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Stabilization of a class of congestion games via intermittent control

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Dear editor,

Congestion games, firstly proposed in [1], have been widely applied in many practical problems such as electric vehicles charging, road pricing, and wireless spectrum sharing. It was proved in [1] that a congestion game admits at least one pure Nash equilibrium (NE).

In reality, players may own different loads, which gives rise to weighted congestion games. In addition, players may also generate extra utilities by themselves, which brings more challenge to analyze the game. We call this model as the weighted congestion game with player-specific utilities (WCGPSU).

In contrast to existing studies on the existence and seeking of NEs of WCGPSU [2], this study focuses on the dynamic behaviors of evolutionary WCGPSU via semi-tensor product (STP) of matrices [3]. STP provides a convenient platform for the analysis and control of finite-valued dynamic systems, such as Boolean networks [4] and finite games [5].

In this study, the intermittent control of evolutionary WCGPSU is investigated, in which some players act as controllers only when some special profiles occur. This kind of control can reduce both the control time and the cost. It should be pointed out that the intermittent control is similar to the event-triggered control in [6, 7], where Ref. [6] considered the networked evolutionary games using external controllers and Ref. [7] focused on Boolean networks.

The main contribution of this study is as follows: (1) Using STP, the algebraic expression of the dynamic equation for the evolutionary WCGPSU is established and the property on NEs is presented. (2) By designing open-loop intermittent control and state feedback intermittent control, respectively, two necessary and sufficient conditions are obtained to stabilize the evolutionary WCGPSU to the NE. In fact, using our method, the game can be globally stabilized to any profile we want.

Notations. For a matrix A, $\mathrm{Col}(A)$ denotes the set of columns of A and $\mathrm{Col}_i(A)$ is the ith column of A. $\delta_k^i := \mathrm{Col}_i(I_k)$ and $\Delta_k := \{\delta_k^i \mid i = 1, 2, \ldots, k\}$. $L = [\delta_m^{i_1}, \delta_m^{i_2}, \ldots, \delta_m^{i_n}]$ is a logical matrix and can be briefly represented as $L = \delta_m[i_1, i_2, \ldots, i_n]$. $\mathcal{L}_{m\times n}$ denotes the set of $m \times n$ logical matrices. $|S|$ denotes the cardinality of the set S.

Throughout this article, STP is the default matrix product, and thus the STP symbol '$\ltimes$' is mostly omitted if no confusion arises. A brief introduction on STP is listed in Appendix A.

Problem formulation. A WCGPSU is defined as a tuple $G = (N, M, S_i, \Xi_j, w_i, r_i, C)$, where $N = \{1, 2, \ldots, n\}$ is the set of players and $M = \{1, 2, \ldots, m\}$ is the set of facilities. The player i chooses k_i facilities to achieve his/her purpose and the number of strategies that player i can choose is $K_i = C_m^{k_i}$. Let s_i denote player i's strategy and $S_i \subset 2^M$ is player i's strategy set. Then $s = (s_1, s_2, \ldots, s_n)$ and $S = \prod_{i=1}^{\mathrm{n}} S_i$ represent the profile and the profile set, respectively. $\Xi_j : \mathbb{R} \to \mathbb{R}$ is the utility function of facility $j \in M$ which only depends on the number of players choosing j and generally a non-decreasing function. w_i is the weight of player i which can influence player i's utility. r_i is the specific utility function of player i. $C = (c_1, c_2, \ldots, c_n)$, where c_i is the whole utility function of player i.

Let $n_j(s) := |\{i \mid j \in s_i\}|$ be the number of players choosing facility j in profile s, and then c_i is defined as

$$c_i(s) := r_i(s_i) + w_i \sum_{j\in s_i} \Xi_j(n_j(s)), \quad i = 1, 2, \ldots, n. \quad (1)$$

Next, using STP, we express (1) into matrix form. Let $\Xi_j(k) := \xi_k^j$ denote the congestion utility of facility j which is chosen by k players. The vector expression of Ξ_j is $\Xi_j = [\xi_1^j, \xi_2^j, \ldots, \xi_n^j]$, $j \in M$.

Stacking all Ξ_j together, one has $\Xi = [\Xi_1, \Xi_2, \ldots, \Xi_m]$.

Construct a matrix $D(s) = \mathrm{diag}(d_1(s), d_2(s), \ldots, d_m(s))$, where

$$d_i(s) = \begin{cases} \delta_n^{n_i(s)}, & n_i(s) \neq 0, \\ \mathbf{0}_n, & \text{otherwise}. \end{cases}$$

Set $E = [E^1, E^2, \ldots, E^{K_i}]$, where $E^{s_i} = [E_1(s_i), E_2(s_i), \ldots, E_m(s_i)]^{\mathrm{T}}$, and $E_j(s_i)$ is a characteristic function.

Put all player-specific utilities of player i together and get $R_i = [r_i(s_i^1), r_i(s_i^2), \ldots, r_i(s_i^{K_i})]$.

Hence, the algebraic expression of c_i is

$$c_i(s) = R_i s_i + w_i \Xi D(s) E s_i, \quad i = 1, 2, \ldots, n, \quad (2)$$

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where $s_i \in \Delta_{K_i}$.

Remark 1. As Ref. [2] pointed out that the WCGPSU is a weighted potential game with the weighted potential function:

$$P(s) = \frac{1}{w_i} \sum_{i \in N} R_i s_i + \sum_{j=1}^{m} \sum_{k=1}^{n_j(s)} \mathrm{Row}_j(\Xi \delta_n^k),$$

then the WCGPSU has at least one pure NE.

Algebraic expression of evolutionary WCGPSU. Let $x_i(t)$ denote the player i's strategy at time t, and then the dynamics of the evolutionary WCGPSU G can be described as

$$x_i(t+1) = f_i(x_1(t), x_2(t), \ldots, x_n(t)),\ i = 1, 2, \ldots, n. \quad (3)$$

In this study, the strategy updating rule (SUR) is chosen as the determined parallel myopic best response adjustment:

$$x_i(t+1) = \begin{cases} x_i(t), & x_i(t) \in \mathcal{O}_i(t), \\ \min \mathcal{O}_i(t), & x_i(t) \notin \mathcal{O}_i(t), \end{cases} \quad (4)$$

where $\mathcal{O}_i(t) = \arg\min_{x_i \in S_i} c_i(x_i(t), x_{-i}(t))$ is the best-response strategy set and $x_{-i} \in \prod_{j \neq i} S_j$.

Using STP, the dynamic equation (3) is converted to

$$x_i(t+1) = L_i x(t),\ i = 1, 2, \ldots, n, \quad (5)$$

where $x(t) = \ltimes_{i=1}^n x_i(t) \in \Delta_K, K = \prod_{i=1}^n K_i$, and L_i is the structure matrix of f_i.

Multiplying all dynamic equations (5), one has

$$x(t+1) = Lx(t), \quad (6)$$

where $L = *_{i=1}^n L_i \in \mathcal{L}_{K \times K}$ and $*$ is the Khari-Rao product.

In fact, the structure matrix L characterizes the NEs of the game.

Theorem 1. Considering the WCGPSU (6) with the SUR (4), then a profile δ_K^q is an NE if and only if $(L)_{qq} = 1$.

Stabilization of evolutionary WCGPSU. Though the WCGPSU possesses at least one NE, the dynamic system (6) may not surely evolve to the NE under the SUR (4). Next we try to stabilize the game to the NE by designing intermittent control, where some players act as controllers only when some special events occur and the game evolves according to the SUR (4), otherwise.

Let Ω denote the state set which can trigger the controllers and can be determined later. For the sake of statement, we look on the first l players as controllers when $x(t) \in \Omega$. Let $x^c(t) = \ltimes_{i=1}^l x_i(t)$, $u(t) = \ltimes_{i=1}^l u_i(t)$ and $x^{uc}(t) = \ltimes_{i=l+1}^n x_i(t)$. Then, one has

$$x^c(t+1) = \begin{cases} u(t), & x(t) \in \Omega, \\ L^1 x(t), & x(t) \notin \Omega, \end{cases}$$

$$x^{uc}(t+1) = L^2 x(t),$$

where $L^1 = *_{i=1}^l L_i$, $L^2 = *_{i=l+1}^n L_i$.

The whole system can be expressed as

$$x(t+1) = \begin{cases} L^* x(t) u(t), & x(t) \in \Omega, \\ Lx(t), & x(t) \notin \Omega, \end{cases} \quad (7)$$

where $\tilde{L} = (I_{K^l} \otimes L^2) W_{[K, K^l]} \in \mathcal{L}_{K \times (KK^l)}$, $K^l = \prod_{i=1}^l K_i$.

Divide $\tilde{L}$ into K parts, that is $\tilde{L} = [\tilde{L}_1, \tilde{L}_2, \ldots, \tilde{L}_K]$. Let $\delta_K^{j^*}$ denote the NE of system (6), and construct

$$S_{uc}^0 = \{\delta_K^j \mid \exists\ k \in\ [0, K-1]\ \text{s.t.}\ L^k \delta_K^j = \delta_K^{j^*}\}$$

as the profile set that can reach the NE freely.

Let $S_c^0 = S_{uc}^0$ and $\tilde{S}^i = S_c^i \cup S_{uc}^i, i \geqslant 1$, where

$$S_c^i = \{\delta_K^j \in \Delta_K \setminus \cup_{\mu=0}^{i-1} \tilde{S}^\mu \mid \tilde{S}^{i-1} \cap \mathrm{Col}(\tilde{L}_j) \neq \emptyset\},$$

$$S_{uc}^i = \{\delta_K^j \in \Delta_K \setminus ((\cup_{\mu=0}^{i-1} \tilde{S}^\mu) \cup S_c^i) \mid \exists\ \tau \in\ [1, K-1] \\ \text{s.t.}\ L^\tau \delta_K^j \in S_c^i\}.$$

The relationship of the above sets is depicted in Figure 1.

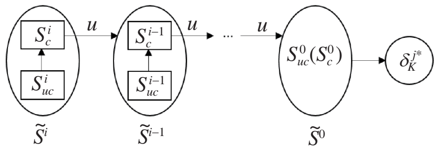

Figure 1 The relationship of S_c^i, S_{uc}^i and $\tilde{S}^i$.

Then one gets the control set $\Omega = \cup_{i=1}^\infty S_c^i$.

First, we consider the open-loop intermittent control. Let $u(t_1), u(t_2), \ldots, u(t_\rho)$ denote the free control sequence, where $t_1, t_2, \ldots, t_\rho,\ \rho \leqslant K$ are the control instants.

Theorem 2. The system (7) can be globally stabilized to the NE $\delta_K^{j^*}$ from any initial state $x(0)$ if and only if there exists a free control sequence $u(t_1), u(t_2), \ldots, u(t_\rho)$ such that

$$\mathrm{Col}_j(\tilde{L} L^{t_\rho - t_{\rho-1} - 1} \cdots \tilde{L} L^{t_2 - t_1 - 1} \tilde{L} L^{t_1} x(0)) \in S_{uc}^0$$

for some $j \in \{1, 2, \ldots, K(K^l)^\rho\}$.

Next, we design the state feedback intermittent control:

$$u(t) = Hx(t), \quad t = t_k, \quad (8)$$

where $H \in \mathcal{L}_{K^l \times K}$ is the designable gain matrix (see the design algorithm in Appendix D) and t_k is the control instant.

Theorem 3. The system (7) can be globally stabilized to the NE $\delta_K^{j^*}$ from any initial state $x(0)$ under the state feedback control (8) if and only if there exists a positive integer $T \in [1, K-1]$ such that $\cup_{i=0}^T \tilde{S}^i = \Delta_K$.

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Supporting information Appendixes A–E. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

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