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Finite-time boundedness analysis and composite anti-disturbance control for uncertain semi-Markovian jump systems with time delay

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Appendix A Proof of Theorem 1

Proof. Choose the Lyapunov-Krasovskii functional for the system in (4) and (5) as

$$V(\xi(t),i)=\sum_{l=1}^{5}V_l(\xi(t),i), \tag{A1}$$

where

$$\begin{aligned}
V_1(\xi(t),i)&=\xi^T(t)P_i\xi(t),\\
V_2(\xi(t),i)&=\int_{t-\tau}^{\tau}\xi^T(s)Q_{1i}\xi(s)ds,\\
V_3(\xi(t),t)&=\int_{-\tau}^{0}\int_{t+\theta}^{t}\xi^T(s)Q_1\xi(s)dsd\theta,\\
V_4(\xi(t),i)&=\int_{-\tau}^{0}\int_{t+\theta}^{t}\dot{\xi}^T(s)Q_{2i}\dot{\xi}(s)dsd\theta,\\
V_5(\xi(t),t)&=\int_{-\tau}^{0}\int_{0}^{\theta}\int_{t+\beta}^{t}\dot{\xi}^T(s)Q_2\dot{\xi}(s)dsd\beta d\theta,
\end{aligned}$$

where $P_i>0$, $Q_g>0$, $Q_{gi}>0$ and satisfy

$$\sum_{j=1}^{N}\lambda_{ij}(h)Q_{gj}\leqslant Q_g. \tag{A2}$$

According to the basic definition, the weak infinitesimal formula of $V(\cdot)$ can be described as:

$$\mathbb{L}V(\xi(t),i)=\lim_{\Delta\to 0}\frac{1}{\Delta}\{E(V(\xi(t+\Delta),\gamma_{t+\Delta})|\xi(t),\gamma_t=i)-V(\xi(t),i)\}, \tag{A3}$$

and then

$$\begin{aligned}
&\mathbb{L}V_1(\xi(t),i)\\
&=\lim_{\Delta\to 0}\frac{1}{\Delta}\{\sum_{j=1,j\neq i}^{N}P_r[\gamma_{t+\Delta}=j|\gamma_t=i]\xi^T(t+\Delta)P_j\xi(t+\Delta)+P_r[\gamma_{t+\Delta}=i|\gamma_t=i]\xi^T(t+\Delta)P_i\xi(t+\Delta)-\xi^T(t)P_i\xi(t)\}\\
&=\lim_{\Delta\to 0}\frac{1}{\Delta}\{\sum_{j=1,j\neq i}^{N}\frac{c_{ij}(G_i(h+\Delta)-G_i(h))}{1-G_i(h)}\xi^T(t+\Delta)P_j\xi(t+\Delta)+\frac{1-G_i(h+\Delta)}{1-G_i(h)}\xi^T(t+\Delta)P_i\xi(t+\Delta)-\xi^T(t)P_i\xi(t)\},
\end{aligned} \tag{A4}$$

where c_{ij} is probability intensity of the system jumping from subsystem i to j, $G_i(h)$ denotes the cumulative distribution function. According to the method the first-order approximation, for $\Delta\to 0$, $\xi(t+\Delta)$ can be written

$$\xi(t+\Delta)=\xi(t)+\dot{\xi}(t)\Delta+o(\Delta),$$

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$$\dot{\xi}(t) = \tilde{A}_i\xi(t) + \tilde{A}_{di}\xi(t-\tau) + \tilde{H}_i\tilde{d}(t) \quad \text{(A5)}$$

Then, we have

$$\xi(t+\Delta) = (I + \tilde{A}_i\Delta)\xi(t) + \tilde{A}_{di}\Delta\xi(t-\tau) + \tilde{H}_i\tilde{d}(t)\Delta + o(\Delta). \quad \text{(A6)}$$

According to (A4) and (A6), for $\lim\limits_{\Delta\to 0}\frac{G_i(h)-G_i(h+\Delta)}{1-G_i(h)} = 0$, we get

$$\mathbb{L}V_1(\xi(t),i) = \Upsilon^T(t)\begin{bmatrix} \Theta(i) & \frac{1-G_i(h+\Delta)}{1-G_i(h)}P_i\tilde{A}_i \\ * & 0 \end{bmatrix}\Upsilon(t), \quad \text{(A7)}$$

where $\Upsilon(t) \triangleq [\xi^T(t) \quad \xi^T(t-\tau) \quad \tilde{d}^T(t)]$,

$$\Theta(i) \triangleq \tilde{A}_i^TP_i + P_i\tilde{A}_i + \lim_{\Delta\to 0}\sum_{j=1,j\neq i}^{N}\frac{c_{ij}G_i(h)-G_i(h+\Delta)}{\Delta(1-G_i(h))}P_j + \lim_{\Delta\to 0}\frac{G_i(h)-G_i(h+\Delta)}{\Delta(1-G_i(h))}P_i.$$

We have the following equations

$$\begin{aligned}&\lim_{\Delta\to 0}\frac{1-G_i(h+\Delta)}{1-G_i(h)} = 1,\\ &\lim_{\Delta\to 0}\frac{G_i(h)-G_i(h+\Delta)}{\Delta(1-G_i(h))} = \lambda_i(h),\end{aligned} \quad \text{(A8)}$$

where $\lambda_{ij}(h) \triangleq c_{ij}\lambda_i(h)$, $i \neq j$, and $\lambda_{ij}(h) \triangleq -\sum\limits_{j=1,j\neq i}^{N}\lambda_{ij}(h)$. Substituting equation (A8) into equation (A7) leads to

$$\begin{aligned}\mathbb{L}V_1(\xi(t),i) = {}&\xi^T(t)(\sum_{j=1}^{N}P_j\lambda_{ij}(h))\xi(t) + \xi^T(t)P_i(\tilde{A}_i\xi(t) + \tilde{A}_{di}\xi(t-\tau) + \tilde{H}_i\tilde{d}(t))\\ &+ (\tilde{A}_i\xi(t) + \tilde{A}_{di}\xi(t-\tau)^T + \tilde{H}_i\tilde{d}(t))P_i\xi(t).\end{aligned} \quad \text{(A9)}$$

And then the same discussion can be given by formula (A3), one has

$$\begin{aligned}\mathbb{L}V_2(\xi(t),i) = {}&\lim_{\Delta\to 0}\frac{1}{\Delta}\{\sum_{j=1,j\neq i}^{N}\frac{c_{ij}G_i(h)-G_i(h+\Delta)}{\Delta(1-G_i(h))}\int_{t-\tau}^{t}\xi^T(s)Q_{1j}\xi(s)ds\\ &+\frac{1-G_i(h+\Delta)}{1-G_i(h)}\int_{t-\tau}^{t}\xi^T(s)Q_{1i}\xi(s)ds + \Delta\xi^T(s)Q_{1i}\xi(s) + o(\Delta) - V_2(\xi(t),i) - \xi^T(t-\tau)Q_{1i}\xi(t-\tau)\}\\ &=\sum_{j=1}^{N}\lambda_{ij}(h)\int_{t-\tau}^{t}\xi^T(s)Q_{1j}\xi(s)ds - \xi^T(t-\tau)Q_{1i}\xi(t-\tau) + \xi^T(t)Q_{1i}\xi(t).\end{aligned} \quad \text{(A10)}$$

Using the same method we can obtain

$$\begin{aligned}\mathbb{L}(\sum_{l=3}^{5}V_l(\xi(t),i)) = {}&\tau\xi^T(t)Q_1\xi(t) - \int_{t-\tau}^{t}\xi^T(s)Q_1\xi(s)ds - \int_{t-\tau}^{t}\dot{\xi}^T(s)Q_{2i}\dot{\xi}(s)ds + \tau\dot{\xi}^T(t)Q_{2i}\dot{\xi}(t)\\ &+\frac{\tau^2}{2}\dot{\xi}^T(t)Q_2\dot{\xi}(t) + \int_{-\tau}^{0}\int_{t+\theta}^{t}\dot{\xi}^T(s)(\sum_{j=1}^{N}\lambda_{ij}(h)Q_{2i} - Q_2)\dot{\xi}(s)dsd\theta.\end{aligned} \quad \text{(A11)}$$

According to the above discussions, the weak infinitesimal operator $V(\cdot)$ is given as

$$\begin{aligned}\mathbb{L}V(\xi(t),i,t) = {}&\xi^T(t)(\sum_{j=1}^{N}P_j\lambda_{ij}(h))\xi(t) + \xi^T(t)P_i(\tilde{A}_i\xi(t) + \tilde{A}_{di}\xi(t-\tau) + \tilde{H}_i\tilde{d}(t)) + (\tilde{A}_i\xi(t) + \tilde{A}_{di}\xi(t-\tau)^T\\ &+\tilde{H}_i\tilde{d}(t))P_i\xi(t) + \int_{t-\tau}^{t}\xi^T(s)(\sum_{j=1}^{N}\lambda_{ij}(h)Q_{1j} - Q_1)\xi(s)ds + \xi^T(t)Q_{1i}\xi(t)\\ &-\xi^T(t-\tau)Q_{1i}\xi(t-\tau) + \tau\xi^T(t)Q_1\xi(t) - \int_{t-\tau}^{t}\dot{\xi}^T(s)Q_{2i}\dot{\xi}(s)ds\\ &+\tau\dot{\xi}^T(t)Q_{2i}\dot{\xi}(t) + \frac{\tau^2}{2}\dot{\xi}^T(t)Q_2\dot{\xi}(t) + \int_{-\tau}^{0}\int_{t+\theta}^{t}\dot{\xi}^T(s)(\sum_{j=1}^{N}\lambda_{ij}(h)Q_{2j} - Q_2)\dot{\xi}(s)dsd\theta.\end{aligned} \quad \text{(A12)}$$

For the J_i, we set $\sum\limits_{j=1}^{N}J_i\lambda_{ij}(h) = 0$. According to lemma1, we can obtain

$$\sum_{j=1}^{N}\lambda_{ij}(h)P_j = \sum_{j=1}^{N}\lambda_{ij}(h)(P_j - J_i)$$

$$= \sum_{j\in S^i_{uk}}^{N} \lambda_{ij}(h)(P_j - J_i) + \sum_{j\in S^i_k}^{N} \lambda_{ij}(P_j - J_i) + \sum_{j\in S^i_k}^{N} \Delta\lambda_{ij}(h)(P_j - J_i)$$
$$= \sum_{j\in S^i_{uk}}^{N} \lambda_{ij}(h)(P_j - J_i) + \sum_{j\in S^i_k}^{N} \lambda_{ij}(P_j - J_i) + \sum_{j\in S^i_k}^{N} \frac{1}{2}\Delta\lambda_{ij}(h)((P_j - J_i) + (P_j - J_i)^T)$$
$$\leqslant \sum_{j\in S^i_{uk}}^{N} \lambda_{ij}(h)(P_j - J_i) + \sum_{j\in S^i_k}^{N} \lambda_{ij}(P_j - J_i) + \sum_{j\in S^i_k}^{N} (\frac{1}{4}k^2_{ij}I + (P_j - J_i)(P_j - J_i)^T). \tag{A13}$$

Similarly, for the J_{gi}, $\sum\limits_{j=1}^{N} J_{gi}\lambda_{ij}(h) = 0$, we can obtain

$$\sum_{j=1}^{N}\lambda_{ij}(h)Q_{gj} \leqslant \sum_{j\in S^i_{uk}}^{N} \lambda_{ij}(h)(Q_{gj} - J_{gi}) + \sum_{j\in S^i_k}^{N} \lambda_{ij}(Q_{gj} - J_{gi}) + \sum_{j\in S^i_k}^{N} (\frac{1}{4}k^2_{ij}I + (Q_{gj} - J_{gi})(Q_{gj} - J_{gi})^T). \tag{A14}$$

For appropriate dimensions N_{1i}, N_{2i}, N_{3i}, the following equations hold

$$2\tilde{\Upsilon}^T(t)N_{Ni}[\xi(t) - \xi(t-\tau) - \int_{t-\tau}^{t} \dot{\xi}(s)ds] = 0,$$
$$2[\xi^T(t)P_i + \dot{\xi}^T(t)P_i][-\dot{\xi}(t) + \tilde{A}_i\xi(t) + \tilde{A}_{di}\xi(t-\tau) + \tilde{H}_i\tilde{d}(t)] = 0, \tag{A15}$$

where

$$\tilde{\Upsilon}(t) = [\xi^T(t) \quad \xi^T(t-\tau) \quad \dot{\xi}^T(t) \quad \tilde{d}^T(t)]^T,$$
$$N_{Ni} = [N^T_{1i} \quad N^T_{2i} \quad N^T_{3i}]^T.$$

According to the analysis, for the J_i, J_{gi}, $\sum\limits_{j=1}^{N} J_i\lambda_{ij}(h) = 0$, $\sum\limits_{j=1}^{N} J_{gi}\lambda_{ij}(h) = 0$, we obtain that

$$\begin{aligned}
&z^T(t)z(t) - \gamma^2\tilde{d}^T(t)\tilde{d}(t) + \Gamma V(\xi(t), i)\\
&\leqslant z^T(t)z(t) - \gamma^2\tilde{d}^T(t)\tilde{d}(t) + \xi^T(t)P_i(\tilde{A}_i\xi(t) + \tilde{A}_{di}\xi(t-\tau) + \tilde{H}_i\tilde{d}(t)) + (\tilde{A}_i\xi(t) + \tilde{A}_{di}\xi(t-\tau)^T\\
&+ \tilde{H}_i\tilde{d}(t))P_i\xi(t)) + \xi^T(t)(\sum_{j\in S^i_{uk}}^{N} \lambda_{ij}(h)(P_j - J_i) + \sum_{j\in S^i_k}^{N} \lambda_{ij}(P_j - J_i) + \sum_{j\in S^i_k}^{N} (\frac{1}{4}k^2_{ij}I + (P_j - J_i)(P_j - J_i)^T))\xi(t)\\
&+ \int_{t-\tau}^{t} \xi^T(s)(\sum_{j=1}^{N}\lambda_{ij}(h)Q_{1j} - Q_1)\xi(s)ds + \xi^T(t)Q_{1i}\xi(t) - \xi^T(t-\tau)Q_{1i}\xi(t-\tau) + \tau\xi^T(t)Q_1\xi(t) - \int_{t-\tau}^{t} \dot{\xi}^T(s)Q_{2i}\dot{\xi}(s)ds\\
&+ \tau\dot{\xi}^T(t)Q_{2i}\dot{\xi}(t) + \frac{\tau^2}{2}\dot{\xi}^T(t)Q_2\dot{\xi}(t) + \int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{\xi}^T(s)(\sum_{j=1}^{N}\lambda_{ij}Q_{2j} - Q_2)\dot{\xi}(s)dsd\theta,\\
&\leqslant \tilde{\Upsilon}^T[\Pi_{2i} + \tau[N_{Ni}, 0]Q^{-1}_{2i}[N_{Ni}, 0]^T]\tilde{\Upsilon},
\end{aligned} \tag{A16}$$

where,

$$\Pi_{2i} = \begin{bmatrix} \Pi^{11}_{2i} & \Pi^{12}_{1i} & \Pi^{13}_{1i} & P_i\tilde{H}_i \\ * & \Pi^{22}_{1i} & \Pi^{23}_{1i} & 0 \\ * & * & \Pi^{33}_{1i} & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix},$$

$$\Pi^{11}_{2i} = P_i\tilde{A}_i + \tilde{A}^T_iP_i + Q_{1i} + \tau Q_1 + N_{1i} + N^T_{1i} + \sum_{j\in S^i_k} \lambda_{ij}(P_j - J_i) + \sum_{j\in S^i_k} (\frac{1}{4}k^2_{ij}I + (P_j - J_i)(P_j - J_i)^T)) + \tilde{C}^T_i\tilde{C}_i,$$

a). From the conditions (6)-(8), if $\forall i \in S^i_k$, and $\lambda_{ij}(h) \geqslant 0(\forall i, j \in S, i \neq j)$, we have

$$\mathbb{L}V(\xi(t), i) < \alpha E\{V(\xi(t), i)\} + \gamma^2\tilde{d}^T(t)\tilde{d}(t) - E\{z(t)^Tz(t)\}, \tag{A17}$$

b). From the conditions (6)-(9), if $\forall i \in S^i_{uk}$, and $\lambda_{ii}(h) = -\sum\limits_{j=1, j\neq i}^{N} \lambda_{ij}(h) < 0$, we also have (A17).

Then multiply the two sides of (A17) by $e^{-\alpha t}$, we can obtain

$$\mathbb{L}(e^{-\alpha t}V(\xi(t), i)) < E\{e^{-\alpha t}[\gamma^2\tilde{d}^T(t)\tilde{d}(t) - z(t)^Tz(t)]\}. \tag{A18}$$

Therefore, for $\forall t \in [0 \quad T]$, integrating the inequality, and we can obtain the following inequality

$$E\{e^{-\alpha t}V(\xi(t), i)\} < E\{\int_0^T e^{-\alpha t}[\gamma^2\tilde{d}^T(t)\tilde{d}(t) - z(t)^Tz(t)]\}ds, \tag{A19}$$

then,

$$E\{e^{-\alpha t}z(t)^T z(t))\} < \gamma^2 \int_0^T e^{-\alpha t}\tilde{d}^T(t)\tilde{d}(t)dt, \tag{A20}$$

it results that

$$E\{\int_0^T z(t)^T z(t)dt\} \leqslant \gamma^2 e^{\alpha T}\{\int_0^T \tilde{d}(t)^T \tilde{d}(t)dt\}. \tag{A21}$$

Therefore, for $\forall t \in [0 \quad T]$, the condition of finite-time boundedness holds, and the performance index of H_∞ is that $\bar{\gamma} = \sqrt{e^{\alpha T}}\gamma$.

Then, we define that $\tilde{P}_i = R_i^{-1/2} P_i R_i^{-1/2}$, $\tilde{Q}_{gi} = R_i^{-1/2} Q_g R_i^{-1/2}$, $\tilde{\tilde{Q}}_{gi} = R_i^{-1/2} Q_{gi} R_i^{-1/2}$. One has

$$\begin{aligned}
&E\xi^T(t)P_i\xi(t) \leqslant EV(\xi(t), i)\\
&< \frac{\gamma^2 d}{\alpha} e^{\alpha t}\int_0^t e^{-\alpha s}ds + e^{\alpha t}EV(\xi_0, \gamma_0)\\
&< e^{\alpha t}[EV(\xi_0, \gamma_0) + \frac{\gamma^2 d}{\alpha}(1 - e^{-\alpha t})]\\
&< e^{\alpha t}\{c_1(\sigma_{max}[\tilde{P}_i] + \tau\sigma_{max}[\tilde{Q}_{1i}] + \frac{1}{2}\tau^2\sigma_{max}[\tilde{\tilde{Q}}_{1i}]) + \tilde{c}_1(\frac{1}{2}\tau^2\sigma_{max}[\tilde{Q}_{2i}] + \frac{1}{6}\tau^3\sigma_{max}[\tilde{\tilde{Q}}_{2i}])\\
&+ \frac{\gamma^2 d}{\alpha}(1 - e^{-\alpha t})\},
\end{aligned} \tag{A22}$$

and

$$E\{\xi^T(t)P_i\xi(t)\} \geqslant \sigma_{min}[\tilde{P}_i]E\{\xi^T(t)R_i\xi(t)\}. \tag{A23}$$

Therefore, according to (10), we can obtain

$$\begin{aligned}
&E\{\xi^T(t)R_i\xi(t)\}\\
&\leqslant e^{\alpha T}\{c_1(\sigma_{max}[\tilde{P}_i] + \tau\sigma_{max}[\tilde{Q}_{1i}] + \frac{1}{2}\tau^2\sigma_{max}[\tilde{\tilde{Q}}_{1i}]) + \tilde{c}_1(\frac{1}{2}\tau^2\sigma_{max}[\tilde{Q}_{2i}] + \frac{1}{6}\tau^3\sigma_{max}[\tilde{\tilde{Q}}_{2i}]) + \frac{\gamma^2 d}{\alpha}(1 - e^{-\alpha T})\}/\sigma_{min}[\tilde{P}_i]\\
&\leqslant c_2,
\end{aligned} \tag{A24}$$

which means that composite control system in (4) and (5) is finite-time boundedness with respect to $(c_1 \quad c_2 \quad T \quad R_i \quad d)$.

The proof is completed.

Appendix B Proof of Theorem 2

For Theorem 2 where

$$\begin{aligned}
&\Pi_{d1i}^{11} = \begin{bmatrix} \Pi_{d1i}^{1111} & \Pi_{d1i}^{1112} \\ * & \Pi_{d1i}^{1122} \end{bmatrix}, \tilde{\Pi}_{d1i}^{11} = \begin{bmatrix} \tilde{\Pi}_{d1i}^{1111} & \Pi_{d1i}^{1112} \\ * & \Pi_{d1i}^{1122} \end{bmatrix},\\
&\Pi_{d1i}^{1111} = A_iX_{1i} + X_{1i}A_i^T + B_iY_i + Y_i^TB_i^T + U_{11ii} + \tau U_{11i} + \tilde{N}_{11i} + \tilde{N}_{11i}^T\\
&\quad + (\lambda_{ii} - \alpha)X_{1i} - \sum_{j\in S_k^i}\lambda_{ij}\tilde{V}_{1i} + \sum_{j\in S_k^i}\frac{1}{4}k_{ij}^2\tilde{X}_{1i} + \beta_iE_{11i}E_{11i}^T,\\
&\tilde{\Pi}_{d1i}^{1111} = A_iX_{1i} + X_{1i}A_i^T + B_iY_i + Y_i^TB_i^T + U_{11ii} + \tau U_{11i} + \tilde{N}_{11i} + \tilde{N}_{11i}^T\\
&\quad - \sum_{j\in S_k^i}\lambda_{ij}\tilde{V}_{1i} + \sum_{j\in S_k^i}\frac{1}{4}k_{ij}^2\tilde{X}_{1i} - \alpha X_{1i} + \beta_iE_{11i}E_{11i}^T,\\
&\Pi_{d1i}^{1112} = B_iV_i,\\
&\Pi_{d1i}^{1122} = P_{2i}W_i + H_{2i}B_iV_i + W_i^TP_{2i} + V_i^TB_i^TH_{2i}^T + U_{21ii} + \tau U_{21i}\\
&\quad + \tilde{N}_{21i} + \tilde{N}_{21i}^T + \sum_{j\in S_k^i}\lambda_{ij}(P_{2j} - \tilde{V}_{2i}) + \sum_{j\in S_k^i}\frac{1}{4}k_{ij}^2I - \alpha P_{2i},\\
&\Pi_{d1i}^{12} = diag\{A_{di}X_{1i} - \tilde{N}_{11i} + \tilde{N}_{12i}^T, -\tilde{N}_{21i} + \tilde{N}_{22i}^T\}, \Pi_{d1i}^{13} = \begin{bmatrix} \Pi_{d1i}^{1311} & \Pi_{d1i}^{1312} \\ * & \Pi_{d1i}^{1322} \end{bmatrix},\\
&\Pi_{d1i}^{1311} = X_{1i}A_i^T + \tilde{N}_{13i} + \beta_iE_{11i}E_{12i}^T, \Pi_{d1i}^{1312} = B_iV_i, \Pi_{d1i}^{1322} = W_i^TP_{2i} + V_i^TB_i^TH_{2i}^T + \tilde{N}_{23i}^T,\\
&\Pi_{d1i}^{14} = \begin{bmatrix} H_i & 0 \\ H_{2i}H_i & P_{2i}M_i \end{bmatrix}, \Pi_{d1i}^{15} = \begin{bmatrix} X_{1i}C_{1i}^T \\ C_{2i}^T \end{bmatrix}, \Pi_{d1i}^{16} = diag\{\tau\tilde{N}_{11i}^T, \tau\tilde{N}_{21i}^T\},\\
&\Pi_{d1i}^{17} = [diag\{X_{1ik_1^i} - \tilde{V}_{1i}, P_{2ik_1^i} - \tilde{V}_{2i}\}, ..., diag\{X_{1ik_1^i} - \tilde{V}_{1i}, P_{2ik_m^i} - \tilde{V}_{2i}\}], \Pi_{d1i}^{18} = diag\{X_{1i}F_{11i}, 0\},
\end{aligned}$$

$$\Pi_{d1i}^{19}=diag\{\Pi_{d1i}^{191},0\},\Pi_{d1i}^{191}=[\sqrt{\lambda_{ik_1^i}}X_{1i},...,\sqrt{\lambda_{ik_{r-1}^i}}X_{1i},\sqrt{\lambda_{ik_{r+1}^i}}X_{1i},...,\sqrt{\lambda_{ik_m^i}}X_{1i}],$$
$$\tilde{\Pi}_{d1i}^{19}=diag\{\tilde{\Pi}_{d1i}^{191},0\},\tilde{\Pi}_{d1i}^{191}=[\sqrt{\lambda_{ik_1^i}}X_{1i},...,\sqrt{\lambda_{ik_m^i}}X_{1i}],$$
$$\Pi_{d1i}^{22}=diag\{-U_{11ii}-\tilde{N}_{12i}-\tilde{N}_{12i}^T,-U_{21ii}-\tilde{N}_{22i}-\tilde{N}_{22i}^T\},\Pi_{d1i}^{23}=diag\{X_{1i}A_{di}^T-\tilde{N}_{13i}^T,-\tilde{N}_{23i}^T\},$$
$$\Pi_{d1i}^{26}=diag\{\tau\tilde{N}_{12i}^T,\tau\tilde{N}_{22i}^T\},\Pi_{d1i}^{28}=diag\{F_{12i},0\},$$
$$\Pi_{d1i}^{33}=diag\{-2X_{1i}+\tau U_{12ii}+\frac{\tau^2}{2}U_{12i}+\beta_iE_{12i}E_{12i}^T,-2P_{2i}+\tau U_{22ii}+\frac{\tau^2}{2}U_{22i}\}$$
$$\Pi_{d1i}^{36}=diag\{\tau\tilde{N}_{13i}^T,\tau\tilde{N}_{23i}^T\},\Pi_{d1i}^{44}=-\gamma^2I,\Pi_{d1i}^{55}=-I,\Pi_{d1i}^{66}=diag\{-\tau U_{12ii},-\tau U_{22ii}\},$$
$$\Pi_{d1i}^{77}=diag\{-\tilde{X}_{ik_1^i},...,-\tilde{X}_{ik_m^i}\},\Pi_{d1i}^{88}=-\beta_iI,\Pi_{d1i}^{99}=diag\{-X_{1k_1^i},...,-X_{1k_{r-1}^i},-X_{1k_{r+1}^i},...,-X_{1k_m^i}\},$$
$$\tilde{\Pi}_{d1i}^{99}=diag\{-X_{1k_1^i},...,-X_{1k_m^i}\},\Pi_{d3i}^{12}=diag\{U_{1gij}-\tilde{V}_{1gi},U_{2gij}-\tilde{V}_{2gi}\},$$
$$\Pi_{d3i}^{11}=diag\{\sum_{j\in S_k^i}\lambda_{ij}(U_{1gij}-\tilde{V}_{1gi})+\sum_{j\in S_k^i}\frac{1}{4}k_{ij}^2\tilde{X}_{1i}-U_{1gi},\sum_{j\in S_k^i}\lambda_{ij}(U_{2gij}-\tilde{V}_{2gi})+\sum_{j\in S_k^i}\frac{1}{4}k_{ij}^2\tilde{X}_{1i}-U_{2gi}\},$$

Proof. Let

$$P_i=\begin{bmatrix}P_{1i}&0\\ *&P_{2i}\end{bmatrix},X_{1i}=P_{1i}^{-1},Y_i=K_iX_{1i},H_{2i}=P_{2i}L_i,X_{1i}X_{1i}=\tilde{X}_{1i},X_i=\begin{bmatrix}X_{1i}&0\\ *&I\end{bmatrix},$$
$$N_{pi}=\begin{bmatrix}N_{1pi}&0\\ *&N_{2pi}\end{bmatrix},\tilde{N}_{pi}=X_iN_{pi}X_i=\begin{bmatrix}\tilde{N}_{1pi}&0\\ *&\tilde{N}_{2pi}\end{bmatrix},\tilde{V}_i=X_iJ_iX_i=\begin{bmatrix}\tilde{V}_{1i}&0\\ *&\tilde{V}_{2i}\end{bmatrix},$$
$$\tilde{V}_{gi}=X_iJ_{gi}X_i=\begin{bmatrix}\tilde{V}_{1gi}&0\\ *&\tilde{V}_{2gi}\end{bmatrix},Q_g=\begin{bmatrix}Q_{1g}&0\\ *&Q_{2g}\end{bmatrix},Q_{gi}=\begin{bmatrix}Q_{1gi}&0\\ *&Q_{2gi}\end{bmatrix},$$
$$U_{gi}=X_iQ_gX_i=\begin{bmatrix}U_{1gi}&0\\ *&U_{2gi}\end{bmatrix},U_{gij}=X_iQ_{gi}X_i=\begin{bmatrix}U_{1gij}&0\\ *&U_{2gij}\end{bmatrix},$$
$$E_{1i}=\begin{bmatrix}E_{11i}&0\\ *&0\end{bmatrix},E_{2i}=\begin{bmatrix}E_{12i}&0\\ *&0\end{bmatrix},F_{1i}=\begin{bmatrix}F_{11i}&0\\ *&0\end{bmatrix},F_{2i}=\begin{bmatrix}F_{12i}&0\\ *&0\end{bmatrix}.$$

Due to the existence of uncertain parameters in inequality (6), we can't solve the disturbance-observer-based controller directly. We divide the inequality (6) into two parts: the part without uncertain parameters and the part with uncertain parameters. Therefore, inequality (6) can be written as

$$\Pi_{ui}+\Delta\Pi_{ui}<0, \tag{B1}$$

where

$$\Pi_{ui}=\begin{bmatrix}\Pi_{ui}^{11}&\Pi_{ui}^{12}&\Pi_{ui}^{13}&P_i\tilde{H}_i&\tilde{C}_i^T&\tau N_{1i}^T&\Pi_{1i}^{17}\\ *&\Pi_{1i}^{22}&\Pi_{ui}^{23}&0&0&\tau N_{2i}^T&0\\ *&*&\Pi_{1i}^{33}&0&0&\tau N_{3i}^T&0\\ *&*&*&-\gamma^2I&0&0&0\\ *&*&*&*&-I&0&0\\ *&*&*&*&*&-\tau Q_{2i}&0\\ *&*&*&*&*&*&\Pi_{1i}^{77}\end{bmatrix},\Delta\Pi_{ui}=\begin{bmatrix}\Delta\Pi_{ui}^{11}&\Delta\Pi_{ui}^{12}&*&0&0&0&0\\ *&0&*&0&0&0&0\\ \Delta\Pi_{ui}^{31}&\Delta\Pi_{ui}^{32}&0&0&0&0&0\\ *&*&*&0&0&0&0\\ *&*&*&*&0&0&0\\ *&*&*&*&*&0&0\\ *&*&*&*&*&*&0\end{bmatrix},$$

$$\Pi_{ui}^{11}=P_iA_{ui}+A_{ui}^TP_i+Q_{1i}+\tau Q_1+N_{1i}+N_{1i}^T+\sum_{j\in S_k^i}\lambda_{ij}(P_j-J_i)+\sum_{j\in S_k^i}\frac{1}{4}k_{ij}^2I-\alpha P_i,$$
$$\Pi_{ui}^{12}=P_iA_{dui}-N_{1i}+N_{2i}^T,\Pi_{ui}^{13}=A_{ui}^TP_i+N_{3i}^T,\Pi_{ui}^{23}=A_{dui}^TP_i-N_{3i}^T,$$
$$A_{ui}=\begin{bmatrix}A_i+B_iK_i&B_iV_i\\0&W_i+L_iB_iV_i\end{bmatrix},A_{dui}=\begin{bmatrix}A_{di}&0\\0&0\end{bmatrix},\Delta\Pi_{ui}^{11}=\begin{bmatrix}P_i\Delta A_i+\Delta A_i^TP_i&0\\0&0\end{bmatrix},\Delta\Pi_{ui}^{12}=\begin{bmatrix}P_i\Delta A_{di}&0\\0&0\end{bmatrix},$$
$$\Delta\Pi_{ui}^{31}=\begin{bmatrix}P_i\Delta A_i&0\\0&0\end{bmatrix},\Delta\Pi_{ui}^{32}=\begin{bmatrix}P_i\Delta A_{di}&0\\0&0\end{bmatrix}.$$

Let $\Gamma_i(t)$ stands for uncertain matrix function whose elements is Lebesgue measurable, and $\Gamma_i^T(t)\Gamma_i(t)\leqslant I$, and we can obtain

$$\Delta\Pi_{ui}=M_{11}\Gamma_i(t)M_{12}+M_{12}^T\Gamma_i^T(t)M_{11}^T<\beta_iM_{11}M_{11}^T+\beta_i^{-1}M_{12}^TM_{12}, \tag{B2}$$

where

$$M_{11} = col[P_iE_{1i} \quad 0 \quad P_iE_{2i} \quad 0 \quad 0 \quad 0 \quad 0],$$
$$M_{12} = [F_{1i} \quad F_{2i} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0].$$

Then, we can redescribe the inequality (B1) as

$$\begin{bmatrix} \Phi_{1i}^{11} & \Pi_{ui}^{12} & \Phi_{1i}^{13} & P_i\tilde{H}_i & \tilde{C}_i^T & \tau N_{1i}^T & \Pi_{1i}^{17} & F_{1i} \\ * & \Pi_{1i}^{22} & \Pi_{ui}^{23} & 0 & 0 & \tau N_{2i}^T & 0 & F_{2i} \\ * & * & \Phi_{1i}^{33} & 0 & 0 & \tau N_{3i}^T & 0 & 0 \\ * & * & * & -\gamma^2 I & 0 & 0 & 0 & 0 \\ * & * & * & * & -I & 0 & 0 & 0 \\ * & * & * & * & * & -\tau Q_{2i} & 0 & 0 \\ * & * & * & * & * & * & \Pi_{1i}^{77} & 0 \\ * & * & * & * & * & * & * & -\beta_i I \end{bmatrix} < 0, \tag{B3}$$

where

$$\Phi_{1i}^{11} = P_iA_{ui} + A_{ui}^TP_i + Q_{1i} + \tau Q_1 + N_{1i} + N_{1i}^T + \sum_{j\in S_k^i}\lambda_{ij}(P_j - J_i) + \sum_{j\in S_k^i}\frac{1}{4}k_{ij}^2 I - \alpha P_i + \beta_i P_iE_{1i}E_{1i}^TP_i,$$

$$\Phi_{1i}^{13} = A_{ui}^TP_i + N_{3i}^T + \beta_iP_iE_{1i}E_{2i}^TP_i, \Phi_{1i}^{33} = -2P_i + \tau Q_{2i} + \frac{\tau^2}{2}Q_2 + \beta_iP_iE_{2i}E_{2i}^TP_i.$$

Simultaneous left- and right- multiply on both sides of the inequality (B3) by $diag\{X_i, X_i, X_i, I, I, X_i, X_i, I\}$, we can obtain

$$\begin{bmatrix} \Pi_{d2i}^{11} & \Pi_{d1i}^{12} & \Pi_{d1i}^{13} & \Pi_{d1i}^{14} & \Pi_{d1i}^{15} & \Pi_{d1i}^{16} & \Pi_{d1i}^{17} & \Pi_{d1i}^{18} \\ * & \Pi_{d1i}^{22} & \Pi_{d1i}^{23} & 0 & 0 & \Pi_{d1i}^{26} & 0 & \Pi_{d1i}^{28} \\ * & * & \Pi_{d1i}^{33} & 0 & 0 & \Pi_{d1i}^{36} & 0 & 0 \\ * & * & * & \Pi_{d1i}^{44} & 0 & 0 & 0 & 0 \\ * & * & * & * & \Pi_{d1i}^{55} & 0 & 0 & 0 \\ * & * & * & * & * & \Pi_{d1i}^{66} & 0 & 0 \\ * & * & * & * & * & * & \Pi_{d1i}^{77} & 0 \\ * & * & * & * & * & * & * & \Pi_{d1i}^{88} \end{bmatrix} < 0, \tag{B4}$$

where

$$\Pi_{d2i}^{11} = \begin{bmatrix} \Pi_{d2i}^{1111} & \Pi_{d1i}^{1112} \\ * & \Pi_{d1i}^{1122} \end{bmatrix},$$

$$\Pi_{d2i}^{1111} = A_iX_{1i} + X_{1i}A_i^T + B_iY_i + Y_i^TB_i^T + U_{11ij} + \tau U_{11i} + \tilde{N}_{11i} + \tilde{N}_{11i}^T + \sum_{j\in S_k^i}\lambda_{ij}(X_{1i}X_{1j}^{-1}X_{1i} - \tilde{V}_{1i}) + \sum_{j\in S_k^i}\frac{1}{4}X_{1i}k_{ij}^2X_{1i} - \alpha X_{1i} + \beta_iE_{1i}E_{1i}^T,$$

and Π_{d1i}^{1112}, Π_{d1i}^{1122}, Π_{d1i}^{12}, Π_{d1i}^{13}, Π_{d1i}^{14}, Π_{d1i}^{15}, Π_{d1i}^{16}, Π_{d1i}^{17}, Π_{d1i}^{18}, Π_{d1i}^{22}, Π_{d1i}^{23}, Π_{d1i}^{26}, Π_{d1i}^{28}, Π_{d1i}^{33}, Π_{d1i}^{36}, Π_{d1i}^{44}, Π_{d1i}^{55}, Π_{d1i}^{66}, Π_{d1i}^{77}, Π_{d1i}^{88} are given in Theorem 2.

For $\forall i \in S$, inequality (B4) holds with the following two cases:

a. if $i \in S_k^i$, use the shur lemma, we obtain (11).

b. if $i \notin S_k^i$, use the shur lemma, we obtain (12).

Simultaneous left- and right- multiply on both sides of the inequality (7) by $diag\{X_i, X_i\}$, we obtain (13).

Simultaneous left- and right- multiply on both sides of the inequality (8),(9) by X_i, we obtain (14), (15).

Consider $\tilde{P}_i = R_i^{-1/2}P_iR_i^{-1/2}$, $\tilde{Q}_{gi} = R_i^{-1/2}Q_gR_i^{-1/2}$, $\tilde{\tilde{Q}}_{gi} = R_i^{-1/2}Q_{gi}R_i^{-1/2}$, and $1 \leqslant \sigma_{min}[\tilde{P}_i]$, $\sigma_{max}[\tilde{P}_i] < \sigma_1$, $\sigma_{max}[Q_{1i}] < \sigma_2$, $\sigma_{max}[Q_1] < \sigma_3$, $\sigma_{max}[Q_{2i}] < \sigma_4$, $\sigma_{max}[Q_2] < \sigma_5$.

According to inequality (10), we can obviously get inequality (16), (17).

That hereby certifies that the theorem has passed proof as above.

Appendix C Numerical Example

Cognitive radio (CR) has been applied in S-MJSs in [1]. Cognitive radio link can be used between sensor and actuator. Many independent channels exist in CR system. Each channel which is selected by the sensor obey special sensing policy.

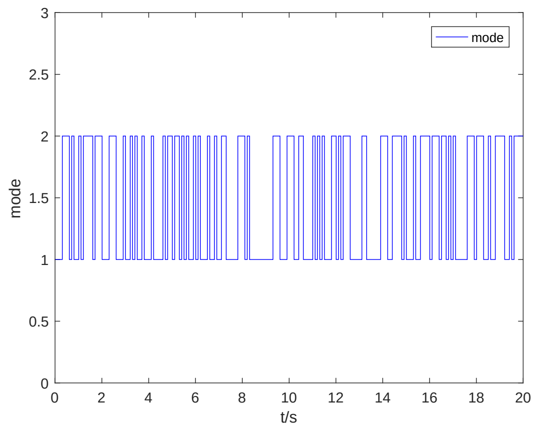

Figure C1 System mode

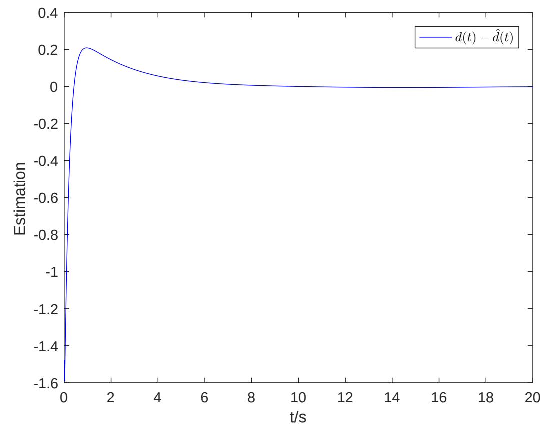

Figure C2 Estimation error

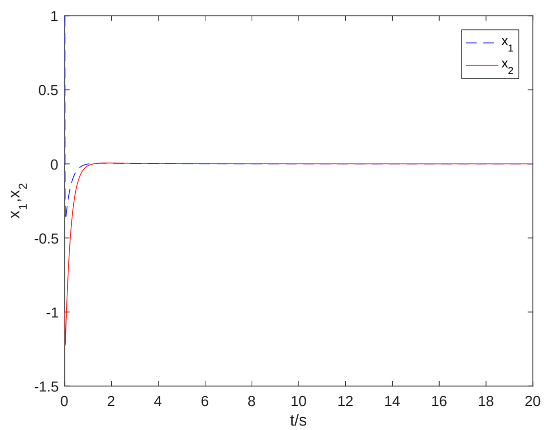

Figure C3 State trajectory

Signal can be transmitted normally when the channel is idle, and it cannot be transmitted to avoid collision when the channel is busy. Each channel of CR system has two states, and which state the channel stays corresponds to semi-Markovian process. The semi-markovian model of the CR system is with the parameters given as

$$A_1=\begin{bmatrix}-4.0 & 1.0\\ -12.0 & -2.0\end{bmatrix}, A_2=\begin{bmatrix}-2.1 & 10.1\\ 5.2 & -4.1\end{bmatrix}, A_{d1}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, A_{d2}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, B_1=\begin{bmatrix}1.2\\ 0.5\end{bmatrix}, B_2=\begin{bmatrix}1.2\\ 0.3\end{bmatrix},$$
$$H_1=H_2=\begin{bmatrix}0.01\\ 0.01\end{bmatrix}, C_{11}=\begin{bmatrix}0.5 & 0.1\end{bmatrix}, C_{12}=\begin{bmatrix}1.2 & 0.1\end{bmatrix}, C_{21}=\begin{bmatrix}0.1 & 0\end{bmatrix}, C_{22}=\begin{bmatrix}0 & 0.1\end{bmatrix}.$$

We assume that the disturbance $d(t)$ is described as Assumption 1, and $d_1(t), d_2(t)$ are the random disturbance whose 2-norm is less than 1. The parameters of $d(t)$ are given as $W_1=W_2=\begin{bmatrix}0 & 0.5\\ -0.5 & 0\end{bmatrix}, V_1=\begin{bmatrix}2.0 & 0\end{bmatrix}, V_2=\begin{bmatrix}1.0 & 0\end{bmatrix}, M_1=\begin{bmatrix}0.1\\ 0.1\end{bmatrix}, M_2=\begin{bmatrix}0.2\\ 0.2\end{bmatrix}$. The uncertain factors are given as $E_{11i}=\begin{bmatrix}0.1\\ 0.1\end{bmatrix}, E_{12i}=\begin{bmatrix}0.2\\ -0.2\end{bmatrix}, F_{11i}=\begin{bmatrix}0.1 & 0.1\end{bmatrix}, F_{12i}=\begin{bmatrix}0.1 & 0\end{bmatrix}$. The TRs matrix is given as$\lambda_{ij}(h)=\begin{bmatrix}? & ?\\ 2.0+\Delta\lambda_{21}(h) & -2.0+\Delta\lambda_{22}(h)\end{bmatrix}$, where, $\Delta\lambda_{ij}(h)\leqslant k_{ij}=|0.1\times\lambda_{ij}(h)|$. Let $\alpha=1$, $c_1=0.5$, $\tilde{c}_1=0.6$, $c_2=2.1$, $T=2$, $d=3$.

The control gains can be obtained by the Theorem 2 as follows:

$$K_1=\begin{bmatrix}-142.5103 & -8.1335\end{bmatrix}, K_2=\begin{bmatrix}-77.0232 & 14.9625\end{bmatrix},$$

the disturbance observer gains are given as follows:

$$L_1=\begin{bmatrix}-3.2207 & -1.5673\\ -1.5673 & -4.2511\end{bmatrix}, L_2=\begin{bmatrix}-3.4903 & -1.6653\\ -1.6653 & -4.0094\end{bmatrix}.$$

Initial condition is given as $x(0) = [1 \quad -1]$, and the optimal value of γ can be obtained $\gamma = 0.2720$. It is shown that Fig. C1 denotes the time sequences of the mode of semi-Markovian based CR system, Fig. C2 denotes the disturbance estimation error, it shown that the observer performance is well, and Fig. C3 denotes the state trajectory of CR system. It is shown that the composite controller work well, and the CR system running normally. These figures verify the correctness of the results in Theorems.

References

1 Ma X, Djouadi S M, Li H. State estimation over a semi-Markov model based cognitive radio system. IEEE Transactions on Wireless Communications, 2012, 11(7): 2391-2401.