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Minimal observability of Boolean networks

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Abstract In this study, the minimum observability of Boolean networks (BNs) is investigated by using the semi-tensor product (STP) of matrices. First, a new system based on the considered BN is obtained to analyze states pair dynamic trajectories, from which a necessary and sufficient condition for the observability of BNs is determined. Second, adding a new observer improves the observability without affecting the observable states. Thus, an algorithm is presented to design an observer for an unobservable BN. In addition, a necessary condition is obtained to determine the minimum number of nodes required to be directly measurable. Further, an algorithm to address the minimal observability is presented. Finally, examples are provided to demonstrate the effectiveness of the obtained results.

Keywords Boolean networks, minimal observability, semi-tensor product

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1 Introduction

Boolean networks (BNs) were first proposed by Kauffman [1] modeling genetic regulatory networks. In BNs, each state variable corresponds to 0 or 1 (i.e., off or on) and is updated based on logical functions at each discrete time instant. BN is a widely used discrete-time system, which is proposed to discover the logical behavior of genetic control networks [2, 3]. It is also widely used in actual biological networks, scientific engineering and neural network modeling. For example, we can infer the semantics encoded in RNA sequences [4], or analyze the transmission ability in a disease outbreak or information transmission [5].

Recently, Cheng et al. [6] proposed a new matrix product, called the semi-tensor product (STP) of matrices. STP as a tool can multiply two matrices with any dimension. This method can easily transform logical functions into algebraic forms, and convert a BN into a standard discrete-time linear system [7]. Several meaningful and interesting results have been obtained, including the dynamic properties [8], observability [9–19], controllability [20–22], reconstructibility [23], stability [24–27], minimum-time control [28], and fault detection [29]. Further, feedback control [30], pining control [31–33], and shift registers [34, 35], have also been considered. For example, a method for identifying reducible state variables is presented in [11], based on which a method for reduced-order observer is designed. In [17], four types of observability of Boolean control networks (BCNs) are proposed, and a unified approach based on weighted pair digraphs is provided to determine all four observability types. In [15], a set controllability method is used to investigate the observability of BNs, and several criteria for determining observability are proposed.

Observability is a critical concept in system science and control theory. It depicts the possibility of uniquely determining the initial system state based on the observed output sequence, which is measured using appropriate sensors. Establishing the observability of a system is the first step in designing the observer of the basic system; that is, all states can be reconstructed based on the output sequence. The controllability and observability of BNs have been extensively studied. For example, from the matrix

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equations perspective, the observability of BNs is studied in [11], and a method for designing reduced-order state observers for BCNs is proposed in [12]. When a given system is not observable with the sensors provided, determining the inclusion of additional sensors to make it observable is desirable. The number of additional sensors required is critical when the system is unobservable in the conventional sense. Thus, the difficulties of minimal observability are attracting considerable interest [36, 37].

Recently, the problem of minimizing controlled nodes has been extensively investigated for a special type of BNs called conjunctive BNs [38–41], such as the minimal controllability [40] and minimal observability [41] of conjunctive BNs. Conjunctive BNs are a special class of BNs whose updating rule is composed of only logic AND operations. An efficient algorithm to solve the minimal observability problem was proposed in [41] using a graph-theory approach. However, the relatively minimal observable problems for BNs remain challenging. In this study, the observability issue (sometimes referred to as initial state estimation) of BNs is investigated based on the STP method and algebraic state-space representation of BNs. In addition, how to determine the minimum outputs leading to an observable BN is addressed.

According to the above analysis, investigating the minimal observability for BNs is necessary as well as interesting. Three problems must be solved for BNs. The conventional approach is to determine whether a given system is observable; the next step is to design a sensor that makes the system observable; the final step is to obtain the minimum number of nodes required for direct observation. For BNs, the observability of systems improves with an increase in the number of nodes that are measured directly; that is, already observable states will not be affected by a new observer. Thus, the main aim of this study is to investigate the dynamic trajectory of indistinguishable states. In detail, a parallel system is presented, and a new system is obtained combining the original and parallel systems. The dynamic trajectory of indistinguishable states for the new system can be investigated. Then, a necessary and sufficient condition is obtained to determine the observability of BNs, and an algorithm is designed to determine the set of indistinguishable states. Certain nodes are obtained using STP of matrices to make indistinguishable states initially distinguishable. A theorem is presented to design the observer for an unobservable BN. A necessary condition is obtained to determine the minimum number of nodes.

The rest of this paper is arranged as follows. Section 2 provides some basic notations and the algebraic form of BNs. The main results on minimal observability are presented in Section 3. A short conclusion is presented in Section 4.

2 Preliminaries

Some basic symbols and notations are given below.

- Let $\mathcal{D} := \{0, 1\}$ and $\mathcal{D}^n = \underbrace{\mathcal{D} \times \cdots \times \mathcal{D}}_{n}$.
- $\Delta_n := \{\delta_n^i | 1 \leqslant i \leqslant n\}$, where δ_n^i is the ith column of the identity matrix I_n.
- $M_{m\times n}$ is the set of $m \times n$ real matrices. R_t is the set of column vectors.
- Let $\mathrm{Col}_i(A)$ denote the ith column of matrix A and $\mathrm{Col}(A)$ denote the set of columns of matrix A. A matrix $A \in M_{m\times n}$ is named a logical matrix if $\mathrm{Col}(A) \subseteq \Delta_n$.
- Let $\mathrm{Row}_i(A)$ denote the ith row of matrix A and $\mathrm{Row}(A)$ denote the set of rows of matrix A.
- $\mathcal{L}_{m\times n}$ is the set of $m \times n$ logical matrices. If $A \in \mathcal{L}_{m\times n}$, it can be shown as $A = [\delta_m^{i_1}\ \delta_m^{i_2}\ \cdots\ \delta_m^{i_n}]$, for notational compactness by $A = \delta_m[i_1\ i_2\ \cdots\ i_n]$. $\Omega(A)$ is a vector set and $\Omega(A) = \{\delta_m^{i_1}, \delta_m^{i_2}, \ldots, \delta_m^{i_n}\}$.
- The (i, j)th element of matrix A is denoted by A_{ij}, and $A^s = \underbrace{A \times \cdots \times A}_{s}$ denotes the product of s matrices.
- $\mathcal{Z}_+$ represents all positive integers.
- Let $\mathbf{1}_k$ or $\mathbf{0}_k$ be the row vector of length k and all entries are equal to 1 or 0.
- For vectors $P = (p_{i,1})_{m\times 1}$, $Q = (q_{i,1})_{m\times 1}$, if $p_{i,1} \leqslant q_{i,1}$ for $i \in [1, m]$, we say that $P \leqslant Q$.
- $W_{[p,q]} = [I_q \otimes \delta_p^1\ I_q \otimes \delta_p^2\ \cdots I_q \otimes \delta_p^p]$.
- $\Phi_n = \delta_{2^{2n}}[1\ 2^n + 2\ \cdots\ (2^n - 2) \cdot 2^n + 2^n - 1\ \ 2^{2n}]$.
- The Khatri-Rao product of $P \in M_{m\times r}$ and $Q \in M_{n\times r}$ is $P * Q = [\mathrm{Col}_1(P) \ltimes \mathrm{Col}_1(Q)\ \cdots\ \mathrm{Col}_r(P) \ltimes \mathrm{Col}_r(Q)]$.

Then, the STP and the algebraic form of BNs are recalled below.

Definition 1. The STP of two matrices $H \in M_{m\times n}$ and $G \in M_{p\times q}$ is defined as $H \ltimes G = (H \otimes I_{\beta/n})(G \otimes I_{\beta/p})$, where $\beta = \mathrm{lcm}(n,p)$ is the least common multiple of n and p, and $\otimes$ is the tensor (or Kronecker) product.

Lemma 1 ([2])**.** If A and B be m and n dimensional column vectors, then $W_{[m,n]}AB = BA$. For given vectors $P \in M_{m\times n}$ and $Q \in R_t$, one has $QP = W_{[m,t]}PW_{[t,n]}Q = (I_t \otimes P)Q$.

Lemma 2. Any logical function $g(x_1,\ldots,x_n)$ with logical arguments $x_1,\ldots,x_n \in \mathcal{D}$ can be expressed in a multi-linear algebraic form as $g(x_1,\ldots,x_n) = M_g \ltimes \boldsymbol{x}_1 \ltimes \boldsymbol{x}_2 \ltimes \cdots \ltimes \boldsymbol{x}_n$, where $\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_n \in \Delta_2$ are the vector forms of logical variables $x_1,x_2,\ldots,x_n \in \mathcal{D}$ and $M_g \in \mathcal{L}_{2\times 2^n}$ is called the structure matrix of g.

A Boolean network can be described as

$$x_i(t+1) = g_i(x_1(t),x_2(t),\ldots,x_n(t)), \quad i = 1,2,\ldots,n \tag{1}$$

with its corresponding outputs,

$$y_j(t) = h_j(x_1(t),x_2(t),\ldots,x_n(t)), \quad j = 1,2,\ldots,m, \tag{2}$$

where $x(t) = (x_1(t),x_2(t),\ldots,x_n(t))^{\mathrm{T}} \in \mathcal{D}^n, y(t) = (y_1(t),y_2(t),\ldots,y_m(t))^{\mathrm{T}} \in \mathcal{D}^m$ ("T" is the transpose of a vector) are states and outputs, and $g_i : \mathcal{D}^n \to \mathcal{D}, i = 1,\ldots,n$, $h_i : \mathcal{D}^n \to \mathcal{D}, i = 1,\ldots,m$ are logical functions.

According to the equivalence between logical variables 1/0 and vector forms δ_2^1/δ_2^2, define $x(t) = x_1(t) \ltimes x_2(t) \cdots \ltimes x_n(t) \in \Delta_{2^n}$. According to Lemma 2, the algebraic form of (1) can be obtained as

$$x_i(t+1) = M_i x(t), \quad i = 1,2,\ldots,n, \tag{3}$$

where $M_i \in \mathcal{L}_{2\times 2^n}, i = 1,2,\ldots,n$ are logical matrices. Similarly, let $y(t) = y_1(t) \ltimes y_2(t) \cdots \ltimes y_m(t)$, and then the outputs can be obtained:

$$y_i(t+1) = N_i x(t), \quad i = 1,2,\ldots,m, \tag{4}$$

where $N_i \in \mathcal{L}_{2\times 2^n}, i = 1,2,\ldots,m$ are logical matrices. Then, BN (1) with outputs (2) can be shown into the algebraic form as follows:

$$\begin{cases} x(t+1) = L \ltimes x(t), \\ y(t) = H \ltimes x(t), \end{cases} \tag{5}$$

where $L = M_1 \ltimes_{i=2}^{n} [(I_{2^n} \otimes M_i)\Phi_n] \in \mathcal{L}_{2^n\times 2^n}$ and L is called the structure matrix, while $H = N_1 \ltimes_{i=2}^{m} [(I_{2^m} \otimes N_i)\Phi_m] \in \mathcal{L}_{2^m\times 2^n}$ and H is called the output matrix.

3 Main results

In the following, let $x(t;x_0)$ and $y(t;x_0)$ denote the state and output vectors of (5) starting from intial state $x(0) = x_0 \in \Delta_{2^n}$ at time $t \in \mathcal{Z}$, respectively. Then, we present the following definitions.

Definition 2 ([9,10])**.** BN (5) is observable if any state $x_0 \in \Delta_{2^n}$ can be uniquely determined with the knowledge of the corresponding output trajectory in some interval $[0,T]$, $T \in \mathcal{Z}_+$. In addition, given two $x_1,x_2 \in \Delta_{2^n}$, x_1 and x_2 are named an initial indistinguishable state pair, if $Hx_1 = Hx_2$; x_1 and x_2 are named an initial distinguishable state pair, if $Hx_1 \neq Hx_2$; x_1 and x_2 are named indistinguishable, if for any $t \in \mathcal{Z}$, $y(t;x_1) = y(t;x_2)$; otherwise they are distinguishable. Then, the BN is called to be observable if each two different states are distinguishable.

Next, we introduce a duplicate system of (5) as $\hat{x}(t+1) = L\hat{x}(t)$, where matrix L is the same with that in (5) and $\hat{x}(t) \in \Delta_{2^n}$. Let $z(t) = x(t) \ltimes \hat{x}(t) \in \Delta_{2^{2n}}$. An augmented system is given below:

$$z(t+1) = x(t+1)\hat{x}(t+1) = Lx(t)L\hat{x}(t) = L(I_{2^n} \otimes L)x(t)\hat{x}(t) = Ez(t), \tag{6}$$

where $E = L(I_{2^n} \otimes L) \in \mathcal{L}_{2^{2n}\times 2^{2n}}$.

Let $z(t;z_0)$ denote the state vectors of the BN (6) starting from initial state $z(0) = z_0$ at time $t \in \mathcal{Z}$. According to $E = L(I_{2^n} \otimes L) \in \mathcal{L}_{2^{2n}\times 2^{2n}}$, given $x(t) = \delta_{2^n}^p, \hat{x}(t) = \delta_{2^n}^q$, then one has that $\mathrm{Col}_{(p-1)2^n+q}(E) = \mathrm{Col}_p(L) \ltimes \mathrm{Col}_q(L)$, where $p,q \in [1,2^n]$.

According to matrix H, the states set $\Delta_{2^{2n}}$ of (6) can be divided into the following three subsets:

$$C_0 = \{\delta_{2^n}^i \delta_{2^n}^i | i = 1, \ldots, n\},\ C_1 = \{\delta_{2^n}^i \delta_{2^n}^j | H\delta_{2^n}^i = H\delta_{2^n}^j, i \neq j\},\ C_2 = \{\delta_{2^n}^i \delta_{2^n}^j | H\delta_{2^n}^i \neq H\delta_{2^n}^j, i \neq j\}.$$

In [16], the issue of observability of BCNs has been studied by dividing the states into three classes: (i) diagonal, (ii) h-distinguishable, and (iii) h-indistinguishable, which are similar to the above proposed three subsets C_0, C_1, C_2. In this paper, new methods have been presented to improve observability of BNs which are not observable, based on the above proposed three subsets C_0, C_1, C_2.

3.1 Necessary conditions on the observability of BNs

In this subsection, we apply the graph-theoretic method to investigate the observability of BNs.

Definition 3. We use $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ to represent a directed graph, where $\mathcal{V}$ is the vertex set, and $\mathcal{E}$ is the edge set. For BN (6), $\mathcal{G}$ is a directed graph (DG) if $\mathcal{V} = \{C_0 \cup C_1 \cup C_2\}$, $\mathcal{E} = \{z, z' | E \ltimes z = z'\}$.

We can find that the maximum space complexity of graph $\mathcal{G}$ is $O(2^{2n})$, because the number of nodes is at most 2^{2n} in graph $\mathcal{G}$ and for every node, there is only an edge. We first come up with two simple necessary conditions for the observability of BN (5).

Definition 4. A BN (5) has property O_1 if its DG has no edge from z_1 to z_2, where $z_1 \in C_1$ and $z_2 \in C_0$.

Theorem 1. If a BN is observable, then it has property O_1.

Proof. Consider a BN that does not satisfy property O_1. There must exist an edge from $\delta_{2^{2n}}^i$ to $\delta_{2^{2n}}^j$, where $\delta_{2^{2n}}^i \in C_1$ and $\delta_{2^{2n}}^j \in C_0$. That is to say $\delta_{2^{2n}}^j = E\delta_{2^{2n}}^i \in C_0$. Obviously, $z(t; z_0)$ for any $t \in \mathcal{Z}$ can be found, where $z_0 \in C_0$. Then $z(t; \delta_{2^{2n}}^i) \in C_0$ for any $t \in \mathcal{Z}_+$. Suppose that $\delta_{2^{2n}}^i = \delta_{2^n}^p \ltimes \delta_{2^n}^q$. By Lemma 2, $x(t; \delta_{2^n}^p) \ltimes x(t; \delta_{2^n}^q) \in C_0$ for any $t \in \mathcal{Z}_+$. That is to say $y(t; \delta_{2^n}^p) = y(t; \delta_{2^n}^q)$ for any $t \in \mathcal{Z}$. Then $\delta_{2^n}^p$ and $\delta_{2^n}^q$ are distinguishable.

Example 1. Consider the following BN:

$$\begin{cases} x_1(t+1) = x_1(t) \wedge x_2(t), \\ x_2(t+1) = \neg(x_1(t) \rightarrow x_2(t)), \\ y_1(t) = x_1(t). \end{cases} \tag{7}$$

Using STP, L and H can be obtained as $L = \delta_4[2, 3, 4, 4], H = \delta_4[1, 1, 2, 2]$. Then, one has the augmented system $z(t + 1) = Ez(t)$, where $E = \delta_{16}[6, 7, 8, 8, 10, 11, 12, 12, 14, 15, 16, 16, 14, 15, 16, 16]$. The DG is shown in Figure 1. Obviously, it holds that $C_0 = \{\delta_{16}^1, \delta_{16}^6, \delta_{16}^{11}, \delta_{16}^{16}\}$ and $C_1 = \{\delta_{16}^2, \delta_{16}^5, \delta_{16}^{12}, \delta_{16}^{15}\}$. There exists an edge from δ_{16}^{12} to δ_{16}^{16}, where $\delta_{16}^{12} \in C_1$ and $\delta_{16}^{16} \in C_0$. Then BN (7) does not have property O_1 and δ_4^3, δ_4^4 are indistinguishable.

Definition 5. A BN is said to have property O_2 if its DG does not have a cycle, composed solely of states in C_1.

Theorem 2. If a BN is observable, it has property O_2.

Proof. Consider a BN that does not satisfy property O_2. There exists a cycle $\{z_1, z_2, \ldots, z_k\}$, where $k \in \mathcal{Z}_+$. Since $z_i \in C_1$ for any $i \in [1, k]$, $z(t; z_0) \in C_1$ for $t \in \mathcal{Z}$. Let $z_0 = x_1 \ltimes x_2$. By Lemma 2, $x(t; x_1) \ltimes x(t; x_2) \in C_1$ for $t \in \mathcal{Z}$. That is $Hx(t; x_1) = Hx(t; x_2)$. By Definition 2, x_1 and x_2 are indistinguishable. The BN is unobservable.

Remark 1. In [41], observability of conjunctive BNs is studied, and note that the state transition space is uniquely determined by the network structure of conjunctive BNs, and thus the problem of minimal observability can be studied by searching the minimal dominating set of network structure of conjunctive BNs. However, in this paper, the state transition space is both determined by the network structure of BNs and also by its update functions, which makes the minimal observability problem more difficult. In the following section, the minimal observability is studied based on properties O_1 and O_2.

Example 2. Consider a BN with three nodes and one output:

$$\begin{cases} x_1(*) = [x_1 \wedge (x_2 \vee x_3)] \vee [x_1 \wedge (\neg x_2 \vee \neg x_3)], \\ x_2(*) = [x_1 \wedge (x_2 \rightarrow x_3)] \vee (x_1 \wedge x_2), \\ x_3(*) = [x_1 \wedge (\neg x_2 \vee \neg x_3)] \vee [x_1 \wedge (\neg x_2 \wedge x_3)], \\ y_1 = x_2, \end{cases} \tag{8}$$

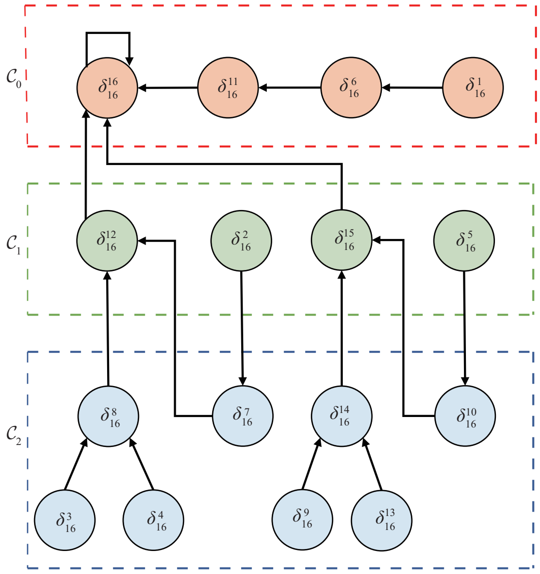

Figure 1 (Color online) The directed graph of BN (7) in Example 1.

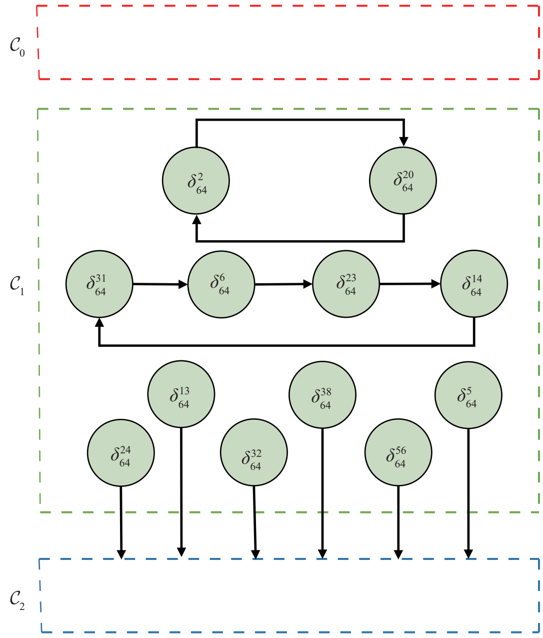

Figure 2 (Color online) The directed graph of BN (8) in Example 2.

where we use $*$ to denote the next time step to save space. Using STP of matrices, L and H can be obtained as $L = \delta_8[3, 4, 2, 1, 6, 7, 6, 4]$, $H = \delta_2[1, 1, 2, 2, 1, 1, 2, 2]$.

Then, one has its augmented system $z(t + 1) = Ez(t)$, where

$$E = \delta_{64}[10, 11, 9, 13, 14, 12, 9, 12, 18, 19, 17, 21, 22, 20, 17, 20, 2, 3, 1, 5, 6, 4, 1, 4, 34, 35, 33, 37, 38, 36, 33, 36,\\ 42, 41, 45, 46, 44, 41, 44, 26, 27, 25, 29, 30, 28, 25, 28, 16, 2, 3, 1, 5, 6, 4, 1, 4, 26, 27, 25, 29, 30, 28, 25, 28].$$

A part of DG is shown in Figure 2. Obviously, there does not exist an edge from z_1 to z_2, where $z_1 \in C_1$ and $z_2 \in C_0$. Then, BN (8) has property O_1. There exist two cycles $\{\delta_{64}^6, \delta_{64}^{23}, \delta_{64}^{14}, \delta_{64}^{31}\}$ and $\{\delta_{64}^2, \delta_{64}^{20}\}$. Then, BN (8) does not have property O_2. Since δ_{64}^6 is an indistinguishable state, and $\delta_{64}^6 = \delta_8^1 \ltimes \delta_8^6$. Then δ_8^1 and δ_8^6 are indistinguishable. Thus, BN (8) is not observable.

3.2 A necessary and sufficient condition for observability

Based on Theorems 1 and 2, this subsection provides a necessary and sufficient condition for observability of the BNs.

Theorem 3. A BN is observable if and only if its DG satisfies properties O_1 and O_2.

Proof. By Theorems 1 and 2, the necessary part is trivial. We shall prove the sufficiency by contradiction. Suppose that BN (5) is unobservable. By Definition 2, there are $x_1, x_2 \in \Delta_{2^n}$ which are indistinguishable and $z_0 = x_1 \ltimes x_2$ is indistinguishable state. That is to say $y(t; x_1) = y(t; x_2)$ for all $t \in \mathcal{Z}$, which means one of the following two situations.

Case 1. There is $t \in \mathcal{Z}_+$ such that $x(t; x_1) = x(t; x_2)$. Then $z(t; z_0) \in C_0$. It can be found that $Ez(t-1; z_0) = z(t; z_0)$. That is to say there exists an edge from $z(t-1; z_0)$ to $z(t; z_0)$, where $z(t-1; z_0) \in C_1$ and $z(t; z_0) \in C_0$. Property O_1 does not hold.

Case 2. There dose not exist $t \in \mathcal{Z}$ such that $x(t; x_1) = x(t; x_2)$. Then $z(t; z_0) \in C_1$ for any $t \in \mathcal{Z}$. Since $0 \leqslant |C_1| < N$, this implies that there is $t_m \in \mathcal{Z}$ such that the trajectories of $x_1(t)$ and $x_2(t)$ are periodic starting from $t_m \in \mathcal{Z}$, and then it can be found that there exists a cycle $(z(t_m), z(t_{m+1}), \ldots, z(t_{m+s}))$, where s is the length of the cycle and $z(t_{m+s}) = x(t; x_1) \ltimes x(t; x_2)$. Obviously, property O_2 does not hold.

3.3 Minimal observability of BNs

In this subsection, the issue is how to add the minimal number of measurements so that an unobservable BN becomes observable. First, a definition regarding the minimal observability problem of BNs is presented.

Minimal observability problem. Given a BN (5) with q outputs, suppose that for any $i \in \{1, \ldots, n\}$ we can make any x_i directly measurable, that is, we can add a new measurement $y_i = x_i$. The problem is to determine a minimal set of indices $\mathcal{I} \in \{1, \ldots, n\}$, which produces an observable BN.

Remark 2. The minimal observability problem has received widespread attention. In [41], a necessary and sufficient observability condition for conjunctive Boolean networks (CBNs) has been derived, and a polynomial-time algorithm has been presented to solve the minimal observability problem in CBNs and observers of observable CBNs are designed. Note that BNs (1) are more general than CBNs, since logical functions and output logical functions include only AND operations in CBNs. Therefore, the results of [41] cannot be applied to BNs. Meanwhile, in [42], an analogous graph-theoretic method provided a sufficient (but not necessary) condition for observability of BCNs and algorithms were obtained to solve the minimal observability problem. However, the algorithms are not optimal anymore, and more measurements than the minimal number should be required. Therefore, it is meaningful to investigate the minimum observability of BNs.

Here, we consider the minimal observability problem of BNs based on the STP method. Let $y'(t) = H_1x(t)$ denote the original outputs (2) of BN. Suppose that $\mathcal{I} = \{j_1, j_2, \ldots, j_i\}$, that is to say, the added measurements are

$$y_{m+1}(t) = x_{j_1}(t), \ldots, y_{m+i}(t) = x_{j_i}(t). \tag{9}$$

Let $y''(t) = y_{m+1} \ltimes y_{m+2} \ltimes \cdots \ltimes y_{m+i}$. Eq. (9) can be shown as $y''(t) = H_2x(t)$ by STP method. With the additional measurements, the new outputs can be described as

$$\begin{cases} y_j(t) = h_j(x_1(t), \ldots, x_n(t)), \quad j = 1, 2, \ldots, m, \\ y_{m+1}(t) = x_{j_1}(t), \ldots, y_{m+i}(t) = x_{j_i}(t). \end{cases} \tag{10}$$

Let $y^*(t) = y_1 \ltimes y_2 \ltimes \cdots \ltimes y_{m+i}$. Eq. (10) can be expressed as $y^*(t) = H_3x(t)$ by STP method, where $H_3 = H_1 * H_2$. Before resolving the minimal observability problem of BNs, we present the following result.

Theorem 4. If two states $x_1, x_2 \in \Delta_{2^n}$ are distinguishable under outputs (2) with output matrix H_1, then $x_1, x_2 \in \Delta_{2^n}$ are distinguishable under the new outputs (10) with output matrix H_3.

Proof. Suppose that $x_1, x_2 \in \Delta_{2^n}$ are distinguishable under outputs (2), and there exists a $k \in \mathcal{Z}_+$ such that $H_1x(k; x_1) \neq H_1x(k; x_2)$. Suppose that $x(k; x_1) = \delta_{2^n}^{i_1}$, $x(k; x_2) = \delta_{2^n}^{i_2}$, and then

$$\mathrm{Col}_{i_1}(H_1) \neq \mathrm{Col}_{i_2}(H_1). \tag{11}$$

Since $H_3 = H_1 * H_2$, $\mathrm{Col}_{i_1}(H_3) = \mathrm{Col}_{i_1}(H_1) \ltimes \mathrm{Col}_{i_1}(H_2)$, $\mathrm{Col}_{i_2}(H_3) = \mathrm{Col}_{i_2}(H_1) \ltimes \mathrm{Col}_{i_2}(H_2)$ can be obtained. Owing to $\mathrm{Col}_{i_1}(H_1) \neq \mathrm{Col}_{i_2}(H_1)$, it is found that $\mathrm{Col}_{i_1}(H_3) \neq \mathrm{Col}_{i_2}(H_3)$. So $H_3x(t; x_1) \neq H_3x(t; x_2)$, and x_1, x_2 are distinguishable under new outputs (10).

Remark 3. It can be found from Theorem 4 that adding the new measurements will improve the observability of the system without affecting the distinguishable state. The distinguishable states will be also distinguishable after certain new measurements are added.

By Theorem 4, we only need to make the indistinguishable states distinguishable under newly added measurements. That is to say that the indistinguishable states of BN (6) need to be considered. The following theorem is presented to determine the added measurement for making an indistinguishable state be initial distinguishable state.

Theorem 5. Suppose that $x', x'' \in \Delta_{2^n}$ are indistinguishable under output (2). Suppose that $x' = \delta_2^{i_1}\delta_2^{i_2}\cdots\delta_2^{i_n}$, $x'' = \delta_2^{j_1}\delta_2^{j_2}\cdots\delta_2^{j_n}$. A measurement $y_s(t) = x_s(t)$ is added, such that x', x'' are initial distinguishable under new output (10), if there exists an $s \in [1, n]$, such that $\delta_2^{i_s} \neq \delta_2^{j_s}$.

Proof. By Lemma 2, a new output $y_1(t) = x_s(t)$ can be described as $y'(t) = H_2x(t)$, where $y'(t) = y_1(t)$, $x(t) = x_1(t) \ltimes x_2(t) \ltimes \cdots \ltimes x_n(t)$. Obviously, when $x' = \ltimes_{i=1}^n x_i(t)$ and $x_s(t) = \delta_2^{i_s}$, $y(t) = y_1(t) = x_s(t) = \delta_2^{i_s}$. That is to say $H_2x' = \delta_2^{i_s}$. Similarly, $x'' = \ltimes_{i=1}^n x_i(t)$ and $x_s(t) = \delta_2^{j_s}$, then $y(t) = y_1(t) = x_s(t) = \delta_2^{i_s}$ and $H_2x'' = \delta_2^{j_s}$. Since $\delta_2^{i_s} \neq \delta_2^{j_s}$, $H_2x' \neq H_2x''$. x', x'' are distinguishable under new output $y'(t) = H_2x(t)$. Based on Theorem 4, x' and x'' are initial distinguishable under new outputs (10).

Based on Theorem 5, an indistinguishable state can be transformed into an initial distinguishable state by adding a measurement. Obviously, all indistinguishable states can be transformed into initial distinguishable states by Theorem 5, which leads an unobservable BN to be an observable BN.

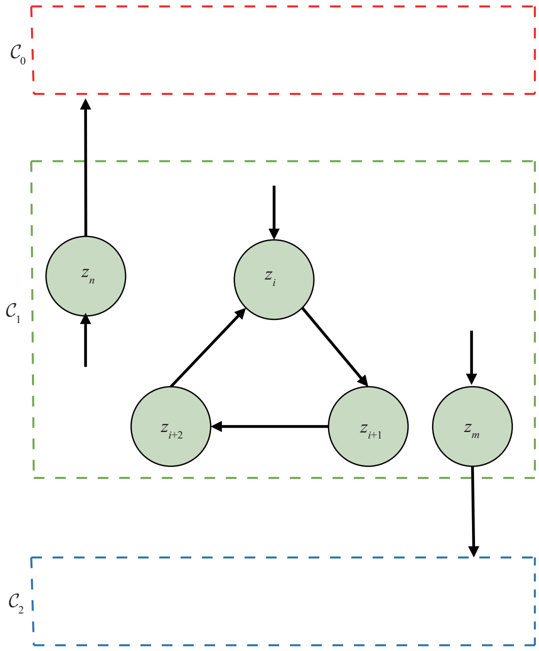

Figure 3 (Color online) The directed graph of an unobservable BN (6) in Remark 5.

Theorem 6. Suppose that $\{z_1, z_2, \dots, z_m\}$ is a cycle in the DG of BN (5) and $E \ltimes z_1 = z_m$, all states in this cycle are distinguishable states if there is an integer $i \in [1, m]$ such that z_i is an initial distinguishable state by Theorem 5.

Proof. Based on Theorem 2, $z_1, z_2, \dots, z_m$ are indistinguishable states. Let $z_1 = x_3 \ltimes x_4$ and $z_i = x_1 \ltimes x_2$. Suppose that $z_i, i \in [1, m]$ is an initial distinguishable state after adding a measurement. Note that $x(t = i-1, x_3) = x_1$ and $x(t = i-1, x_4) = x_2$. Since z_i is an initial distinguishable state, x_3, x_4 are distinguishable and z_1 is distinguishable state. Similarly, other states are distinguishable states.

Based on Theorems 5 and 6, the following algorithm (Algorithm 1) can be proposed to obtain a set of states, and we can add measurements to the states in the set to make these states become initial distinguishable states, thereby obtaining an observable BN.

Algorithm 1 This algorithm is given to obtain a set of states, and adding these measurements can make those states be initial distinguishable states

Input: A BN (3) with m outputs.
Output: The newly added measurement.
1: Generate the directed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ of BN (5).
2: Establish a list L_1 of all states that reach C_0 in next time.
3: Establish a list L_c of cycles composed, where cycles are consisted only of states in C_1 for every cycle $m_j \in L_c$.
4: Copy L_1 into a list $\mathcal{S}$, select an element from every cycle $m_j \in L_c$, and add these elements to L_2.
5: Assign $\mathcal{S} = L_1 \cup L_2$.
6: Make all states of $\mathcal{S}$ initial distinguishable according to Theorem 5.
7: Return the newly added measurements.

Remark 4. As shown in Figure 3, it can be found that all vertexes $z \in C_1$ can be divided into three categories, one will reach C_0 and it means that property O_1 does not hold. Another will constitute a cycle and it means that property O_2 does not hold. The last one will reach C_2. That is to say, the properties O_1 and O_2 do not hold for the above two cases. Next, we apply Algorithm 1 to obtain the set $\mathcal{S} = \{z_{i+1}, z_n\}$. According to Theorem 5, z_{i+1} and z_n can be transformed into initial distinguishable states. After adding the states in set S as new measurements, the directed graph is shown in Figure 4(a). Obviously, properties O_1 and O_2 hold at this moment. The main idea of Algorithm 1 is to find all the cycles and states reaching C_0, and thus the total computational complexity of Algorithm 1 is $O(2^n \times n)$.

Example 3. Reconsider BN (7). The algorithm thus returns $\mathcal{S} = \{\delta_{16}^{12}, \delta_{16}^{15}\}$. H_1 denotes the added output matrix. Since $\delta_{16}^{12} = \delta_4^3 \ltimes \delta_4^4$, $\delta_4^3 = \delta_2^2 \ltimes \delta_2^1, \delta_4^4 = \delta_2^2 \ltimes \delta_2^2$. By Theorem 5, we can add $y_2(t) = x_2(t)$

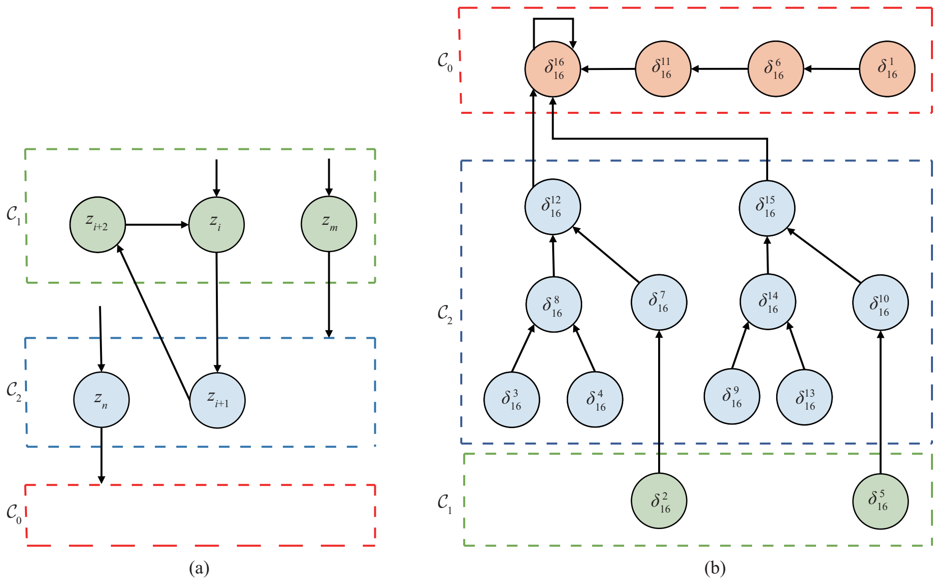

Figure 4 (Color online) (a) The directed graph of an observable BN (6) after adding new measurements in Remark 5; (b) the directed graph of BN (7) in Example 1 after adding new measurement $y_2(t) = x_2(t)$.

to make this indistinguishable states be distinguishable yielding an observable BN:

$$x_1(t+1) = x_1(t) \wedge x_2(t),$$
$$x_2(t+1) = \neg(x_1(t) \to x_2(t)),\ y_1(t) = x_1(t),\ y_2(t) = x_2(t).$$

After adding measurement $y_2(t) = x_2(t)$, the directed graph of BN in Example 1 is shown as Figure 5. This BN is indeed observable, since the DG has properties O_1 and O_2.

Example 4. Reconsider BN (8) in Example 2. Apply Algorithm 1 to this BN, the algorithm thus returns $\mathcal{S} = \{\delta_{64}^{14}, \delta_{64}^{20}\}$. Since $\delta_{64}^{14} = \delta_8^2 \ltimes \delta_8^6$, $\delta_8^2 = \delta_2^1 \ltimes \delta_2^1 \ltimes \delta_2^2$, and $\delta_8^6 = \delta_2^2 \ltimes \delta_2^1 \ltimes \delta_2^2$, by Theorem 5, we can add $y_2 = x_1$ to make this indistinguishable state be initial distinguishable state. Similarly, $y_3 = x_3$ can be added to make δ_{64}^{20} be initial distinguishable state yielding an observable BN shown below:

$$\begin{cases} x_1(*) = [x_1 \wedge (x_2 \vee x_3)] \vee [x_1 \wedge (\neg x_2 \vee \neg x_3)], \\ x_2(*) = [x_1 \wedge (x_2 \to x_3)] \vee (x_1 \wedge x_2), \\ x_3(*) = [x_1 \wedge (\neg x_2 \vee \neg x_3)] \vee [x_1 \wedge (\neg x_2 \wedge x_3)], \\ y_1 = x_2, \quad y_2 = x_1, \quad y_3 = x_3. \end{cases}$$

After adding new measurements $y_2(t) = x_1(t)$ and $y_3(t) = x_3(t)$, part of the directed graph of BN (8) in Example 2 is shown in Figure 4(b). This BN (8) is indeed observable, since the DG has properties O_1 and O_2.

Algorithm 1 provides a simple way to design the measurements. However, the measurements may be not the optimal solution for the minimal observability problem. Set $\mathcal{S}$ is comprised of L_1 and L_2. L_1 is a certain set, while the element of L_2 is chosen randomly from each cycle. Then the measurements in Example 4 are not optimal solution for the minimal observability problem. Next, the optimal solution of minimal observability problem is considered.

Lemma 3. Assume that $y_1 = x_i$ is the added measurement, and then any state $z \in \mathrm{Col}(\Theta)$ is distinguishable, where $\Theta = \Theta_1 \ltimes \Theta_2$, $\Theta_1 = W_{[2,2^{i-1}]}(I_2 \otimes W_{[2,2^{n+i-2}]})x_i x_i'$, $\Theta_2 = x_1 \cdots x_{i-1} x_{i+1} \cdots x_n x_1' \cdots x_{i-1}' x_{i+1}' \cdots x_n'$.

Proof. Suppose that $y_1 = x_i$ is the added measurement. According to Theorem 5, x, x' are distinguishable under new measurement $y_i = x_i$, if $x_i \neq x_i'$. Assume that $x = x_1 \cdots x_i \cdots x_n$, $x' = x_1' \cdots x_i' \cdots x_n'$.

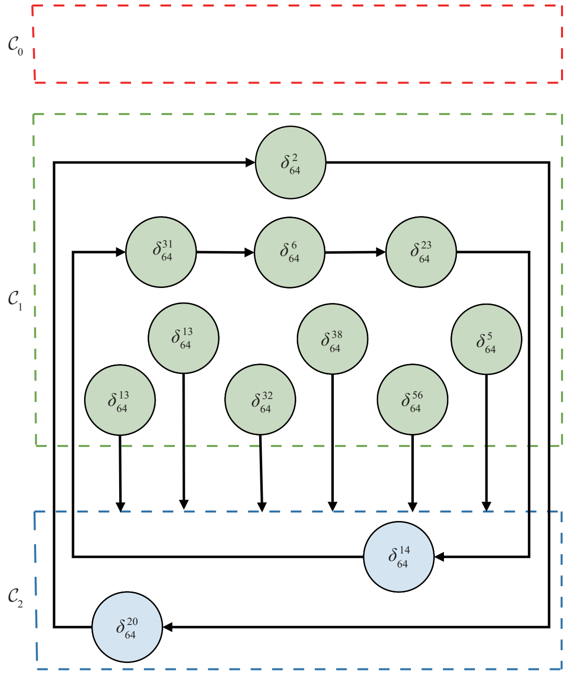

Figure 5 (Color online) Part of the directed graph of BN (8) in Example 2 after adding new measurements $y_2(t) = x_1(t)$ and $y_3(t) = x_3(t)$.

Let $z_1 = x \ltimes x' = x_1 \cdots x_i \cdots x_n x'_1 \cdots x'_i \cdots x'_n$. The following can be obtained:

$$\begin{aligned} z &= x_1 \cdots x_i, \cdots x_n x'_1 \cdots x'_i \cdots x'_n \\ &= W_{[2,2^{i-1}]} x_i x_1 \cdots x_{i-1} x_{i+1} \cdots x_n x'_1 \cdots x'_i \cdots x'_n \\ &= W_{[2,2^{i-1}]} (I_2 \otimes W_{[2,2^{n+i-2}]}) x_i x'_i x_1 \cdots x_{i-1} x_{i+1} \cdots x_n x'_1 \cdots x'_{i-1} x'_{i+1} \cdots x'_n. \end{aligned}$$

Because x_i, x'_i can only take values from $\{\delta_2^1, \delta_2^2\}$, $x_i x'_i = \delta_4^2$ or $x_i x'_i = \delta_4^3$ can be obtained. For the convenience of writing, let $\Theta_1 = W_{[2,2^{i-1}]}(I_2 \otimes W_{[2,2^{n+i-2}]}) x_i x'_i$, $\Theta_2 = x_1 \cdots x_{i-1} x_{i+1} \cdots x_n x'_1 \cdots x'_{i-1} x'_{i+1} \cdots x'_n$. Obviously, Θ_1 is a $2^{n+i} \times 2^{n+i-2}$ matrix, and Θ_2 is a $2^{2n-2} \times 1$ vector. By Definition 1, $\Theta = \Theta_1 \ltimes \Theta_2 = (\Theta_1 \otimes I_{2^{n-i}}) \times \Theta_2$. On the basis of properties of STP calculation, any states $z \in \mathrm{Col}(\Theta)$ is distinguishable.

Lemma 3 can be used to determine a set D_i whose states are initial distinguishable states under new measurements $y_i = x_i$. The states that reach C_0 in next time can be seen as a cycle with length 1. Suppose there are k cycles in BN (6), and the set of elements in each cycle is denoted as R_j, $j \in [1, k]$. The following definition can be obtained.

Definition 6. A cycle r_j, $j \in [1, k]$ is called distinguishable under new measurements $y_i = x_i$, if $D_i \cap R_j \neq \emptyset$, where R_j is a set of all elements in the cycle r_j.

Next, we give a necessary and sufficient condition for minimal observability. Define a matrix Γ_1, where $(\Gamma_1)_{ij} = 1$, if cycle r_j, $j \in [1, k]$ is distinguishable under $y_1 = x_i$, else $(\Gamma_1)_{ij} = 0$. Then, the following theorem is obtained.

Theorem 7. If there exists a sequence $\{i_1, i_2, \ldots, i_s\} \subset \{1, 2, \ldots, n\}$ with the minimal cardinality s such that $\mathrm{Row}_{i_1}(\Gamma_1) + \cdots + \mathrm{Row}_{i_s}(\Gamma_1) \geqslant \mathbf{1}_k$, then $x_{i_1}, x_{i_2}, \ldots, x_{i_s}$ is the minimal states which need to be directly measurable.

Proof. Suppose that there exists $s = 1$ such that $\mathrm{Row}_{i_1}(\Gamma_1) \geqslant \mathbf{1}_k$. By the definition of Γ_1, $\mathrm{Row}_{i_1}(\Gamma_1) \geqslant \mathbf{1}_k$ means that all cycles are distinguishable. Obviously, BN (6) is observable and this is a minimal solution for s. Suppose that $s > 1$, and there exists an $s \in [1, n]$, such that $\mathrm{Row}_{i_1}(\Gamma_1) + \cdots + \mathrm{Row}_{i_s}(\Gamma_1) \geqslant \mathbf{1}_k$. Then cycle r is distinguishable. Obviously, BN (6) is observable. Assume that s is the minimal number, that is to say, there does not exist a $k < s$ such that $\mathrm{Row}_{i_1}(\Gamma_1) + \cdots + \mathrm{Row}_{i_s}(\Gamma_1) \geqslant \mathbf{1}_k$. Obviously, it holds for Theorem 7.

Remark 5. Note that BN (6) will be observable, when $s = n$, that is to say, all nodes are directly measurable, i.e., $\mathrm{Row}_1(\Gamma_1) + \mathrm{Row}_1(\Gamma_2) + \cdots + \mathrm{Row}_n(\Gamma_1) \geqslant \mathbf{1}_k$. In [43], a strategy has been developed to allocate sensors by determining the output equations, and all possible forms of observation matrices can be derived. In this paper, for a BN (1) with output (2), the observability can be determined by Theorem 3 and the optimal measurements can also be designed by Theorem 7 if BN (6) is unobservable. For a BN (1) without output, the optimal measurements can also be designed by Theorem 7, since states set $\Delta_{2^{2^n}}$ of (6) can be divided into C_0 and C_1.

Remark 6. Algorithm 2 below solves the minimal observability problem based on Theorem 7. In [42], an efficient algorithm has been presented for solving the minimal observability problem of BCNs. However, more measurements than the minimal number are indeed required. While in this paper, for the minimal observability of BNs, the minimal number of measurements can be determined. The total computational complexity proposed in Algorithm 2 is divided into two parts: the first part is a polynomial using the number of nodes of BNs (5), and the second part is a polynomial with the number k of cycles of BN (6). Therefore, the total computational complexity of Algorithm 2 is $O(m^2n2^{n-1})$.

Algorithm 2 This algorithm is given to determine the minimal number of nodes that are directly measurable and also the specific nodes $O(\cdot,\{\cdot\})$

1: **Initially mark a set** $S=\emptyset$;
2: **for** $i=1$ **do**
3:   Initialize $\mathrm{Row}_1(\Gamma_1),\mathrm{Row}_2(\Gamma_1),\ldots,\mathrm{Row}_n(\Gamma_1)$;
4:   $j=1$;
5:   **if** $\mathrm{Row}_j(\Gamma_1)\geqslant 1_m$ **then**
6:     Return $O(1,\{j\})$;
7:   **else**
8:     $j=j+1$;
9:   **end if**
10:   **if** $j=n+1$ **then**
11:     Go to 14;
12:   **else**
13:     Go to 4;
14:   **end if**
15:   $i=i+1$;
16:   **for** $i=i+1$ **do**
17:     Initialize an order from 1 to C_n^i for matrix Γ_1 as $\{w_j\}$: $\mathrm{Row}_{s_1}(\Gamma_1)+\mathrm{Row}_{s_2}(\Gamma_1)+\cdots+\mathrm{Row}_{s_i}(\Gamma_1), j=1,2,\ldots,\frac{n!}{i!(n-i)!}$;
18:     $j=1$;
19:     **if** $w_j:\mathrm{Row}_{s_1}(\Gamma_1)+\mathrm{Row}_{s_2}(\Gamma_1)+\cdots+\mathrm{Row}_{s_i}(\Gamma_1)\geqslant \mathbf{1}_k$ **then**
20:       Return $O(i,\{s_1,s_2,\ldots,s_i\})$ and $S=S\cup\{s_1,s_2,\ldots,s_i\}$;
21:     **else**
22:       $j=j+1$;
23:     **end if**
24:     **if** $j=\frac{n!}{i!(n-i)!}+1$ **then**
25:       Go to 14;
26:     **end if**
27:   **end for**
28: **end for**

Example 5. Reconsider BN (8) in Example 2. Based on Figure 5, there are two cycles m_1 and m_2, where $m_1=\{\delta_{64}^6,\delta_{64}^{23},\delta_{64}^{14},\delta_{64}^{31}\}$ and $m_2=\{\delta_{64}^2,\delta_{64}^{20}\}$. By Lemma 3, three sets $D_i, i=\{1,2,3\}$ whose states are distinguishable states under new measurement $y_1=x_i$ can be obtained. Obviously, $m_1\cap D_i\neq\emptyset$, $i=\{1,3\}$ and $m_1\cap D_3\neq\emptyset$. Then Γ_1 can be obtained as $\Gamma_1=[\delta_3^1+\delta_3^3,\delta_3^3]$. By Theorem 7, there exists $s=1$ such that $\mathrm{Row}_3(\Gamma_1)\geqslant\mathbf{1}_2$. Then making x_3 directly measurable yielding an observable BN.

Example 6. Consider a BN with four nodes and one output:

$$\begin{cases} x_1(*)=\neg x_1\wedge[x_2\vee(\neg x_2\wedge x_4)],\\ x_2(*)=x_2,\\ x_3(*)=[x_1\wedge((x_2\wedge x_3)\vee(\neg x_2\wedge(x_3\vee x_4)))]\vee[\neg x_1\wedge((x_2\wedge x_3)\vee(\neg x_2\wedge(x_3\wedge x_4)))],\\ x_4(*)=[x_1\wedge((x_2\wedge x_4)\vee(\neg x_2\wedge(x_3\vee x_4)))]\vee[\neg x_1\wedge((x_2\wedge x_4)\vee(\neg x_2\wedge x_3))],\\ y_1=x_1. \end{cases}\tag{12}$$

By Lemma 2, matrices L and H are obtained: $L=\delta_{16}[9,10,11,12,13,14,14,16,2,1,4,3,5,15,8,16]$, $H=\delta_1[1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2]$. Then matrix E can be obtained. Seven cycles are obtained, whose part of the directed graph of (12) in Example 6 is shown as Figure 6. Each cycle corresponds to the number m_1, m_2, m_3, m_4, m_5, m_6, m_7. By Lemma 3, four sets whose states are distinguishable under new measurement $y_{m+1}=x_i$, $i\in[1,4]$, can be obtained, and Γ_1 is given as $\Gamma_1=[\delta_4^3,\delta_4^4,\delta_4^3+\delta_4^4,\delta_4^2+\delta_4^4,\delta_4^4,\delta_4^2+\delta_4^3+\delta_4^4,\delta_4^3+\delta_4^4]$.

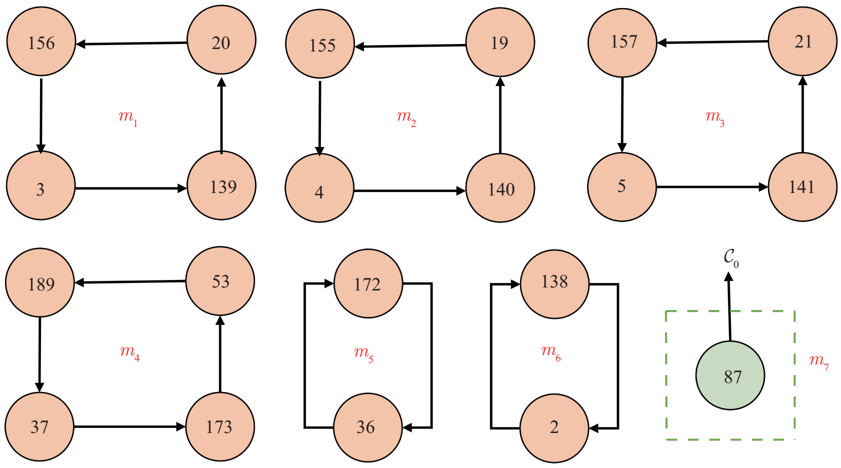

Figure 6 (Color online) Part of the directed graph of BN (12) in Example 6. The number i represents the index of state δ_{256}^i, and $m_i, i \in [1, 7]$ is the name of cycles.

Based on Algorithm 2, it can be found that $\text{Row}_3(\Gamma_1) + \text{Row}_4(\Gamma_1) \geqslant \mathbf{1}_7$. Then making x_3, x_4 directly measurable yielding the following BN:

$$\begin{cases} x_1(*) = \neg x_1 \wedge [x_2 \vee (\neg x_2 \wedge x_4)], x_2(*) = x_2, \\ x_3(*) = [x_1 \wedge ((x_2 \wedge x_3) \vee (\neg x_2 \wedge (x_3 \vee x_4)))] \vee [\neg x_1 \wedge ((x_2 \wedge x_3) \vee (\neg x_2 \wedge (x_3 \wedge x_4)))], \\ x_4(*) = [x_1 \wedge ((x_2 \wedge x_4) \vee (\neg x_2 \wedge (x_3 \vee x_4)))] \vee [\neg x_1 \wedge ((x_2 \wedge x_4) \vee (\neg x_2 \wedge x_3))], \\ y_1(t) = x_1(t), y_2(t) = x_3(t), y_3(t) = x_4(t). \end{cases} \tag{13}$$

It is straightforward to verify that BN (13) is indeed observable, and the measurement is a solution of the minimal observability problem.

4 Conclusion

In this study, we use the STP of matrices to investigate the minimum observability of BNs. The necessary and sufficient conditions for determining the observability of BNs were obtained. In addition, the minimum observability problem is defined. A procedure for designing an observer for unobservable BNs was also proposed, and a sufficient and necessary condition to determine the minimum number of nodes was obtained, which must be directly measurable. Finally, the effectiveness of the main results is illustrated using examples. The proposed method for determining observability will be considered for general BNs in future studies. In addition, we will consider the observability of BNs with disturbances.

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