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Exponential stabilization of ODE system with Euler-Bernoulli beam actuator dynamics

Xiao-Hui Wu & Hongyinping Feng*

School of Mathematical Sciences, Shanxi University, Taiyuan 030006*, P.R. China*

Appendix A

In order to represent the solution of the following equations (eq.(3) and (5) in this study)

$$\begin{cases} H_{2xxxx}(x) + AH_1(x) = 0, \\ H_1(x) - AH_2(x) = 0, \\ H_{2xxx}(1) = c_2 H_2(1) + AH_3, \\ H_{2xx}(1) = -c_1 H_{2x}(1) - AH_4, \\ B_1 + H_{2xx}(0) = 0, \quad B_2 - H_{2xxx}(0) = 0, \\ H_3 = k_2 H_2(1), \quad H_4 = k_1 H_{2x}(1), \end{cases} \tag{A1}$$

we first introduce some auxiliary functions. For any $(z, s) \in [-1, 1] \times \mathbb{C}$ and $k_1,\ k_2,\ c_1,\ c_2 > 0$, define

$$\begin{cases} \Theta_0(z,s) = \sum\limits_{j=0}^{\infty} \dfrac{(-s^2)^j z^{4j}}{(4j)!},\ \Theta_1(z,s) = \sum\limits_{j=0}^{\infty} \dfrac{(-s^2)^j z^{4j+1}}{(4j+1)!}, \\ \Theta_2(z,s) = \sum\limits_{j=0}^{\infty} \dfrac{(-s^2)^j z^{4j+2}}{(4j+2)!},\ \Theta_3(z,s) = \sum\limits_{j=0}^{\infty} \dfrac{(-s^2)^j z^{4j+3}}{(4j+3)!}, \\ M_{11}(s) = -s^2\Theta_3(1,s) + \Theta_0(1,s)(c_1 + k_1 s),\ M_{12}(s) = s^2\Theta_2(1,s) + \Theta_1(1,s)(c_2 + k_2 s), \\ M_{21}(s) = -s^2\Theta_2(1,s) - s^2\Theta_3(1,s)(c_1 + k_1 s),\ M_{22}(s) = s^2\Theta_1(1,s) + \Theta_0(1,s)(c_2 + k_2 s). \end{cases} \tag{A2}$$

Since

$$\cosh\alpha - \cos\alpha = 2\sum_{j=0}^{\infty} \frac{\alpha^{4j}}{(4j)!}, \quad \forall \ \ \alpha \in \mathbb{R}, \tag{A3}$$

a simple computation shows that, for any given $z \in [-1, 1]$, $\Theta_0(z, \cdot)$ is a convergent power series in $\mathbb{C}$ and hence is analytic in $\mathbb{C}$. Similarly, the function $\Theta_i(z, \cdot)$ is also analytic in $\mathbb{C}$, $i = 1, 2, 3$. Therefore, for any $A \in \mathbb{R}^{n\times n}$, the matrix-valued function $\Theta_i(z, A)$ is well defined due to [1, Definition 1.2, p.3].

Let $c_1, c_2 > 0$ and $\mathcal{H} = H^2(0, 1) \times L^2(0, 1)$ be a Hilbert space equipped with inner product

$$\begin{aligned} \langle (f_1, g_1), (f_2, g_2)\rangle_{\mathcal{H}} &= \langle f_1'', f_2''\rangle_{L^2(0,1)} + \langle g_1, g_2\rangle_{L^2(0,1)} \\ &+ c_2 f_1(1)\overline{f_2(1)} + c_1 f_1'(1)\overline{f_2'(1)}, \quad \forall\ (f_i, g_i) \in \mathcal{H},\ i = 1, 2. \end{aligned} \tag{A4}$$

Lemma 1. Suppose that $A \in \mathbb{R}^{n\times n}$, the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is given by

$$\begin{cases} \mathcal{A}(f, g) = (g, -f^{(4)}), \quad \forall\ (f, g) \in D(\mathcal{A}), \\ D(\mathcal{A}) = \big\{(f, g) \mid f \in H^4(0, 1), g \in H^2(0, 1), \\ \qquad f'''(0) = f''(0) = 0, f''(1) = -c_2 f'(1) - k_2 g(1),\ f'''(1) = c_1 f(1) + k_1 g(1)\big\} \end{cases} \tag{A5}$$

* Corresponding author (email: fhyp@sxu.edu.cn)

and

$$M(A) = \begin{pmatrix} M_{11}(A) & M_{12}(A) \\ M_{21}(A) & M_{22}(A) \end{pmatrix}, \tag{A6}$$

where M_{ij} is given by (A2), $i, j = 1, 2$. If

$$\sigma(A) \cap \sigma(\mathcal{A}) = \emptyset, \tag{A7}$$

then the matrix $M(A) \in \mathbb{R}^{2n\times 2n}$ is invertible. As a result, system (A1) admits a unique solution given by

$$\begin{cases} H_2(x) = \Theta_0(x-1, A)H_2(1) + \Theta_1(x-1, A)H_{2x}(1) \\ \qquad -\Theta_2(x-1, A)(c_1 I + k_1 A)H_{2x}(1) + \Theta_3(x-1, A)(c_2 I + k_2 A)H_2(1), \\ \begin{pmatrix} H_{2x}(1) \\ H_2(1) \end{pmatrix} = [M(A)]^{-1}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \\ H_1(x) = AH_2(x),\ H_3 = k_2 H_2(1), H_4 = k_1 H_{2x}(1), \end{cases} \tag{A8}$$

where Θ_j, $j = 0, 1, 2, 3$ are defined by (A2) and $M(A)$ is given by (A6).

Proof. Consider the characteristic equation $\mathcal{A}(f, g) = \lambda(f, g)$ where $(f, g) \in D(\mathcal{A})$. A simple computation shows that $\lambda \in \sigma(\mathcal{A})$ if and only if (f, g) satisfies

$$\begin{cases} f^{(4)}(x) + \lambda^2 f(x) = 0, \\ g(x) = \lambda f(x),\ f''(0) = f'''(0) = 0, \\ f''(1) = -(c_1 + k_1\lambda)f'(1),\ f'''(1) = (c_2 + k_2\lambda)f(1). \end{cases} \tag{A9}$$

Then we obtain $\lambda \in \sigma(\mathcal{A})$ if and only if the characteristic function of $\mathcal{A}$ satisfies

$$\begin{aligned} \Delta(\lambda) = & [-\lambda^2\Theta_3(1, \lambda) + \Theta_0(1, \lambda)(c_1 + k_1\lambda)][\lambda^2\Theta_1(1, \lambda) + \Theta_0(1, \lambda)(c_2 + k_2\lambda)] \\ & +[\lambda^2\Theta_2(1, \lambda) + \lambda^2\Theta_3(1, \lambda)(c_1 + k_1\lambda)][\lambda^2\Theta_2(1, \lambda) + \Theta_1(1, \lambda)(c_2 + k_2\lambda)] \\ = & \ 0, \end{aligned} \tag{A10}$$

where $\Theta_j, j = 0, 1, 2, 3$ are defined in (A2). Owing to (A6) and (A7), we have

$$\det \begin{pmatrix} M_{11}(\lambda) & M_{12}(\lambda) \\ M_{21}(\lambda) & M_{22}(\lambda) \end{pmatrix} = \Delta(\lambda) \neq 0, \quad \forall\ \lambda \in \sigma(A). \tag{A11}$$

This implies that all the eigenvalues of $M(A)$ are nonzero and hence $M(A)$ is invertible.

By the classical ordinary equation theory, the general solution (H_1, H_2) of (A1) satisfies

$$\begin{cases} H_2(x) = \Theta_0(x-1, A)H_2(1) + \Theta_1(x-1, A)H_{2x}(1) \\ \qquad +\Theta_2(x-1, A)H_{2xx}(1) + \Theta_3(x-1, A)H_{2xxx}(1), \\ H_1(x) = AH_2(x). \end{cases} \tag{A12}$$

Using the boundary conditions in (A1), we obtain

$$M(A)\begin{pmatrix} H_{2x}(1) \\ H_2(1) \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \tag{A13}$$

which, together with the invertibility of $M(A)$, leads to

$$\begin{pmatrix} H_{2x}(1) \\ H_2(1) \end{pmatrix} = [M(A)]^{-1}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}. \tag{A14}$$

The proof is complete. □

Remark 1. The kernel functions H_1 and H_2 can also be written in the form of hyperbolic functions. For example,

$$\begin{aligned} H_2(x) = & \ \tfrac{1}{2}[\cosh(\sqrt{Ai}(x-1)) + \cos(\sqrt{Ai}(x-1))]H_2(1) \\ & \tfrac{1}{2}(\sqrt{Ai})^{-1}[\sinh(\sqrt{Ai}(x-1)) + \sin(\sqrt{Ai}(x-1))]H_{2x}(1) \\ & -\tfrac{1}{2}(\sqrt{Ai})^{-2}[\cosh(\sqrt{Ai}(x-1)) - \cos(\sqrt{Ai}(x-1))](c_1 I + k_1 A)H_{2x}(1) \\ & +\tfrac{1}{2}(\sqrt{Ai})^{-3}[\sinh(\sqrt{Ai}(x-1)) - \sin(\sqrt{Ai}(x-1))](c_2 I + k_2 A)H_2(1), \end{aligned} \tag{A15}$$

where $\sqrt{Ai}$ denotes any square root of Ai (see [1, Chapter 2, p.35]). We give power series expression just for easy computation because it is not easy to compute or approximate $\sinh(\sqrt{Ai})$ and $\cosh(\sqrt{Ai})$ for any given $A \in \mathbb{R}^{n\times n}$.

Appendix B Proof of the Theorem 1

Lemma 2. Let $\mathcal{A}$ be given by (A5). Then $\mathcal{A}^{-1}$ exists and is compact on $\mathcal{H}$. Hence, $\sigma(\mathcal{A})$, the spectrum of $\mathcal{A}$, consists of isolated eigenvalues of finite algebraic multiplicity only. Moreover, $\mathcal{A}$ is dissipative on $\mathcal{H}$ and hence generates a C_0-semigroup of contractions $e^{\mathcal{A}t}$ on $\mathcal{H}$.

Proof. For any given $(f_1, g_1) \in \mathcal{H}$, we solve $\mathcal{A}(f,g) = (g, -f^{(4)}) = (f_1, g_1)$ to obtain

$$\begin{cases} g(x) = f_1(x), \\ f(x) = f(1) + f'(1)(x-1) + \displaystyle\int_1^x \int_1^{s_1} f''(s_2)ds_2ds_1, \\ f''(x) = -\displaystyle\int_0^x \int_0^{s_1} g_1(s_2)ds_2ds_1, \end{cases} \tag{B1}$$

where

$$f(1) = \frac{1}{c_2}\left[-k_2f_1(1) - \int_0^1 g_1(s)ds\right], \quad f'(1) = \frac{1}{c_1}\left[-k_1f_1'(1) + \int_0^1 \int_0^{s_1} g_1(s_2)ds_2ds_1\right]. \tag{B2}$$

By Sobolev trace embedding theorem, $\mathcal{A}^{-1}$ is compact on $\mathcal{H}$. Therefore, $\sigma(\mathcal{A})$ consists of isolated eigenvalues of finite algebraic multiplicity only. For any $(f,g) \in D(\mathcal{A})$, a simple computation gives

$$\mathrm{Re}\,\langle \mathcal{A}(f,g), (f,g)\rangle_{\mathcal{H}} = -k_1 g'^2(1) - k_2 g^2(1) \leqslant 0, \tag{B3}$$

which implies that $\mathcal{A}$ is dissipative on $\mathcal{H}$ and $D(\mathcal{A})$ is dense in $\mathcal{H}$. By the Lumer-Phillips theorem, $\mathcal{A}$ generates a contraction C_0-semigroup $e^{\mathcal{A}t}$ on $\mathcal{H}$. □

Lemma 3. The operator $\mathcal{A}$ given by (A5) generates an exponentially stable C_0-semigroup on $\mathcal{H}$.

Proof. Consider the following system

$$\begin{cases} w_{tt}(x,t) + w_{xxxx}(x,t) = 0, \ x \in (0,1), \ t > 0, \\ w_{xxx}(0,t) = 0, \ w_{xx}(0,t) = 0, \quad t \geqslant 0, \\ w_{xx}(1,t) = -c_1w_x(1,t) - k_1w_{xt}(1,t), \quad t \geqslant 0, \\ w_{xxx}(1,t) = c_2w(1,t) + k_2w_t(1,t), \quad t \geqslant 0, \end{cases} \tag{B4}$$

where $c_1, \ c_2, \ k_1, \ k_2 > 0$. First, we define the following function

$$\begin{cases} E(t) = \dfrac{1}{2}\displaystyle\int_0^1 w_t^2(x,t) + w_{xx}^2(x,t)dx + \frac{c_1}{2}w_x^2(1,t) + \frac{c_2}{2}w^2(1,t) \\ \rho_1(t) = \displaystyle\int_0^1 xw_x(x,t)w_t(x,t)dx, \\ \rho_2(t) = \displaystyle\int_0^1 z(x,t)w_t(x,t)dx, \quad z(x,t) = w(1,t) + (x-1)w_x(1,t). \end{cases} \tag{B5}$$

We will estimate the time derivatives of $E(t)$ and $\rho_i(t), \ i = 1,2$ along the solution of system (B4), respectively. By a simple computation, it follows that

$$\dot{E}(t) = -k_1w_{xt}^2(1,t) - k_2w_t^2(1,t), \tag{B6}$$

$$\begin{aligned} \dot{\rho}_1(t) &= \int_0^1 xw_x(x,t)w_{tt}(x,t) + xw_{xt}(x,t)w_t(x,t)dx \\ &= -w_{xxx}(1,t)w_x(1,t) + w_{xx}(1,t)w_x(1,t) + \frac{1}{2}w_{xx}^2(1,t) \\ &\quad + \frac{1}{2}w_t^2(1,t) - \frac{1}{2}\int_0^1 3w_{xx}^2(x,t) + w_t^2(x,t)dx \end{aligned} \tag{B7}$$

and

$$\begin{aligned} \dot{\rho}_2(t) &= \int_0^1 z_t(x,t)w_t(x,t) - z(x,t)w_{xxxx}(x,t)dx \\ &= -w(1,t)w_{xxx}(1,t) + w_x(1,t)w_{xx}(1,t) + \int_0^1 z_t(x,t)w_t(x,t)dx. \end{aligned} \tag{B8}$$

Taking the boundary condition of (B4) into account, we have

$$\begin{cases} w_{xx}^2(1,t) = -[c_1w_x(1,t)]^2 + [k_1w_{xt}(1,t)]^2 - 2c_1w_{xx}(1,t)w_x(1,t), \\ w_{xxx}^2(1,t) = -[c_2w(1,t)]^2 + [k_2w_t(1,t)]^2 + 2c_2w_{xxx}(1,t)w(1,t). \end{cases} \tag{B9}$$

Inserting (B9) into (B7) gives

$$\begin{aligned}\dot{\rho}_1(t) \leqslant\ & \frac{\delta_2}{2c_2}w_{xxx}^2(1,t)+\frac{1}{2}w_x^2(1,t)+w_{xx}(1,t)w_x(1,t)+\frac{1+\delta_2}{2c_1}w_{xx}^2(1,t)\\ & +\frac{1}{2}w_t^2(1,t)-\frac{1}{2}\int_0^1 3w_{xx}^2(x,t)+w_t^2(x,t)dx\\ \leqslant\ & -\frac{\delta_2 c_2}{2}w^2(1,t)+\frac{\delta_2 k_2^2}{2c_2}w_t^2(1,t)+\delta_2 w_{xxx}(1,t)w(1,t)\\ & -\frac{(1+\delta_2)c_1-1}{2}w_x^2(1,t)+\frac{1+\delta_2}{2c_1}[k_1w_{xt}(1,t)]^2-\delta_2 w_{xx}(1,t)w_x(1,t)\\ & +\frac{1}{2}w_t^2(1,t)-\frac{1}{2}\int_0^1 3w_{xx}^2(x,t)+w_t^2(x,t)dx,\end{aligned} \tag{B10}$$

where $\delta_2\in\mathbb{R}$ satisfies $\frac{\delta_2}{c_2}\geqslant 1$, $\frac{1+\delta_2}{c_1}\geqslant 1$. Then, (B8) and (B10) yield

$$\begin{aligned}\dot{\rho}_1(t)+\delta_2\dot{\rho}_2(t) \leqslant\ & -\frac{\delta_2 c_2}{2}w^2(1,t)+\frac{\delta_2 k_2^2+c_2}{2c_2}w_t^2(1,t)\\ & -\frac{(1+\delta_2)c_1-1}{2}w_x^2(1,t)+\frac{1+\delta_2}{2c_1}k_1^2w_{xt}^2(1,t)\\ & -\int_0^1 \frac{3}{2}w_{xx}^2(x,t)+\frac{1}{4}w_t^2(x,t)dx+\delta_2^2\int_0^1 z_t^2(x,t)dx,\end{aligned} \tag{B11}$$

where $\delta_2\in\mathbb{R}$ satisfies $\delta_2>0$, $(1+\delta_2)c_1>1$. In view of (B5), we have

$$\int_0^1 z_t^2(x,t)dx\leqslant 2\int_0^1 w_t^2(1,t)+(x-1)^2w_{xt}^2(1,t)dx\leqslant 2w_t^2(1,t)+\frac{2}{3}w_{xt}^2(1,t). \tag{B12}$$

Let $\omega_0=\min\{\frac{(1+\delta_2)c_1-1}{c_1},\delta_2,\frac{1}{2}\}$ and $\delta_0=\max\left\{\frac{\delta_2k_2^2+c_2}{2c_2}+2\delta_2^2,\frac{1+\delta_2}{2c_1}k_1^2+\frac{2}{3}\delta_2^2\right\}$. The inequality (B11) becomes

$$\dot{\rho}_1(t)+\delta_2\dot{\rho}_2(t)\leqslant -\omega_0E(t)+\delta_0[w_{xt}^2(1,t)+w_t^2(1,t)]. \tag{B13}$$

Define the following Lyapunov function

$$F(t)=E(t)+\delta_1\left[\rho_1(t)+\delta_2\rho_2(t)\right], \tag{B14}$$

where

$$\begin{cases}0<\delta_1<\min\left\{\frac{k_1}{\delta_0},\frac{k_2}{\delta_0},\left[4+\frac{2}{c_1}+\delta_2\left(1+\frac{2}{c_2}+\frac{2}{3c_1}\right)\right]^{-1}\right\},\\ \delta_2>\max\left\{c_2,c_1-1,\frac{1}{c_1}-1\right\}.\end{cases} \tag{B15}$$

Then, it follows from the Poincaré inequality that

$$|\rho_1(t)+\delta_2\rho_2(t)|\leqslant\left[4+\frac{2}{c_1}+\delta_2\left(1+\frac{2}{c_2}+\frac{2}{3c_1}\right)\right]E(t),\forall\ t\geqslant 0. \tag{B16}$$

As a result, we deduce that

$$0\leqslant\left[1-\delta_1\left(4+\frac{2}{c_1}+\delta_2\left(1+\frac{2}{c_2}+\frac{2}{3c_1}\right)\right)\right]E(t)\leqslant F(t)\leqslant\left[1+\delta_1\left(4+\frac{2}{c_1}+\delta_2\left(1+\frac{2}{c_2}+\frac{2}{3c_1}\right)\right)\right]E(t),\forall\ t\geqslant 0. \tag{B17}$$

By (B6), (B13) and (B17), the time derivative of $F(t)$ is

$$\begin{aligned}\dot{F}(t)\ =\ & \dot{E}(t)+\delta_1\left[\dot{\rho}_1(t)+\delta_2\dot{\rho}_2(t)\right]\\ \leqslant\ & -(k_1-\delta_1\delta_0)w_{xt}^2(1,t)-(k_2-\delta_1\delta_0)w_t^2(1,t)-\delta_1\omega_0E(t)\\ \leqslant\ & -\delta_1\omega_0\left[1+\delta_1\left(4+\tfrac{2}{c_1}+\delta_2\left(1+\tfrac{2}{c_2}+\tfrac{2}{3c_1}\right)\right)\right]^{-1}F(t)\end{aligned} \tag{B18}$$

where $0<\delta_1<\min\{\frac{k_1}{\delta_0},\frac{k_2}{\delta_0}\}$. The proof is complete. □

Remark 2. The exponential stability of $e^{\mathcal{A}t}$ can also be obtained by the Riesz basis method (e.g. [2,3]).

Next, we consider the system

$$\begin{cases}\dot{\tilde{X}}(t)=\left[A+H_{2x}(1)K_1^\top-H_2(1)K_2^\top\right]\tilde{X}(t),\\ w_{tt}(x,t)+w_{xxxx}(x,t)=0,\\ w_{xx}(0,t)=0,\quad w_{xxx}(0,t)=0,\\ w_{xx}(1,t)=-c_1w_x(1,t)-k_1w_{xt}(1,t)+K_1^\top\tilde{X}(t),\\ w_{xxx}(1,t)=c_2w(1,t)+k_2w_t(1,t)+K_2^\top\tilde{X}(t).\end{cases} \tag{B19}$$

Define the operator $\mathcal{B} = [\mathcal{B}_1\ \mathcal{B}_2] : \mathbb{R}^2 \to \mathcal{H}$ by

$$\mathcal{B}_1 s = (0, s\delta'(x-1)), \quad \mathcal{B}_2 s = (0, -s\delta(x-1)),\ \forall\ s \in \mathbb{R}, \tag{B20}$$

where $\delta(\cdot)$ is the Dirac distribution. Letting

$$K = [K_1\ K_2] \ \text{ and } \ \tilde{A} = A + H_{2x}(1)K_1^\top - H_2(1)K_2^\top, \tag{B21}$$

system (B19) can be written abstractly as

$$\begin{cases} \dot{\tilde{X}}(t) = \tilde{A}\tilde{X}(t), \\ \dfrac{d}{dt}(w(\cdot,t), w_t(\cdot,t)) = \mathcal{A}(w(\cdot,t), w_t(\cdot,t)) + \mathcal{B}K^\top \tilde{X}(t), \end{cases} \tag{B22}$$

where $\mathcal{A}$ is defined by (A5).

Lemma 4. Let $\mathcal{A}$, $\mathcal{B}$ and K be given by (A5), (B20) and (B21), respectively. Suppose that $\tilde{A} \in \mathbb{R}^{n\times n}$ is Hurwitz. Then for any $(\tilde{X}(0), w(\cdot,0), w_t(\cdot,0)) \in \mathbb{R}^n \times \mathcal{H}$, system (B22) admits a unique solution $(\tilde{X}, w, w_t) \in C([0,+\infty); \mathbb{R}^n \times \mathcal{H})$ that decays to zero exponentially in $\mathbb{R}^n \times \mathcal{H}$ as time t goes to infinity.

Proof. It follows from Lemma 3 in Appendix B that the operator $\mathcal{A}$ generates an exponentially stable C_0-semigroup $e^{\mathcal{A}t}$ on $\mathcal{H}$. By a simple computation, $\mathcal{B}$ is admissible for $e^{\mathcal{A}t}$ by (B6) (see, e.g., [4,5]). Since $\tilde{A}$ is Hurwitz, we can obtain the desired results by applying Lemma 5 in Appendix C. The proof is complete. □

Proof of the Theorem 1. By Lemma 3 in Appendix B, the operator $\mathcal{A}$ defined by (A5) generates an exponentially stable C_0-semigroup $e^{\mathcal{A}t}$ on $\mathcal{H}$. Hence, assumption $\sigma(A) \subset \{\lambda \in \mathbb{C} \mid \mathrm{Re}\lambda \geqslant 0\}$ implies that (A7) holds. It follows from Lemma 1 that $M(A)$ defined by (A6) is invertible and system (A1) admits a unique solution given by (A8).

Owing to the second of equation (A8), (A11) and the controllability of pair $(A, [B_1\ B_2])$, we can conclude that the pair $(A, [-H_{2x}(1)\ H_2(1)])$ is also controllable, as given in [6, Proposition 9.1]. Indeed, by [1, Theorem 1.18.(d), p.13], we have

$$[M_{ij}(A)]^* = M_{ij}(A^*), i, j = 1, 2, \tag{B23}$$

since $M_{ij}(\mathbb{R}) \subset \mathbb{R}$ where the analytic functions M_{ij} are defined by (A2). By (A13) and (B23), we obtain that for any $v \in \ker(\overline{\lambda} - A^*)$ with $\lambda \in \sigma(A)$ and $\overline{\lambda} \in \sigma(A^*)$,

$$\begin{cases} M_{11}(\overline{\lambda})H_{2x}^*(1)v + M_{12}(\overline{\lambda})H_2^*(1)v = B_1^* v, \\ M_{21}(\overline{\lambda})H_{2x}^*(1)v + M_{22}(\overline{\lambda})H_2^*(1)v = B_2^* v. \end{cases} \tag{B24}$$

By (A11), we have

$$\det \begin{pmatrix} M_{11}(\bar{\lambda}) & M_{12}(\bar{\lambda}) \\ M_{21}(\bar{\lambda}) & M_{22}(\bar{\lambda}) \end{pmatrix} = \overline{\det \begin{pmatrix} M_{11}(\lambda) & M_{12}(\lambda) \\ M_{21}(\lambda) & M_{22}(\lambda) \end{pmatrix}} \neq 0. \tag{B25}$$

And hence $H_{2x}^*(1)v = H_2^*(1)v = 0$ if and only if $B_1^* v = B_2^* v = 0$. By Hautus's lemma, the pair $(A^*, [B_1\ B_2]^*)$ is observable if and only if $(A^*, [H_{2x}(1)\ H_2(1)]^*)$ is observable. Therefore, the pair $(A, [H_{2x}(1)\ H_2(1)])$ is controllable if and only if the pair $(A, [B_1\ B_2])$ is controllable. By pole placement theorem, we can select K_1, K_2 such that $A + H_{2x}(1)K_1^\top - H_2(1)K_2^\top$ is Hurwitz.

Now, we prove the well-posedness and the stability of the closed-loop system. To this end, we introduce the following transformation:

$$\left(\tilde{X}(t), w(\cdot,t), w_t(\cdot,t)\right)^\top = (I + \mathbb{P})\left(X(t), w(\cdot,t), w_t(\cdot,t)\right)^\top, \tag{B26}$$

where I is the identity operator on $\mathbb{R}^n \times \mathcal{H}$,

$$\mathbb{P}\left(X(t), w(\cdot,t), w_t(\cdot,t)\right)^\top = \left(\int_0^1 [H_1(x)w(x,t) + H_2(x)w_t(x,t)]dx + H_3 w(1,t) + H_4 w_x(1,t), 0, 0\right)^\top. \tag{B27}$$

It is trivial that the transformation $I + \mathbb{P}$ is invertible and its inverse is $I - \mathbb{P}$. Therefore, the solution of closed-loop system is well-posed and exponentially stable in $\mathbb{R}^n \times \mathcal{H}$ if and only if the solution of transformed system (B19) is well-posed and exponentially stable in $\mathbb{R}^n \times \mathcal{H}$. By Lemma 4, system (B19) admits a unique solution $(\tilde{X}, w, w_t) \in C([0,\infty); \mathbb{R}^n \times \mathcal{H})$ that decays to zero exponentially in $\mathbb{R}^n \times \mathcal{H}$ as time t goes to infinity. The proof is complete. □

Appendix C

Let S_i be a densely defined operator in Hilbert space X_i, $i = 1, 2$, $C \in \mathcal{L}(D(S_2), Y)$, and $B \in \mathcal{L}(Y, [D(S_1^*)]')$, where Y is Hilbert space and $[D(S_1^*)]'$ is the dual space of $D(S_1)$ with respect to the pivot space X_1. Consider the following cascade system

$$\begin{cases} \dot{x}_1(t) = S_1 x_1(t) + BCx_2(t), \\ \dot{x}_2(t) = S_2 x_2(t). \end{cases} \tag{C1}$$

Lemma 5. Suppose that S_i generates an exponentially stable C_0-semigroup $e^{S_i t}$ on X_i, $i = 1, 2$, $B \in \mathcal{L}(Y, [D(S_1^*)]')$ is admissible for $e^{S_1 t}$, and $C \in \mathcal{L}(D(S_2), Y)$ is admissible for $e^{S_2 t}$. Then, for any initial state $(x_1(0), x_2(0)) \in X_1 \times X_2$, system (C1) admits a unique solution $(x_1, x_2) \in C([0,\infty); X_1 \times X_2)$ that decays to zero exponentially in $X_1 \times X_2$ as time t goes to infinity.

Proof. Since system (C1) is a cascade, the solution to x_2-subsystem is always well defined provided the initial state $x_2(0)$ is given. Owing to the admissibility of C for $e^{S_2 t}$, $Cx_2 \in L^2_{\rm loc}([0,\infty);Y)$. Moreover, it follows from [7, Proposition 4.3.6, p.124] and the exponential stability of C_0-semigroup $e^{S_2 t}$ that there exist two positive constants L_2 and ω_2 such that

$$\|x_2(t)\|_{X_2} \leqslant L_2 e^{-\omega_2 t}, \quad \forall\, t \geqslant 0, \tag{C2}$$

and

$$v \in L^2([0,\infty);Y), \quad v(t) := e^{\omega_2 t} C x_2(t). \tag{C3}$$

On the other hand, the admissibility of B for $e^{S_1 t}$ implies that, for any $x_1(0) \in X_1$, the x_1-subsystem of (C1) admits a solution $x_1 \in C([0,\infty);X_1)$ that is given by

$$x_1(t) = e^{S_1 t} x_1(0) + \int_0^t e^{S_1(t-s)} BCx_2(s) ds. \tag{C4}$$

Hence, when the initial state $(x_1(0), x_2(0)) \in X_1 \times X_2$ is given, such a defined $(x_1, x_2) \in C([0,\infty); X_1 \times X_2)$ is a solution of system (C1). Owing to the linearity of (C1), this solution is unique.

Since $e^{S_1 t}$ is exponentially stable, there exist two positive constants ω_1 and L_1 such that

$$\left\| e^{S_1 t} \right\|_{X_1} \leqslant L_1 e^{-\omega_1 t}, \quad \forall\, t \geqslant 0. \tag{C5}$$

Moreover, by [8, Remark 2.6], there exists a constant $M > 0$ independent of v such that

$$\left\| \int_0^t e^{S_1(t-s)} Bv(s) ds \right\|_{X_1} \leqslant M \|v\|_{L^2([0,\infty);Y)}, \quad \forall\, t \geqslant 0. \tag{C6}$$

Let $0 < \theta < 1$, it follows from (C3), (C5) and (C6) that

$$\begin{aligned} \left\| \int_0^t e^{S_1(t-s)} BCx_2(s) ds \right\|_{X_1} &\leqslant \left\| \int_0^{\theta t} e^{S_1(t-s)} BCx_2(s) ds \right\|_{X_1} + \left\| \int_{\theta t}^t e^{S_1(t-s)} BCx_2(s) ds \right\|_{X_1} \\ &\leqslant \left\| e^{S_1(1-\theta)t} \int_0^{\theta t} e^{S_1(\theta t - s)} BCx_2(s) ds \right\|_{X_1} + \left\| e^{-\omega_2 \theta t} \int_{\theta t}^t e^{S_1(t-s)} Bv(s) ds \right\|_{X_1} \\ &\leqslant L_1 e^{-\omega_1(1-\theta)t} M \|Cx_2\|_{L^2([0,\infty);Y)} + e^{-\omega_2 \theta t} M \|v\|_{L^2([0,\infty);Y)}. \end{aligned} \tag{C7}$$

Finally, the exponential stability of (x_1, x_2) can be obtained by (C2), (C4), (C5) and (C7). The proof is complete. □

Appendix D Numerical Simulations

In this section, we make some simulations for the closed-loop system (1) and (9) to validate the theoretical results. For the sake of simplicity, we consider a second order ODE system. The corresponding matrices and the initial state are chosen as

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{D1}$$

and $w(x,0) = x,\ w_t(x,0) = 0$. It is evident that (A,B) is controllable. According to Theorem 1, the kernel functions $H_j, j = 1,2,3,4$ given by (A8) are well defined. We approximate the kernel functions by setting

$$\begin{cases} \Theta_0(z,A) = \displaystyle\sum_{j=0}^{20} \frac{(-A^2)^j z^{4j}}{(4j)!}, \quad \Theta_1(z,A) = \displaystyle\sum_{j=0}^{20} \frac{(-A^2)^j z^{4j+1}}{(4j+1)!}, \\ \Theta_2(z,A) = \displaystyle\sum_{j=0}^{20} \frac{(-A^2)^j z^{4j+2}}{(4j+2)!}, \quad \Theta_3(z,A) = \displaystyle\sum_{j=0}^{20} \frac{(-A^2)^j z^{4j+3}}{(4j+3)!}. \end{cases} \tag{D2}$$

The the gain constants are selected as $c_i = 1,\ k_i = 2,\ i = 1,2$. We assign the pole by Matlab to get the gain vectors

$$K_1 = \begin{pmatrix} -1.3626 \\ 3.3767 \end{pmatrix}, \quad K_2 = \begin{pmatrix} -4.5905 \\ 1.7007 \end{pmatrix} \tag{D3}$$

which leads to $\sigma(\tilde{A}) = \{-1,-1\}$ with $\tilde{A} = A + H_{2x}(1)K_1^\top - H_2(1)K_2^\top$.

The finite difference scheme is adopted in discretization. The numerical results are programmed in Matlab. The time step and the space step are taken as 0.0002 and 0.02, respectively. System (1) with $u_1 = u_2 = 0$ is plotted in Figure D1 which shows that the system without control is unstable. In Figure D2, we give the trajectories of state feedback law u_1, u_2. The closed-loop system is plotted in Figure D3. From these simulations, we see visually that the state of closed-loop system decay to zero smoothly. The convergences are satisfactory. Therefore, the controllers are effective and hence the theory results are validated.

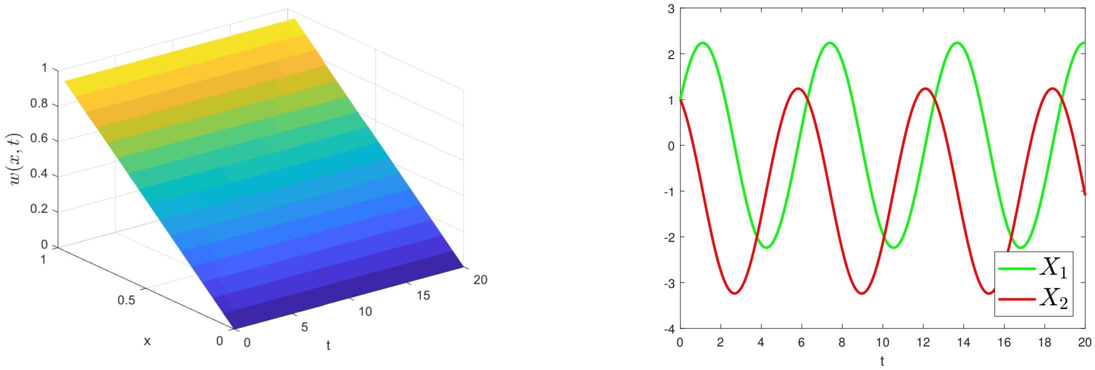

Figure D1 The state of uncontrolled system.

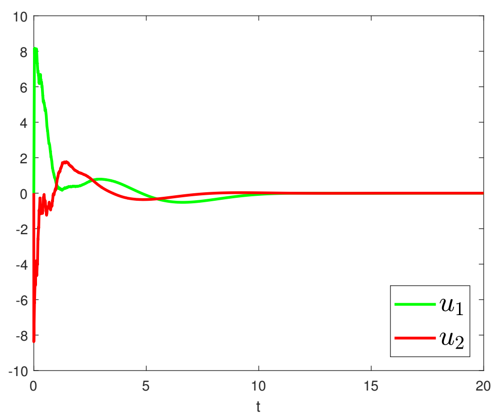

Figure D2 State feedback laws.

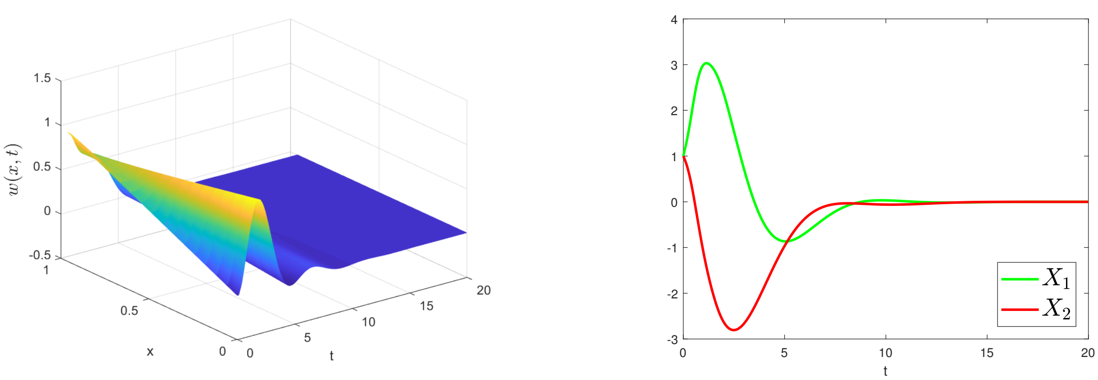

Figure D3 The state of closed-loop system.

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