic point if the associated directed graph is precisely connected using this proposed method. Furthermore, under fixed heterogeneous networks, Ref. [8] solved the state consensus problems of double-integrator MAS based on edge-event-triggered control. However, in many practical cases, the outputs of a system cannot be determined precisely. For example, the binary-valued temperature sensor can only determine whether the temperature exceeds a certain threshold or not. Such information is defined as set-valued information [9]. Thus, many novel studies have been done in the field of set-valued observations. For example, Ref. [10] has studied the problem of tracking control and parameter identification with quantized ARMAX (the stochastic autoregressive moving-average model with auxiliary input) systems. In consideration of a fixed graph with set-valued information, a consensus algorithm with

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an empirical measuring method was proposed and some good results were obtained [11–13]. The recursive projection identification (RPI) algorithm [14, 15] was initially used to design the control law [16].

In actual complex networks, the topologies of the MAS are usually changing; on the contrary, many external factors, such as communication cutoff, transmission delay, and lack of input, can result in the variety of the topologies; thus, there is a need to investigate the switched MAS. For precise measurements, the novel Krasovskii-LaSalle theorem for switched networks was proposed to study the consensus of discrete MAS [17]. Also, Ref. [18] presented the coordination consensus using nearest neighbors' precise states considering the jointly connected graphs. When accurate information could not be obtained, Ref. [19] demonstrated that average convergence at an exponential rate under the condition of bounded switching time can be achieved with an MAS with five quantizers by selecting the appropriate control gain and proportional function. Ref. [20] studied the distributed average consensus of MAS with directed time-variant topologies and quantized communication. Ref. [21] studied the quantization consensus of an MAS with directed switching topologies and proposed an effective distribution protocol.

As discussed above, there are no less than three quantizers in the observation system in the study of the consensus problem of MAS with switching topologies and quantized communication. However, those methods are no longer suitable in systems with two quantizers. Therefore, the phenomenon of switched MAS with binary-valued observations is still unclear and demands further research. Consequently, this paper studies a simple case that consists of switching undirected graphs with binary-valued information and each graph emerges at a non-zero probability. Because of binary-valued communications between agents, meaning each agent can only get limited information from its neighbors, to design the control law, we first estimated all the neighbors' states using the binary-valued information. Many binary-valued identification algorithms have been successfully applied in the study of binary-valued MAS with fixed topologies; for example, an empirical estimation method was used in [12], and an estimation method based on recursive projection algorithm was applied in [15]. The former is an off-line estimation that requires a waiting time to collect enough data, and the latter is an on-line estimation that can be conducted as soon as one piece of data is obtained. In this paper, for the estimation of systems with random switching, we could not afford a waiting time to collect the required data; as a result, the empirical estimation method was not utilized.

Thus, we applied the RPI algorithm to design the two-step control law. First, we designed the estimation update algorithm for neighbors' states based on the RPI algorithm; then, we constructed a random approximation control law based on the estimations.

The main contributions are as follows.

(1) A two-step control law based on the RPI algorithm was designed. Unlike the binary-valued MAS with fixed topology, the binary-valued communication relationship between the agents may be time-variant in the case of switching topology. Thus, the corresponding adjacent and degree matrices are not constant matrices but random ones making the designation of the estimation and updating algorithms more complicated. Similarly, the randomness of the matrices makes it difficult to interpret some of their properties and derive the Lyapunov function. Different from the quantized consensus with switched topologies, the goal of this paper is to design a control law using the binary-valued observations with switched topologies. It is the binary-valued observations with switched topologies that results in very little obtainable information and brings more difficulties in designing the control law.

(2) In the closed-loop system, the corresponding Lyapunov functions collectively analyzed for the estimation and control were strongly integrated. It was proven that the estimations can converge to the true states and the agents can be consistent. Additionally, it was also proven that both the estimation convergence rate and the agent's consensus rate can reach $O(1/t)$. Besides, the results also imply that the larger the value of the lowest probability that a graph emerges with, the more easily the consensus could be achieved.

The remaining sections of this paper are organized as follows: the problem formulation is presented in Section 2; the control law and consensus algorithm are presented in Section 3; the main results are discussed in Section 4; the simulation is demonstrated in Section 5; while the conclusion and future work are presented in Section 6.

2 Problem formulation

Consider the following MAS with n agents at time t:

$$x_i(t+1) = x_i(t) + u_i(t), \quad i = 1, \ldots, n,$$

where $x_i(t) \in \mathbb{R}$ is the state of agent i, $u_i(t) \in \mathbb{R}$ is the control input of agent i. The above MAS can be written as vector form:

$$x(t+1) = x(t) + u(t), \quad i = 1, \ldots, n,$$

where $x(t) = [x_1(t), \ldots, x_n(t)]'$, $u(t) = [u_1(t), \ldots, u_n(t)]'$.

Let $G_{m(t)}$ denote the topology of the n agents at time t, $G_{m(t)} \in \{G_1, \ldots, G_h\}$. Let $G_{m(t)} = (N^{m(t)}, E^{m(t)})$, where $N^{m(t)} \equiv \{1, \ldots, n\}$ denote the set of agents and $E^{m(t)} (\in \{E_1, \ldots, E_h\})$ denote the ordered edges set of the topology $G_{m(t)}$. Let $N_i^{m(t)}$ ($m(t) \in \{1, \ldots, h\}$) denote the neighbors set of agent i in the topology $G_{m(t)}$. $j \in N_i^{m(t)}$ means agent j is the neighbor of agent i, and the edge $(i, j) \in E^{m(t)}$. In turn, agent i is the neighbor of agent j, and the edge $(j, i) \in E^{m(t)}$. It is worth mentioning that the edges (i, j) and (j, i) are not the same edge. Let $A_{m(t)}$ and $D_{m(t)}$ denote the adjacency matrix and the degree matrix of the n agents, respectively, where $D_{m(t)} = \{d_1^{m(t)}, \ldots, d_n^{m(t)}\}$ and $d_i^{m(t)}$ is the degree of agent i. Then the Laplace matrix of $G_{m(t)}$ is $L_{m(t)} = D_{m(t)} - A_{m(t)}$.

The agent i receives the binary-valued information from its neighbor j and there are noises during the communicating process.

$$\begin{cases} y_{ij}(t) = x_j(t) + e_{ij}(t), \\ s_{ij}(t) = I_{\{y_{ij}(t) \leqslant C\}}, \end{cases}$$

where $j \in N_i^{m(t)}$, $x_j(t)$ is the state of agent j at time t, $e_{ij}(t)$ is the communicating noise, $y_{ij}(t)$ is the hidden output, C is the threshold value, $I_{\{\cdot\}}$ is the indicative function, $s_{ij}(t)$ is the binary-valued information that agent i collects from its neighbor j.

Assumption 1. $\{G_1, \ldots, G_h\}$ are jointly connected and G_γ emerges at time t with a probability $p_\gamma > 0$, where $\gamma \in \{1, \ldots, h\}$.

Assumption 2. The noise $e_{ij}(t)$ is independent identically normally distributed as $N(\mu, \delta^2)$ for i, j, t, where $F(\cdot)$ and $f(\cdot)$ are distribution function and density function of $e_{ij}(t)$, respectively.

Assumption 3. $L_{m(t)}$ and $e_{ij}(t)$ are independent. For $t \neq l$, $L_{m(t)}$ and $L_{m(l)}$ are independent; $e_{ij}(t)$ and $e_{ij}(l)$ are independent.

3 Algorithm design

The goal of this paper is to estimate the neighbors' states of each agent based on collected information and design control law to reach consensus. The consensus algorithm and control law will be given as follows.

(i) Initiation: $x_i(1) = x_i^0$ is the initial state of agent i. $\hat{x}_{ij}(0) = x_{ij}^0$ is the initial value of the agent j's state estimated by agent i. $|x_i^0| \leqslant M$, $|x_i^0| \leqslant M$, where $M > 0$ is a given constant.

(ii) Observation: the agent i observes its neighbor agent j's binary-valued information,

$$\begin{cases} y_{ij}(t) = x_j(t) + e_{ij}(t), \\ s_{ij}(t) = I_{\{y_{ij}(t) \leqslant C\}}, \end{cases} \tag{1}$$

where $j \in N_i^{m(t)}$, $i = 1, \ldots, n$ and $m(t) = 1, \ldots, h$.

(iii) Estimation: the agent i estimates its neighbor j's state at time t,

$$\hat{x}_{ij}(t) = \pi_M \left\{ \hat{x}_{ij}(t-1) + \frac{\beta}{t} \left(F\left(C - \hat{x}_{ij}(t-1)\right) - s_{ij}(t) \right) \right\}, \tag{2}$$

where β is the updating step-size and $\pi_M\{\cdot\}$ is projection operator defined as below:

$$\pi_M\{x\} = \mathop{\arg\min}_{|\zeta|\leqslant M} \quad |x-\zeta| = \begin{cases} -M, & x < -M; \\ x, & |x| \leqslant M; \\ M, & x > M. \end{cases}$$

(iv) Update: the agent i estimates its neighbor j's state and designs the control law to update its own state based on the estimations,

$$x_i(t+1) = x_i(t) - \frac{1}{(t+1)d_{\max}} \sum_{j\in N_i^{m(t)}} \left(x_i(t) - \hat{x}_{ij}(t)\right),$$

where $d_{\max} = \max\{d_i^{m(t)}, i = 1,\ldots,n, m(t) = 1,\ldots,h\}$, and $d_i^{m(t)}$ means the agent has $d_i^{m(t)}$ neighbors.

(v) Repetition: $t = t+1$.

For convenience, defining the above estimation and update as vector form, some matrices will be defined first.

Firstly, we rewrite the estimations into vector form. The undirected graphs $G_1, G_2, \ldots, G_h$ constitute a jointly connected graph denoted as G and $G = (N, E)$, where $N = \{1, 2, \ldots, n\}$ is the set of agents, E is the set of G's edges. We consider the agent i in jointly connected graph G: assume there are d_i neighbors of agent i and $\hat{x}_{ij}(t)$ is the estimation of agent j estimated by agent i, where $j \in N_i$, N_i is the set of neighbors of agent i, $i = 1, 2, \ldots, n$. Hence, the estimation vector is

$$\begin{aligned}\hat{x}(t) = [&\hat{x}_{1r_1}(t), \ldots, \hat{x}_{1r_{d_1}}(t), \ldots, \hat{x}_{ir_{d_1+\cdots+d_{i-1}+1}}(t), \ldots, \hat{x}_{ir_{d_1+\cdots+d_i}}(t),\\ &\ldots, \hat{x}_{nr_{d_1+\cdots+d_{n-1}+1}}(t), \ldots, \hat{x}_{nr_{d_1+\cdots+d_n}}(t)]',\end{aligned}$$

where '$'$' means transposition, $r_{d_1+\cdots+d_{i-1}+1}, \ldots, r_{d_1+\cdots+d_i} \in N_i$, $i = 1, 2, \ldots, n$, and assume that $\sum_{i=1}^n d_i = m$.

Matrix $P_{m(t)}$: let $G_{m(t)}$ denote the MAS's topology at time t. Without loss of generality, assume the edge between agent i and its neighbor j is consistent with $r_{d_1+\cdots+d_i} \in N_i$ described previously. Thus, $P_{m(t)}(d_1 + \cdots + d_i) = [0, \ldots, 0, 1, 0, \ldots, 0] \in \mathbb{R}^{1\times m}$, the element of which in the $(d_1 + \cdots + d_i)$th position is 1 when $(i, j) \in E_{m(t)}$, and $P_{m(t)}(d_1 + \cdots + d_i) = [0, \ldots, 0]$ when $(i, j) \notin E_{m(t)}$. Then $P_{m(t)} = [P'_{m(t)}(1), \ldots, P'_{m(t)}(m)]'$ and $\sum_{\gamma=1}^h P_\gamma \geqslant I_{m\times m}$, where P_γ presents the corresponding $P_{m(t)}$ of G_γ at time t. The $P_{m(t)}$ ensures that $\hat{x}_{ij}(t)$ can be updated with obtaining new information $s_{ij}(t)$, and let $\hat{x}_{ij}(t) = \hat{x}_{ij}(t-1)$ otherwise.

Matrix Q: similarly, by continually using the assumption of the edge (i, j) in the introduction of matrix $P_{m(t)}$, $Q(d_1 + \cdots + d_i) = [0, \ldots, 0, 1, 0, \ldots, 0] \in \mathbb{R}^{1\times n}$ (the element in the jth position is 1) and $Q = [Q'(1), \ldots, Q'(m)]'$.

Matrix $W_{m(t)}$: in the graph $G_{m(t)}$, there are $d_i^{m(t)}$ neighbors of agent i, which are in $\{r_{d_1+\cdots+d_{i-1}+1}, \ldots, r_{d_1+\cdots+d_i}\} \in N_i$. Without loss of generality, we extract the first $d_i^{m(t)}$ agents $\in N_i$. Then $W_{m(t)}(i) = [0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0] \in \mathbb{R}^{1\times m}$ in which there are $d_i^{m(t)}$ elements that are 1 and the corresponding positions are $d_1 + \cdots + d_{i-1} + 1, \ldots, d_1 + \cdots + d_{i-1} + d_i^{m(t)}$; $W_{m(t)}(i) = [0, \ldots, 0] \in \mathbb{R}^{1\times m}$ with the agent i no neighbors. Thus, we get $W_{m(t)} = [W'_{m(t)}(1), \ldots, W'_{m(t)}(n)]' \in \mathbb{R}^{n\times m}$.

After matrices have been defined, the estimations and updates are given as follows:

(1) Estimation:

$$\hat{x}(t) = \prod_M \left\{\hat{x}(t-1) + P_{m(t)}\frac{\beta}{t}(F(\mathbb{C} - \hat{x}(t-1)) - s(t))\right\}, \tag{3}$$

where $\prod_M\{\cdot\}$ is the m-dimensional projection operator vector with each dimension denoted as $\pi_M\{\cdot\}$, $\mathbb{C} = C\mathbb{1}_{m\times 1}$ and

$$\begin{aligned}s(t) = [&s_{1r_1}(t), \ldots, s_{1r_{d_1}}(t), \ldots, s_{ir_{d_1+\cdots+d_{i-1}+1}}(t), \ldots, s_{ir_{d_1+\cdots+d_i}}(t),\\ &\ldots, s_{nr_{d_1+\cdots+d_{n-1}+1}}(t), \ldots, s_{nr_{d_1+\cdots+d_n}}(t)]'.\end{aligned}$$

If $d_1 + \cdots + d_i = j$, then $s_{ir_{d_1+\cdots+d_i}}$ presents that the agent i can receive binary-valued information from neighbor j. $s_{ir_{d_1+\cdots+d_i}}$ updates itself as the new information received lately if the agent i can obtain new data at time t; otherwise, $s_{ir_{d_1+\cdots+d_i}}$ should remain the old value as it was at time $t-1$.

(2) Update:

$$x(t+1) = x(t) - \frac{1}{(t+1)d_{\max}} L_{m(t)} x(t) + \frac{1}{(t+1)d_{\max}} W_{m(t)} \varepsilon(t), \tag{4}$$

where $\varepsilon(t) = \hat{x}(t) - Qx(t)$,

$$\begin{aligned} \varepsilon(t) = [&\varepsilon_{1r_1}(t), \ldots, \varepsilon_{1r_{d_1}}(t), \ldots, \varepsilon_{ir_{d_1+\cdots+d_{i-1}+1}}(t), \ldots, \varepsilon_{ir_{d_1+\cdots+d_i}}(t), \\ &\ldots, \varepsilon_{nr_{d_1+\cdots+d_{n-1}+1}}(t), \ldots, \varepsilon_{nr_{d_1+\cdots+d_n}}(t)]'. \end{aligned}$$

If $d_1 + \cdots + d_i = j$, then $\varepsilon_{ir_{d_1+\cdots+d_i}}(t) = \hat{x}_{ir_{d_1+\cdots+d_i}}(t) - x_j(t)$.

4 Main results

4.1 Two convergence indexes

Let $\mathbb{E}[\cdot]$ denote the expectation. The weak consensus ($\mathbb{E}[|x_i(t) - x_j(t)|^2] \to 0$) of agents can be described by

$$V(t) = \mathbb{E}\left[x'(t) L_{m(t)} x(t)\right] = \mathbb{E}\left[x'(t) \left(\sum_{\gamma=1}^{h} p_\gamma L_\gamma\right) x(t)\right]. \tag{5}$$

The square error $R(t)$ of estimations is defined as follows:

$$R(t) = \mathbb{E}[\varepsilon'(t)\varepsilon(t)]. \tag{6}$$

We will see later that the two convergence indexes are Lyapunov functions depending on each other and we will use this quality to discuss the consensus problem of the MAS.

4.2 Theory preparation

Corollary 1. Under Assumption 1, the Laplace matrix $\sum_{\gamma=1}^{h} L_\gamma$ is a nonnegative definite matirx with rank $n-1$.

Lemma 1 ([22], Theorem 7.2.2)**.** Assume A, $B \in \mathbb{R}^{n\times n}$ are Hermitian matrices with the eigenvalues ascending as below:

$$\begin{gathered} \lambda_1(A) \leqslant \lambda_2(A) \leqslant \cdots \leqslant \lambda_n(A), \\ \lambda_1(B) \leqslant \lambda_2(B) \leqslant \cdots \leqslant \lambda_n(B), \\ \lambda_1(A+B) \leqslant \lambda_2(A+B) \leqslant \cdots \leqslant \lambda_n(A+B); \end{gathered}$$

then

$$\lambda_i(A+B) \geqslant \begin{cases} \lambda_i(A) + \lambda_1(B), \\ \lambda_{i-1}(A) + \lambda_2(B), \\ \qquad\vdots \\ \lambda_1(A) + \lambda_i(B). \end{cases}$$

Lemma 2. Under Assumption 1, the matrix $\sum_{\gamma=1}^{h} p_\gamma L_\gamma$ is a nonnegative definite matrix with rank $n-1$.

Proof. Let $p_{\min} = \min\{p_1, \ldots, p_h\}$, $\Delta p_\gamma = p_\gamma - p_{\min} \geqslant 0$, $\gamma \in \{1, \ldots, h\}$; then

$$\sum_{\gamma=1}^{h} p_\gamma L_\gamma = p_{\min} \sum_{\gamma=1}^{h} L_\gamma + \sum_{\gamma=1}^{h} \Delta p_\gamma L_\gamma.$$

It is obvious that $\lambda(p_{\min} \sum_{\gamma=1}^{h} L_\gamma) \geqslant 0$ by Lemma 1. Because L_γ is a Laplace matrix and is positive and semi-definite, the matrix $\sum_{\gamma=1}^{h} \Delta p_\gamma L_\gamma$ is also a positive semi-definite matrix, meaning $\lambda(\sum_{\gamma=1}^{h} \Delta p_\gamma L_\gamma) \geqslant 0$. By Lemma 1, we can get that $\lambda(\sum_{\gamma=1}^{h} p_\gamma L_\gamma) \geqslant 0$ and no more than one eigenvalue is 0. For each L_γ that is a Laplace matrix with an eigenvector being $\mathbb{1}_n$, there must be one 0-eigenvalue of $\sum_{\gamma=1}^{h} p_\gamma L_\gamma$. The lemma can be obtained.

Corollary 2. Under Assumption 1, let λ_n denote the largest eigenvalue and λ_2 denote the smallest positive one of $\sum_{\gamma=1}^{h} p_\gamma L_\gamma$. Set $c = \frac{\lambda_2^2}{\lambda_n}$; then $(\sum_{\gamma=1}^{h} p_\gamma L_\gamma)^2 \geqslant c \sum_{\gamma=1}^{h} p_\gamma L_\gamma$.

Proof. The proof is similar to Theorem 5 in [23].

Lemma 3 ([13], Lemma 3)**.** For arbitrary $\nu \in \mathbb{R}$,

(i)

$$\prod_{i=1}^{k}\left(1-\frac{\nu}{i}\right) = O\left(\frac{1}{k^\nu}\right), \quad k \to \infty.$$

(ii) For arbitrary $\delta > 0$,

$$\sum_{l=1}^{k}\prod_{i=l+1}^{k}\left(1-\frac{\nu}{i}\right)\frac{1}{l^{1+\delta}} = \begin{cases} O\left(\dfrac{1}{k^\delta}\right), & 0<\delta<\nu; \\ O\left(\dfrac{\ln k}{k^\nu}\right), & \delta=\nu; \\ O\left(\dfrac{1}{k^\nu}\right), & \delta>\nu. \end{cases}$$

Lemma 4. The agent state $x_i(t)$ and the estimation $\hat{x}_{ij}(t)$ are all bounded, meaning that there exists a bounded positive constant M such that $|x_i(t)| \leqslant M$ and $|\hat{x}_{ij}(t)| \leqslant M$, where $i = 1, 2, \ldots, n$, $j \in N_i^{m(t)}$.

Proof. Let $\Omega = \{x | x \in \mathbb{R}, |x| \leqslant M\}$. Then $x_i(0) \in \Omega$, $\hat{x}_i(0) \in \Omega$. Owing to the definition of the projection operator, we can get $|\hat{x}_{ij}(t)| \leqslant M$, meaning $\hat{x}_{ij}(t) \in \Omega$.

(1) When there is no neighbor of the agent i at time t, meaning $d_i^{m(t)} = 0$, the state of agent i does not update, meaning $x_i(t+1) = x_i(t)$.

(2) When there exists a neighbor of the agent i, meaning $d_i^{m(t)} > 0$, we have

$$\begin{aligned} x_i(t+1) &= x_i(t) - \frac{1}{(t+1)d_{\max}}\sum_{j\in N_i^{m(t)}}(x_i(t) - \hat{x}_{ij}(t)) \\ &= \left(1 - \frac{d_i^{m(t)}}{(t+1)d_{\max}}\right)x_i(t) + \frac{1}{(t+1)d_{\max}}\sum_{j\in N_i^{m(t)}}\hat{x}_{ij}(t) \\ &= \left(1 - \frac{d_i^{m(t)}}{(t+1)d_{\max}}\right)x_i(t) + \frac{d_i^{m(t)}}{(t+1)d_{\max}}\sum_{j\in N_i^{m(t)}}\frac{1}{d_i^{m(t)}}\hat{x}_{ij}(t). \end{aligned}$$

For $\sum_{j\in N_i^{m(t)}} \frac{1}{d_i^{m(t)}} = 1$ and $\frac{1}{d_i^{m(t)}} > 0$, we have $\sum_{j\in N_i^{m(t)}} \frac{1}{d_i^{m(t)}}\hat{x}_{ij}(t) \in \Omega$; for $0 < \frac{d_i^{m(t)}}{(t+1)d_{\max}} < 1$ and $(1 - \frac{d_i^{m(t)}}{(t+1)d_{\max}}) + \frac{d_i^{m(t)}}{(t+1)d_{\max}} = 1$, we have $x_i(t+1) \in \Omega$. Both cases indicate that $|x_i(t+1)| \leqslant M$. Combining the items (1) and (2), $x_i(t+1) \in \Omega$ for arbitrary $t > 0$.

Lemma 5 ([24], the remark in Subsection 2.6.2)**.** Let X be a variable and $\boldsymbol{\nu}$ be a σ-algebra; thus

$$\mathbb{E}[\mathbb{E}[X|\boldsymbol{\nu}]] = \mathbb{E}[X].$$

Lemma 6. The recursive inequality of Lyapunov function $V(t)$ is

$$V(t+1) \leqslant \left(1 - \frac{3c/2}{(t+1)d_{\max}}\right)V(t) + \frac{2\lambda_W/c}{(t+1)d_{\max}}R(t) + \frac{B}{(t+1)^2}, \tag{7}$$

where $B = B_1 + B_2 + B_3$, $0 < B_i < \infty$, $i = 1, \ldots, h$, $c = \frac{\lambda_2^2}{\lambda_n}$, λ_n is the largest eigenvalue of $\sum_{\gamma=1}^{h} p_\gamma L_\gamma$, and λ_2 is the smallest positive eigenvalue of $\sum_{\gamma=1}^{h} p_\gamma L_\gamma$, $\lambda_W = \max\left\{\lambda_{\max}\{W_1' \check{L} W_1\}, \ldots, \lambda_{\max}\{W_h' \check{L} W_h\}\right\}$.

Proof. See the proof in Appendix A.

Lemma 7. The recursive inequality of Lyapunov function $R(t)$ is

$$R(t) \leqslant \left(1 - \frac{2p_{\min}\beta f_M d_{\max} - \frac{\lambda_{QL}}{\alpha} - 2\sqrt{\lambda_Q \lambda_{\check{W}}}}{t d_{\max}}\right)R(t-1) + \frac{\alpha}{t d_{\max}}V(t-1) + \frac{\tilde{B}}{t^2}, \tag{8}$$

where $\tilde{B} = B_4 + B_5$, $0 < B_i < \infty$, $i = 1, 2$, $p_{\min} = \min\{p_1, \ldots, p_h\}$, $f_M = f(|C| + M)$, β is the constant given in (2), $\lambda_{QL} = \max\{\lambda_{\max}\{QL_1Q'\}, \ldots, \lambda_{\max}\{QL_hQ'\}\}$, $\lambda_Q = \lambda_{\max}\{QQ'\}$, $0 < \alpha < \infty$, $\lambda_{\breve{W}} = \max\{\lambda_{\max}\{W_1'W_1\}, \ldots, \lambda_{\max}\{W_h'W_h\}\}$.

Proof. See proof in Appendix B.

4.3 The conditions of consensus and consensus speed of $O(\frac{1}{t})$

Put the two Lyapunov functions into a vector and matrix form:

$$\begin{cases} V(t) \leqslant \left(1 - \dfrac{3c/2}{td_{\max}}\right) V(t-1) + \dfrac{2\lambda_W/c}{td_{\max}} R(t-1) + \dfrac{B}{t^2}, \\ R(t) \leqslant \left(1 - \dfrac{2p_{\min}\beta f_M d_{\max} - \dfrac{\lambda_{QL}}{\alpha} - 2\sqrt{\lambda_Q \lambda_{\breve{W}}}}{td_{\max}}\right) R(t-1) + \dfrac{\alpha}{td_{\max}} V(t-1) + \dfrac{\tilde{B}}{t^2}. \end{cases} \tag{9}$$

Let

$$U = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}, \quad Z(t) = \begin{pmatrix} V(t) \\ R(t) \end{pmatrix}, \quad H = \begin{pmatrix} B \\ \tilde{B} \end{pmatrix},$$

where $u_1 = \frac{3c}{2d_{\max}}$, $u_2 = -\frac{2\lambda_W}{cd_{\max}}$, $u_3 = -\frac{\alpha}{d_{\max}}$, $u_4 = \frac{2p_{\min}\beta f_M d_{\max} - \frac{\lambda_{QL}}{\alpha} - 2\sqrt{\lambda_Q \lambda_{\breve{W}}}}{d_{\max}}$. Then by (9) we obtain

$$||Z(t)|| \leqslant \left\| \left(I - \frac{1}{t} U\right) Z(t-1) \right\| + \frac{1}{t^2} ||H||, \tag{10}$$

and we have the following theorem.

Theorem 1. Under Assumptions 1–3, we have

$$||Z(t)|| = \begin{cases} O\left(\dfrac{1}{t^{\lambda_{\min}(U)}}\right), & \lambda_{\min}(U) < 1; \\ O\left(\dfrac{\ln t}{t}\right), & \lambda_{\min}(U) = 1; \\ O\left(\dfrac{1}{t}\right), & \lambda_{\min}(U) > 1. \end{cases}$$

Proof. Let $u_2 = u_3$; then the matrix U is symmetry. By $c = \frac{\lambda_2^2}{\lambda_n}$, we get $\alpha = \frac{2\lambda_n \lambda_W}{\lambda_2^2}$. Because U is symmetry,

$$\lambda\left(I - \frac{2U}{t} + \frac{U^2}{t^2}\right) = \left(1 - \frac{\lambda(U)}{t}\right)^2 \leqslant \left(1 - \frac{\lambda_{\min}(U)}{t}\right)^2.$$

Thus

$$\left\| I - \frac{U}{t} \right\| = \sqrt{\lambda_{\max}\left(\left(I - \frac{U}{t}\right)^2\right)} = \sqrt{\lambda_{\max}\left(I - \frac{2U}{t} + \frac{U^2}{t^2}\right)} \leqslant 1 - \frac{\lambda_{\min}(U)}{t}.$$

Hence

$$\begin{aligned} ||Z(t)|| &\leqslant \left(I - \frac{\lambda_{\min}(U)}{t}\right) ||Z(t-1)|| + \frac{1}{t^2} ||H|| \\ &\leqslant \prod_{i=\lceil \lambda_{\max}(U) \rceil + 1}^{t} \left(I - \frac{\lambda_{\min}(U)}{i}\right) ||Z(\lceil \lambda_{\max}(U) \rceil)|| + \sum_{i=\lceil \lambda_{\max}(U) \rceil + 1}^{t} \prod_{l=i+1}^{t} \left(I - \frac{\lambda_{\min}(U)}{l}\right) \frac{||H||}{i^2}, \end{aligned}$$

where $\lceil \cdot \rceil$ means the biggest integer which is no more than '$\cdot$'. Finally, we can obtain the theorem by Lemma 3.

Remark 1. As $V(t) \leqslant ||Z(t)||$ and $R(t) \leqslant ||Z(t)||$, $V(t)$ and $R(t)$ have the order no more than that of $||Z(t)||$, where the vector $Z(t)$ and the matrix U are described as above.

Theorem 2. The binary-valued switched MAS reaches weak consensus, and the estimations of neighbors converge to the real states, for $\beta > \frac{\kappa_1}{2p_{\min}f_M d_{\max}}$, where

$$\kappa_1 = \frac{8\lambda_n^3\lambda_W^2}{3\lambda_2^6} + \frac{\lambda_2^2\lambda_{QL}}{2\lambda_n\lambda_W} + 2\sqrt{\lambda_Q\lambda_{\breve{W}}},$$

and $p_{\min}$, λ_Q, λ_{QL}, λ_W, $\lambda_{\breve{W}}$ are defined in Lemma 6 or Lemma 7, λ_2, λ_n are defined in Lemma 2.

Proof. Let $|\lambda I - U| = (\lambda - u_1)(\lambda - u_4) - u_2^2 = 0$; then

$$\lambda_{\min}(U) = \frac{u_1 + u_4 - \sqrt{(u_1+u_4)^2 - 4(u_1u_4 - u_2^2)}}{2}.$$

If $\beta > \frac{\kappa_1}{2p_{\min}f_M d_{\max}}$, then $u_1u_4 > u_2^2$. For $u_1 > 0$, $u_4 > 0$, then $u_1 + u_4 > 0$. Thus the smallest eigenvalue of matrix U follows:

$$\lambda_{\min}(U) = \frac{u_1 + u_4 - \sqrt{(u_1+u_4)^2 - 4(u_1u_4 - u_2^2)}}{2} > 0.$$

By Theorem 1 and Remark 1, we can obtain

$$\mathbb{E}[|x_i(t) - x_j(t)|^2] \to 0, \quad \forall i, j = 1, 2, \ldots, n, i \neq j,$$

and

$$\mathbb{E}[|\hat{x}_{ij}(t) - x_j(t)|^2] \to 0, \quad j \in N_i, i = 1, 2, \ldots, n.$$

Theorem 3. The binary-valued switched MAS reaches weak consensus at a speed of $O(\frac{1}{t})$, and the estimations of neighbors converge to the real states at a speed of $O(\frac{1}{t})$, for $u_1 = \frac{3\lambda_2^2}{2\lambda_n} > 1$ and $\beta > \frac{\kappa_2}{2p_{\min}f_M d_{\max}}$, where

$$\kappa_2 = \frac{8\lambda_n^3\lambda_W^2}{\lambda_2^4(3\lambda_2^2 - 2\lambda_n d_{\max})} + \frac{\lambda_2^2\lambda_{QL}}{2\lambda_n\lambda_W} + 2\sqrt{\lambda_Q\lambda_{\breve{W}}} + d_{\max},$$

and $p_{\min}$, λ_Q, λ_{QL}, λ_W, $\lambda_{\breve{W}}$ are defined in Lemma 6 or Lemma 7, λ_2, λ_n are defined in Lemma 2.

Proof. If $\beta > \frac{\kappa_2}{2p_{\min}f_M d_{\max}}$, then

$$\begin{aligned} u_4 &= 2p_{\min}\beta f_M d_{\max} - \frac{\lambda_{QL}}{\alpha} - 2\sqrt{\lambda_Q\lambda_{\breve{W}}} \\ &> \frac{8\lambda_n^3\lambda_W^2}{\lambda_2^4(3\lambda_2^2 - 2\lambda_n)} + 1 \\ &= \frac{u_2^2}{u_1 - 1} + 1. \end{aligned}$$

If $u_1 > 1$, then $u_4(u_1 - 1) > u_2^2 + (u_1 - 1)$. Consequently,

$$(u_1+u_4)^2 - 4(u_1u_4 - u_2^2) < (u_1 + u_4 - 2)^2.$$

For $u_1 > 1$ and $u_4 > 1$, then $u_1 + u_4 - 2 > 0$, and

$$\sqrt{(u_1+u_4)^2 - 4(u_1u_4 - u_2^2)} < u_1 + u_4 - 2.$$

Hence, the smallest eigenvalue of matrix U follows

$$\lambda_{\min}(U) = \frac{u_1 + u_4 - \sqrt{(u_1+u_4)^2 - 4(u_1u_4 - u_2^2)}}{2} > 1.$$

By Theorem 1 and Remark 1, we can obtain Theorem 3.

Remark 2. Theorems 2 and 3 mean that the larger the $p_{\min}$, the smaller the corresponding $\frac{\kappa_1}{2p_{\min}f_M d_{\max}}$ and $\frac{\kappa_2}{2p_{\min}f_M d_{\max}}$, and the easier the β is to obtain, which is consistent with perceptual intuition: when the value of the lowest probability that a graph emerges with is not too small, the information among agents can be transmitted in time so that the consensus could be reached more easily.

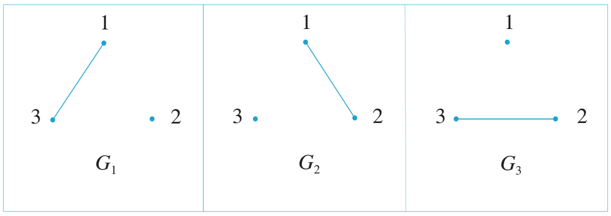

Figure 1 (Color online) A switched MAS with three agents.

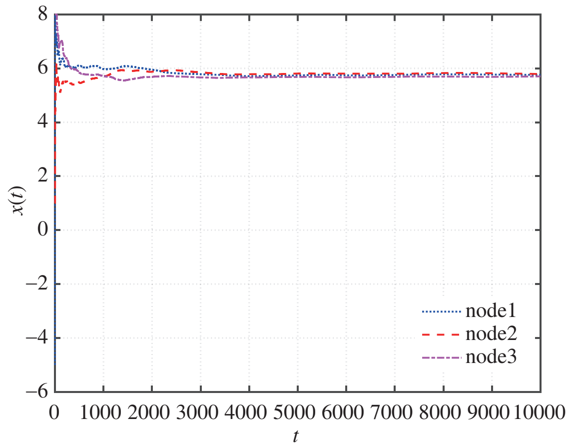

Figure 2 (Color online) Updates of agents' states.

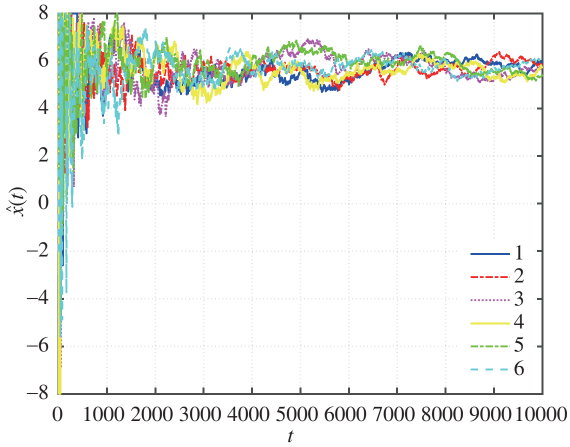

Figure 3 (Color online) Estimations of the neighbors.

5 Simulation

This part will give a simulation with a three-agent system and there are three switched graphs, which are depicted in Figure 1.

Set the initial states as $x(0) = [-5, 1, 8]'$ and the initial estimations as $\hat{x}(0) = [0, 2, 2, -3, 1, -2]'$. Take G_1 for example and we give the corresponding matrices:

$$Q = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}', \quad L_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad P_1 = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix},$$

where

$$P_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$P_{12} = P_{21} = P_{22} = \mathbf{0}_{3\times3}$. Then $d_{\max} = 1$, $\lambda_2 = 0.9278$, $\lambda_3 = 1.0722$, $\lambda_{QL} = \max\{\lambda_{\max}\{QL_1Q'\}, \lambda_{\max}\{QL_2Q'\}, \lambda_{\max}\{QL_3Q'\}\} = 4$, $\lambda_W = \max\{\lambda_{\max}\{W_1'\check{L}W_1\}, \lambda_{\max}\{W_2'\check{L}W_2\}, \lambda_{\max}\{W_3'\check{L}W_3\}\} = 1.0631$, $p_{\min} = p_1 = \frac{7}{24}$, $p_2 = \frac{1}{3}$, $p_{\max} = p_3 = \frac{3}{8}$, $\lambda_{\check{W}} = \max\{\lambda_{\max}\{W_1'W_1\}, \lambda_{\max}\{W_2'W_2\}, \lambda_{\max}\{W_3'W_3\}\} = 1$, $\lambda_Q = 2$. Let $M = 8$, $\mu = 0$ and $\delta = 6$ of the noises; then $f_M = 0.0273$. Thus $\kappa_1 = 10.1628$, and the consensus can be obtained when $\beta > 639$ by Theorem 2. Compute $\kappa_2 = 39.6737$ and $u_1 = 1.2043 > 1$, and the consensus speed of $O(\frac{1}{t})$ can be reached with $\beta > 2492$ by Theorem 3. Finally, we let $\beta = 2500$ and the following are the simulation results.

Figure 2 depicts the updating process of three agents and we can see that the three agents converge to the same state. Figure 3 presents the convergence tendency of estimations of agents' neighbors and we also obtain that the estimations are converge to the same one. Figure 4 demonstrates the trajectory of consensus index $V(t)$, converging to 0. Figure 5 describes the trajectory of the logarithm of $V(t)$ compared with $12 - \log(t)$ and $-6 - \log(t)$, and we see that the three trajectories are almost parallel, meaning that the consensus speed of $O(\frac{1}{t})$ can be obtained when we give β a proper value.

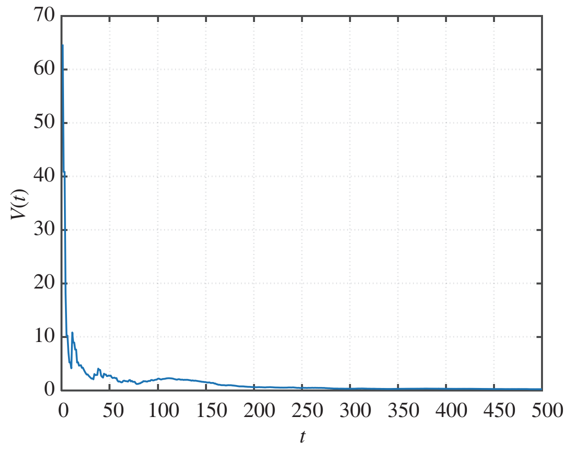

Figure 4 (Color online) The trajectory of $V(t)$.

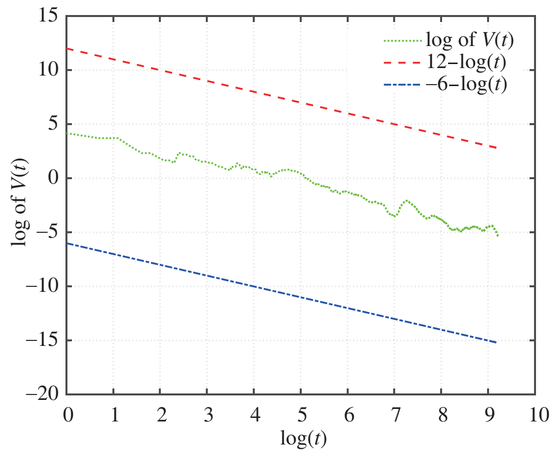

Figure 5 (Color online) The trajectory of the log of $V(t)$.

6 Conclusion

This paper presents preliminary studies on the consensus of random switched MAS based on the RPI algorithm. It is proved that by properly selecting the iteration step β in the RPI algorithm, the estimates can converge to the real states and the agents can achieve consensus. Moreover, both the convergence speed of the estimation and the consensus speed of the agents are proved to be $O(1/t)$. And the simulation result is consistent with the theoretical analysis.

In this work, we assumed that each graph emerges randomly with a positive non-zero probability. The consensus problem of switched binary-valued MAS is a more interesting and challenging topic for future research. We assumed a convergence in the analysis of the agents' final states to study whether the final states converge to or are approximate to the average initial states. In this paper, we considered that each graph switches with a non-zero probability. But is the consensus algorithm and control law still sufficient to make the agents attain consensus when the graphs are under more general switching conditions?

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Appendix A Derivation of $V(t)$

From Lemma 6,

$$\begin{aligned}V(t+1) &= \mathbb{E}\left[x'(t+1)L_{m(t+1)}x(t+1)\right]\\ &= \mathbb{E}\left[x'(t)\left(I-\frac{1}{(t+1)d_{\max}}L_{m(t)}\right)L_{m(t+1)}\left(I-\frac{1}{(t+1)d_{\max}}L_{m(t)}\right)x(t)\right]\\ &\quad+\frac{2}{(t+1)d_{\max}}\mathbb{E}\left[x'(t)\left(I-\frac{1}{(t+1)d_{\max}}L_{m(t)}\right)L_{m(t+1)}W_{m(t)}\varepsilon(t)\right]\\ &\quad+\frac{1}{(t+1)^2d_{\max}^2}\mathbb{E}\left[\varepsilon'(t)W'_{m(t)}L_{m(t+1)}W_{m(t)}\varepsilon(t)\right].\end{aligned}\tag{A1}$$

By Lemma 5, we get the first term on the right of (A1) as follows:

$$\begin{aligned}&\mathbb{E}\left[x'(t)\left(I-\frac{1}{(t+1)d_{\max}}L_{m(t)}\right)L_{m(t+1)}\left(I-\frac{1}{(t+1)d_{\max}}L_{m(t)}\right)x(t)\right]\\ &=\mathbb{E}\left[\mathbb{E}\left[x'(t)\left(I-\frac{1}{(t+1)d_{\max}}L_{m(t)}\right)L_{m(t+1)}\left(I-\frac{1}{(t+1)d_{\max}}L_{m(t)}\right)x(t)\middle|x(t)\right]\right]\\ &=\mathbb{E}\left[x'(t)\mathbb{E}\left[\left(I-\frac{1}{(t+1)d_{\max}}L_{m(t)}\right)L_{m(t+1)}\left(I-\frac{1}{(t+1)d_{\max}}L_{m(t)}\right)\right]x(t)\right]\\ &=\mathbb{E}\left[x'(t)\mathbb{E}\left[\mathbb{E}\left[\left(I-\frac{1}{(t+1)d_{\max}}L_{m(t)}\right)L_{m(t+1)}\left(I-\frac{1}{(t+1)d_{\max}}L_{m(t)}\right)\middle|L_{m(t)}\right]\right]x(t)\right]\\ &=\mathbb{E}\left[x'(t)\mathbb{E}\left[\left(I-\frac{1}{(t+1)d_{\max}}L_{m(t)}\right)(\sum_{\gamma=1}^{h}p_\gamma L_\gamma)\left(I-\frac{1}{(t+1)d_{\max}}L_{m(t)}\right)\right]x(t)\right]\\ &=\mathbb{E}\left[x'(t)\left[(\sum_{\gamma=1}^{h}p_\gamma L_\gamma)-\frac{2}{(t+1)d_{\max}}(\sum_{\gamma=1}^{h}p_\gamma L_\gamma)^2+\frac{1}{(t+1)^2d_{\max}^2}\mathbb{E}\left[L_{m(t)}(\sum_{\gamma=1}^{h}p_\gamma L_\gamma)L_{m(t)}\right]\right]x(t)\right].\end{aligned}$$

Let $L=\sum_{\gamma=1}^{h}p_\gamma L_\gamma(\sum_{\gamma'=1}^{h}p_\gamma L_{\gamma'})L_\gamma$, $\check{L}=\sum_{\gamma=1}^{h}p_\gamma L_\gamma$, and by Lemma 2 we get

$$\begin{aligned}&\mathbb{E}\left[x'(t)\left(I-\frac{1}{(t+1)d_{\max}}L_{m(t)}\right)L_{m(t+1)}\left(I-\frac{1}{(t+1)d_{\max}}L_{m(t)}\right)x(t)\right]\\ &\leqslant\mathbb{E}\left[x'(t)\left\{\left(1-\frac{2c}{(t+1)d_{\max}}\right)\check{L}+\frac{L}{(t+1)^2d_{\max}^2}\right\}x(t)\right]\\ &\leqslant\left(1-\frac{2c}{(t+1)d_{\max}}\right)V(t)+\frac{B_1}{(t+1)^2},\end{aligned}\tag{A2}$$

where $0<B_1<\infty$.

By Lemma 5, we obtain the second term on the right of (A1) as follows:

$$\begin{aligned}&\frac{2}{(t+1)d_{\max}}\mathbb{E}\left[x'(t)\left(I-\frac{1}{(t+1)d_{\max}}L_{m(t)}\right)L_{m(t+1)}W_{m(t)}\varepsilon(t)\right]\\ &=\frac{2}{(t+1)d_{\max}}\mathbb{E}\left\{\mathbb{E}\left[x'(t)\left(I-\frac{1}{(t+1)d_{\max}}L_{m(t)}\right)L_{m(t+1)}W_{m(t)}\varepsilon(t)\right]\middle|x(t),\hat{x}(t),L_{m(t)}\right\}\\ &=\frac{2}{(t+1)d_{\max}}\mathbb{E}\left[x'(t)\left(I-\frac{1}{(t+1)d_{\max}}L_{m(t)}\right)\check{L}W_{m(t)}\varepsilon(t)\right]\\ &=\frac{2}{(t+1)d_{\max}}\mathbb{E}\left[x'(t)\left(I-\frac{1}{(t+1)d_{\max}}L_{m(t)}\right)\tilde{L}'\tilde{L}W_{m(t)}\varepsilon(t)\right]\\ &\leqslant\frac{2}{(t+1)d_{\max}}\sqrt{\mathbb{E}\left[x'(t)\left(I-\frac{1}{(t+1)d_{\max}}L_{m(t)}\right)\tilde{L}'\tilde{L}\left(I-\frac{1}{(t+1)d_{\max}}L_{m(t)}\right)x(t)\right]\mathbb{E}\left[\varepsilon'(t)W'_{m(t)}\check{L}W_{m(t)}\varepsilon(t)\right]}.\end{aligned}$$

For $m(t) \in \{1, \ldots, h\}$, thus we can get the maximum eigenvalue of the matrix $W'_{m(t)}\check{L}W_{m(t)}$, defined as $\lambda_W = \max\{\lambda_{\max}\{W'_1\check{L}W_1\}, \ldots, \lambda_{\max}\{W'_h\check{L}W_h\}\}$. We have

$$\begin{aligned}
&\frac{2}{(t+1)d_{\max}}\mathbb{E}\left[x'(t)\left(I - \frac{1}{(t+1)d_{\max}}L_{m(t)}\right)L_{m(t+1)}W_{m(t)}\varepsilon(t)\right] \\
&\leqslant \frac{2}{(t+1)d_{\max}}\sqrt{\mathbb{E}\left[x'(t)\left(\check{L} - \frac{2c}{(t+1)d_{\max}}\check{L} + \frac{L}{(t+1)^2d_{\max}^2}\right)x(t)\right]\mathbb{E}\left[\varepsilon'(t)W'_{m(t)}\check{L}W_{m(t)}\varepsilon(t)\right]} \\
&\leqslant \frac{2}{(t+1)d_{\max}}\sqrt{\left[\left(1 - \frac{2c}{(t+1)d_{\max}}\right)V(t)\right][\lambda_W R(t)]} + \frac{B_2}{(t+1)^2} \\
&\leqslant \frac{2}{(t+1)d_{\max}}\sqrt{V(t)\lambda_W R(t)} + \frac{B_2}{(t+1)^2} \\
&= \frac{2}{(t+1)d_{\max}}\sqrt{\frac{c}{2}V(t)\cdot\frac{2\lambda_W}{c}R(t)} + \frac{B_2}{(t+1)^2} \\
&\leqslant \frac{1}{(t+1)d_{\max}}\left[\frac{c}{2}V(t) + \frac{2\lambda_W}{c}R(t)\right] + \frac{B_2}{(t+1)^2} \\
&\leqslant \frac{c/2}{(t+1)d_{\max}}V(t) + \frac{2\lambda_W/c}{(t+1)d_{\max}}R(t) + \frac{B_2}{(t+1)^2}, \quad t > \frac{2c}{d_{\max}} - 1 \quad \text{and} \quad t > 0,
\end{aligned} \tag{A3}$$

where $0 < B_2 < \infty$ and c is as defined in Lemma 2.

The third term on the right of (A1) is

$$\frac{1}{(t+1)^2d_{\max}^2}\mathbb{E}\left[\varepsilon'(t)W'_{m(t)}L_{m(t+1)}W_{m(t)}\varepsilon(t)\right] \leqslant \frac{B_3}{(t+1)^2}, \tag{A4}$$

where $0 < B_3 < \infty$.

By (A2)–(A4), the Lyapunov function $V(t)$ can be obtained:

$$V(t+1) \leqslant \left(1 - \frac{3c/2}{(t+1)d_{\max}}\right)V(t) + \frac{2\lambda_W/c}{(t+1)d_{\max}}R(t) + \frac{B}{(t+1)^2}, \tag{A5}$$

where $B = B_1 + B_2 + B_3$.

Appendix B Derivation of $R(t)$

From Lemma 7, by the property ([14], Proposition 6) of the projection operator, we have

$$\begin{aligned}
&R(t) \\
&= \mathbb{E}\left[\varepsilon'(t)\varepsilon(t)\right] \\
&= \mathbb{E}\Bigg[\left[\prod_M\left\{\hat{x}(t-1) + P_{m(t)}\frac{\beta}{t}(F(\mathbb{C} - \hat{x}(t-1)) - s(t))\right\} - Qx(t)\right]' \\
&\quad \cdot \left[\prod_M\{\hat{x}(t-1) + P_{m(t)}\frac{\beta}{t}(F(\mathbb{C} - \hat{x}(t-1)) - s(t))\} - Qx(t)\right]\Bigg] \\
&\leqslant \mathbb{E}\Bigg[\left[\hat{x}(t-1) + P_{m(t)}\frac{\beta}{t}(F(\mathbb{C} - \hat{x}(t-1)) - s(t)) - Qx(t)\right]' \\
&\quad \cdot \left[\hat{x}(t-1) + P_{m(t)}\frac{\beta}{t}(F(\mathbb{C} - \hat{x}(t-1)) - s(t)) - Qx(t)\right]\Bigg] \\
&= \mathbb{E}\Bigg[\left[\varepsilon(t-1) + \frac{\beta}{t}P_{m(t)}(F(\mathbb{C} - \hat{x}(t-1)) - s(t)) + \frac{1}{td_{\max}}Q(L_{m(t-1)}x(t-1) - W_{m(t-1)}\varepsilon(t-1))\right]' \\
&\quad \cdot \left[\varepsilon(t-1) + \frac{\beta}{t}P_{m(t)}(F(\mathbb{C} - \hat{x}(t-1)) - s(t)) + \frac{1}{td_{\max}}Q(L_{m(t-1)}x(t-1) - W_{m(t-1)}\varepsilon(t-1))\right]\Bigg] \\
&= R(t-1) + \frac{2\beta}{t}\mathbb{E}[\varepsilon'(t-1)P_{m(t)}(F(\mathbb{C} - \hat{x}(t-1)) - s(t))] \\
&\quad + \frac{2}{td_{\max}}\mathbb{E}[\varepsilon'(t-1)Q(L_{m(t-1)}x(t-1) - W_{m(t-1)}\varepsilon(t-1))] + \frac{B_4}{t^2},
\end{aligned} \tag{B1}$$

where $0 < B_4 < \infty$.

The second term on the right of (B1) is

$$\begin{aligned}
&\frac{2\beta}{t}\mathbb{E}\left[\varepsilon'(t-1)P_{m(t)}(F(\mathbb{C} - \hat{x}(t-1)) - s(t))\right] \\
&\quad = \frac{2\beta}{t}\mathbb{E}\left[\varepsilon'(t-1)P_{m(t)}F(\mathbb{C} - \hat{x}(t-1))\right] - \frac{2\beta}{t}\mathbb{E}\left[\mathbb{E}\left[\varepsilon'(t-1)P_{m(t)}s(t)|s(1), \ldots, s(t-1), \hat{x}(t-1), x(t-1), L_{m(t)}\right]\right] \\
&\quad = \frac{2\beta}{t}\mathbb{E}\left[\varepsilon'(t-1)P_{m(t)}\left(F(\mathbb{C} - \hat{x}(t-1)) - F(\mathbb{C} - Qx(t))\right)\right]
\end{aligned}$$

$$= \frac{2\beta}{t}\mathbb{E}\left[\mathbb{E}\left[\varepsilon'(t-1)P_{m(t)}\left(F(\mathbb{C}-\hat{x}(t-1))-F(\mathbb{C}-Qx(t))\right)|\hat{x}(t-1),x(t-1),x(t)\right]\right]$$
$$= \frac{2\beta}{t}\mathbb{E}\left[\varepsilon'(t-1)\left(\sum_{\gamma=1}^{h}p_\gamma P_\gamma\right)\left(F(\mathbb{C}-\hat{x}(t-1))-F(\mathbb{C}-Qx(t))\right)\right].$$

Let $\check{P}=\sum_{\gamma=1}^{h}P_\gamma$. For $F(C-\hat{x}_{ij}(t-1))-F(C-x_j(t))$, there exists $\xi_{ij}(t)\in(C-x_j(t),C-\hat{x}_{ij}(t-1))$ or $(C-\hat{x}_{ij}(t-1),C-x_j(t))$, such that

$$F(C-\hat{x}_{ij}(t-1))-F(C-x_j(t))=-f(\xi_{ij}(t))(\hat{x}_{ij}(t-1)-x_j(t)).$$

Let $\xi(t)=(\xi_1(t),\ldots,\xi_r(t),\ldots,\xi_m(t))$. r is the number of the edge (i,j), and then

$$\begin{aligned}
&F(\mathbb{C}-\hat{x}(t-1))-F(\mathbb{C}-Qx(t))\\
&\quad=-\mathrm{diag}(f(\xi(t)))(\hat{x}(t-1)-Qx(t))\\
&\quad=-\mathrm{diag}(f(\xi(t)))\left(\hat{x}(t-1)-Q\left(x(t-1)-\frac{1}{td_{\max}}L_{m(t-1)}x(t-1)+\frac{1}{td_{\max}}W_{m(t-1)}\varepsilon(t-1)\right)\right)\\
&\quad=-\mathrm{diag}(f(\xi(t)))\left(\varepsilon(t-1)+\frac{1}{td_{\max}}Q\left(L_{m(t-1)}x(t-1)-W_{m(t-1)}\varepsilon(t-1)\right)\right),
\end{aligned}$$

where

$$\mathrm{diag}(f(\xi(t)))=\begin{bmatrix} f(\xi_1(t)) & & \mathbf{0}\\ & \ddots & \\ \mathbf{0} & & f(\xi_m(t))\end{bmatrix}.$$

Hence

$$\begin{aligned}
&\frac{2\beta}{t}\mathbb{E}\left[\varepsilon'(t-1)\left(\sum_{\gamma=1}^{h}p_\gamma P_\gamma\right)\left(F(\mathbb{C}-\hat{x}(t-1))-F(\mathbb{C}-Qx(t))\right)\right]\\
&\quad=-\frac{2\beta}{t}\mathbb{E}\left[\varepsilon'(t-1)\left(\sum_{\gamma=1}^{h}p_\gamma P_\gamma\right)\mathrm{diag}\left(f(\xi(t))\right)\varepsilon(t-1)\right]+\frac{B_5}{t^2},
\end{aligned}\tag{B2}$$

where $0<B_5<\infty$. For $|\hat{x}_{ij}(t)|\leqslant M$, $|\hat{x}_i(t)|\leqslant M$, $\mathrm{d}F/\mathrm{d}x=f$; then $f_M\leqslant f(\xi_{ij}(t))\leqslant 1$, where $f_M=f(|C|+M)$, $\mathrm{diag}(f(\xi(t)))\geqslant f_M I_{m\times m}$. And by $\check{P}\geqslant I_{m\times m}$ we have

$$\begin{aligned}
&-\frac{2\beta}{t}\mathbb{E}\left[\varepsilon'(t-1)\left(\sum_{\gamma=1}^{h}p_\gamma P_\gamma\right)\mathrm{diag}(f(\xi(t)))\varepsilon(t-1)\right]\\
&\quad\leqslant-\frac{2\beta}{t}\mathbb{E}\left[\varepsilon'(t-1)p_{\min}I_{m\times m}f_M I_{m\times m}\varepsilon(t-1)\right]\\
&\quad\leqslant-\frac{2p_{\min}\beta f_M}{t}R(t-1).
\end{aligned}\tag{B3}$$

By (B2) and (B3) we can get the second item on the right of (B1) as follows:

$$\frac{2\beta}{t}\mathbb{E}\left[\varepsilon'(t-1)P_{m(t)}(F(\mathbb{C}-\hat{x}(t-1))-s(t))\right]\leqslant-\frac{2p_{\min}\beta f_M}{t}R(t-1)+\frac{B_5}{t^2}.\tag{B4}$$

The third item on the right of (B1) is

$$\begin{aligned}
&\frac{2}{td_{\max}}\mathbb{E}\left[\varepsilon'(t-1)Q(L_{m(t-1)}x(t-1)-W_{m(t-1)}\varepsilon(t-1))\right]\\
&\quad=\frac{2}{td_{\max}}\mathbb{E}\left[\varepsilon'(t-1)QL_{m(t)}x(t-1)\right]-\frac{2}{td_{\max}}\mathbb{E}\left[\varepsilon'(t-1)QW_{m(t)}\varepsilon(t-1)\right].
\end{aligned}\tag{B5}$$

Let $\lambda_{QL}=\max\left\{\lambda_{\max}\{QL_1Q'\},\ldots,\lambda_{\max}\{QL_hQ'\}\right\}$ and $0<\alpha<\infty$. Then the first item on the right of (B5) is

$$\begin{aligned}
&\frac{2}{td_{\max}}\mathbb{E}\left[\varepsilon'(t-1)QL_{m(t-1)}x(t-1)\right]\\
&\quad\leqslant\frac{2}{td_{\max}}\sqrt{\mathbb{E}[\varepsilon'(t-1)Q\tilde{L}'_{m(t-1)}\tilde{L}_{m(t-1)}Q'\varepsilon(t-1)]\mathbb{E}[x'(t-1)\tilde{L}'_{m(t-1)}\tilde{L}_{m(t-1)}x(t-1)]}\\
&\quad\leqslant\frac{2}{td_{\max}}\sqrt{\lambda_{QL}R(t-1)V(t-1)}\\
&\quad=\frac{2}{td_{\max}}\sqrt{\left[\frac{\lambda_{QL}}{\alpha}R(t-1)\right][\alpha V(t-1)]}\\
&\quad\leqslant\frac{1}{td_{\max}}\left[\frac{\lambda_{QL}}{\alpha}R(t-1)+\alpha V(t-1)\right].
\end{aligned}\tag{B6}$$

Let $\lambda_{\bar{W}}=\max\{\lambda_{\max}\{W_1'W_1\},\ldots,\lambda_{\max}\{W_h'W_h\}\}$ and $\lambda_Q=\lambda_{\max}\{QQ'\}$. Then the second item on the right of (B5) is

$$-\frac{2}{td_{\max}}\mathbb{E}\left[\varepsilon'(t-1)QW_{m(t-1)}\varepsilon(t-1)\right]$$

$$
\begin{aligned}
&\leqslant \frac{2}{td_{\max}}\sqrt{\mathbb{E}\left[\varepsilon'(t-1)QQ'\varepsilon(t-1)\right]\mathbb{E}[\varepsilon'(t-1)W'_{m(t-1)}W_{m(t-1)}\varepsilon(t-1)]}\\
&\leqslant \frac{2}{td_{\max}}\sqrt{\lambda_Q R(t-1)\lambda_{\breve{W}} R(t-1)}\\
&= \frac{2\sqrt{\lambda_Q\lambda_{\breve{W}}}}{td_{\max}}R(t-1).
\end{aligned} \tag{B7}
$$

By (B5)–(B7) we can obtain the third item on the right of (B1) as below:

$$
\begin{aligned}
&\frac{2}{td_{\max}}\mathbb{E}\left[\varepsilon'(t-1)Q(L_{m(t-1)}x(t-1)-W_{m(t-1)}\varepsilon(t-1))\right]\\
&\quad\leqslant \frac{1}{td_{\max}}\left[\frac{\lambda_{QL}}{\alpha}R(t-1)+\alpha V(t-1)\right]+\frac{2\sqrt{\lambda_Q\lambda_{\breve{W}}}}{td_{\max}}R(t-1).
\end{aligned} \tag{B8}
$$

Thus, by (B1), (B4) and (B8), the Lyapunov function $R(t)$ has the following form:

$$
R(t)\leqslant\left(1-\frac{2p_{\min}\beta f_M d_{\max}-\frac{\lambda_{QL}}{\alpha}-2\sqrt{\lambda_Q\lambda_{\breve{W}}}}{td_{\max}}\right)R(t-1)+\frac{\alpha}{td_{\max}}V(t-1)+\frac{\tilde{B}}{t^2}, \tag{B9}
$$

where $\tilde{B}=B_4+B_5$.