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Randomized difference-based gradient-free algorithm for distributed resource allocation

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Abstract This paper considers a distributed resource allocation problem over time-varying networks. The objective of each agent in the network is to optimize the sum of separable convex functions subjected to resource constraints by observing its local objective function and the information exchanged with its adjacent neighbors. Thus, the problem lies in a distributed framework. In existing literature dealing with similar problems, the measurement of the gradients/subgradients of the objective functions has been applied in the algorithm design. In this paper, by adding stochastic dithers to the local objective functions and constructing randomized differences, we propose a distributed gradient-free algorithm for solving the problem, and show that the algorithm is strongly convergent; that is, the estimates generated from each agent almost certainly converge to the optimal resource allocation solution of the network. Finally, the effectiveness of the algorithm is validated by conducting numerical experiments.

Keywords resource allocation, distributed algorithm, randomized difference

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1 Introduction

A resource allocation problem involves a group of agents cooperatively utilizing a finite resource to minimize or maximize a global objective function, which is usually formulated as the sum of the local objective functions of all agents. Resource allocation problems have wide applications, such as the economic dispatch in a smart grid [1–3], load control problem in power systems [4], and energy management in communication networks [5, 6], and have been studied in diverse fields.

A pioneer theoretical paper on resource allocation is [7], where the problem of allocating limited resources to two agents, called "activities" in this paper, was considered, and several mathematical models were proposed to address this problem. Since then, this problem has been extensively studied (e.g., [8–10] and the references therein). The algorithms used in the abovementioned papers were formulated in a centralized manner; that is, the information regarding the entire system was sent to an actual/virtual center, where these algorithms performed allocation, and then, the results were sent to each resource demand unit of the system. However, with the recent, rapid development and special demands of networked systems, distributed resource allocation has received much attention and many algorithms have been proposed [11–15]. Compared to centralized algorithms, distributed algorithms do not possess any processing center, and thus, have the merits of privacy protection, scalability and robustness for large-scale networks. In [11], a distributed algorithm was proposed to estimate the saddle point of a dual problem related to primal resource sharing. In [12], an asynchronous gradient-descent algorithm was introduced for distributed resource allocation. Considering communication noises and various uncertainties, in a recent study [14], a stochastic approximation (SA)-based distributed resource allocation algorithm was introduced, and the ordinary differential equation (ODE) method was used for the convergence analysis of the SA. In [13], the relation between the decentralized resource allocation problem and decentralized

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consensus optimization problem was studied and a novel class of decentralized schemes for solving resource allocation problems was proposed. In [15], a distributed dual gradient tracking algorithm was proposed to solve the resource allocation problems over an unbalanced network, where the algorithm was found to converge linearly for strongly convex and Lipschitz smooth cost functions and sublinearly without Lipschitz smoothness. In [16], a distributed continuous-time algorithm for solving the resource allocation problem was considered. In the aforementioned papers and almost all other existing literature on resource allocation, the gradient information of the objective functions was applied in the algorithm design. However, as shown in [17,18], in many cases, the gradient or subgradient is unavailable or computationally expensive, and even impracticable. Hence, algorithms without gradients or subgradients, also known as derivative-free methods in optimization areas, are in high demand in both theoretical research and practical applications.

In fact, many studies have been conducted on gradient-free algorithms under both centralized and distributed settings. For example, as a centralized algorithm, a randomized difference-based gradient-free algorithm, called the Kiefer-Wolfowitz (KW) algorithm, was introduced in [19] and its modified version with randomly varying truncations was proposed in [20]. In addition, a simultaneous perturbation SA (SPSA) algorithm was introduced in [21]. Based on a pre-smoothing technique, a gradient-free algorithm was applied to nonsmooth optimization in [18]. Please refer to [17] for a unified view of various gradient-free optimization procedures. Meanwhile, for achieving distributed settings on networked systems, some gradient-free algorithms have been recently proposed (e.g., [22–26]). These studies focused on the distributed optimization of a global objective function over networks with/without constraints and the algorithms were mostly found to be of both consensus and convergence types (i.e., the estimates for all agents in the networks converged to the same optimal point, or its neighborhood, of the objective function). In the resource allocation problem, each agent has its own operating variable, and the optimal solution to each agent is not expected to converge to the same value, and thus, the above methods cannot be directly applied. Meanwhile, a few studies have also used gradient-free algorithms for distributed resource allocation. For example, in [27], based on the concept of extremum seeking control (ESC), a distributed continuous-time algorithm for real-time resource allocation was introduced and its convergence was established using the classical results of ESC. In [28], using the so-called robust box-constrained gradient fairness [29] with SPSA [21], a distributed discrete-time algorithm was proposed and was found to converge to a neighborhood of the optimal solution with a constant step size. To the best of our knowledge, so far, no strongly consistent results on gradient-free algorithms for distributed resource allocation of discrete-time systems with communication noises has been reported. Strong consistency means that the algorithms converge to the optimal solution of resource allocation with probability one.

Therefore, this paper considers a gradient-free algorithm for distributed resource allocation over undirected and randomly switching networks with communication noises among agents. The contributions of this paper are listed as follows.

(i) Motivated by the algorithms proposed in [14], this paper transforms the original resource allocation into the saddle-point-seeking of an augmented Lagrangian duality problem. Then, based on the local observation of each agent in the network and the information exchanged with adjacent neighbors, a distributed algorithm is proposed to solve the augmented Lagrangian duality problem. The key difference in the proposed algorithm design and that proposed in [14] is that, by introducing stochastic perturbation to each local objective function, and then constructing randomized differences to replace its gradient information, the proposed algorithms is completely gradient-free.

(ii) Under the uncertainty environment including the time-varying network topology and the communication and resource-sharing noises among the agents, by applying the ODE method for convergence analysis of SA (e.g., [30]), we first prove that the estimates generated from the algorithm converge to an equilibrium point of an ODE associated with the problem, and then establish an almost certain convergence of the estimates to the optimal solution of resource allocation. Compared to [14], as the gradient information of the objective functions is not directly available, the convergence analysis conducted in this study is more involved. As a bottleneck of distributed algorithms is the high-dimensional gradient information exchanged among agents, the method used in this study is suitable for the cases in which the direct measurements of gradients are not available or are costly and the objective functions have high dimensions. To the best of our knowledge, this is the first study conducted on a strongly consistent convergence of gradient-free algorithms for a distributed resource allocation of discrete-time systems with imperfect communications.

The remainder of this paper is organized as follows. Section 2 presents the problem formulation for distributed resource allocation and introduces a randomized difference-based gradient-free algorithm for solving this problem. Section 3, under certain mild conditions, establishes an almost certain convergence of estimates generated from the algorithm. Section 4 provides some simulation examples, and Section 5 presents the concluding remarks. Some technical results applied in this paper are presented in Appendix A.

Notations. The vectors used in this paper are deemed as column vectors, and the transpose of a vector x is denoted as x^{T}. $\mathrm{dom} f$ denotes the domain of a function $f(\cdot)$. $A_k = [a_k^{ij}]$ denotes the adjacent matrix of a time-varying network at time k and a_k^{ij} denotes its (i,j)-entry. The Kronecker product of matrices A and B is denoted as $A \otimes B$. The projection mapping of x onto a set Ω is denoted by $\mathcal{P}_{\Omega}(x) \triangleq \arg\min_{y\in\Omega} \|x-y\|_2$, where $\|\cdot\|_2$ is the 2-norm in the real space. For a convex set $\Omega \subseteq \mathbb{R}^p$ and $x \in \Omega$, $N_{\Omega}(x) \triangleq \{v \in \mathbb{R}^p | \langle v, y-x\rangle \leqslant 0, \forall y \in \Omega\}$ is defined as the normal cone of Ω at x, where $\langle u, w\rangle$ denotes the inner product of u and w. $X \triangleq \mathrm{col}\{x^1,\ldots,x^n\} = [(x^1)^{\mathrm{T}},\ldots,(x^n)^{\mathrm{T}}]^{\mathrm{T}}$ denotes a stacked column vector with $x^1,\ldots,x^n$.

2 Gradient-free algorithm for distributed resource allocation

2.1 Problem formulation

Let us consider a time-varying network $\mathcal{G}_k = (\mathcal{N}, \mathcal{E}_k)$, where $k \geqslant 1$ indicates the time indices, $\mathcal{N} = \{1,2,\ldots,n\}$ indicates the agent set, and $\mathcal{E}_k \subseteq \mathcal{N}\times\mathcal{N}$ indicates the edge set. When $(j,i) \in \mathcal{E}_k$, we assume that agent i can receive information from agent j at time k. We define the neighbor set of agent i at time k as $\mathcal{N}_k^i$, i.e., $\mathcal{N}_k^i = \{j \in \mathcal{N} | (j,i) \in \mathcal{E}_k\}$. In addition, we define the adjacency matrix of the network at time k by $A_k = [a_k^{ij}]$ with $a_k^{ij} = 1$ when $j \in \mathcal{N}_k^i$, and otherwise, $a_k^{ij} = 0$. Its Laplacian matrix is denoted as $L_k = D_k - A_k$, where $D_k = \mathrm{diag}[\sum_{j=1}^n a_k^{1j},\ldots,\sum_{j=1}^n a_k^{nj}]$.

We consider a resource allocation problem over a time-varying network $\mathcal{G}_k, k \geqslant 1$. Each agent $i,\ i \in \mathcal{N}$ possesses a local resource variable $x^i \in \mathbb{R}^p$, which can only be operated by itself throughout the allocation process. Meanwhile, each agent i has a local objective function $f^i(\cdot): \mathbb{R}^p \to \mathbb{R}$ and can access a fixed local resource r^i, but aims at cooperating with other agents to minimize the global objective function of the network with resource constraint $\sum_{i\in\mathcal{N}} x^i = \sum_{i\in\mathcal{N}} r^i$. Without loss of generality, the global objective function of the network is denoted as the sum of all local objective functions $f^i(\cdot),\ i = 1,\ldots,n$. Thus, the problem lies in a distributed framework and can be formulated as follows:

$$\begin{aligned} &\min_{x^i\in\mathbb{R}^p,\ i\in\mathcal{N}} \sum_{i\in\mathcal{N}} f^i(x^i), \\ &\text{s.t.} \sum_{i\in\mathcal{N}} x^i = \sum_{i\in\mathcal{N}} r^i,\ x^i \in \Omega^i,\ \Omega^i \subset \mathbb{R}^p, \end{aligned} \tag{1}$$

where $\Omega^i \triangleq \{x : h^{ij}(x) \leqslant 0, j = 1,\ldots,m_i\}$, with $h^{ij}(\cdot): \mathbb{R}^p \to \mathbb{R}, j = 1,\ldots,m_i$ being a continuously differentiable convex function and Ω^i being available to agent i only.

The existing distributed algorithms for solving problem (1) usually require the gradient information on $f^i(\cdot),\ i \in \mathcal{N}$. As stated in Section 1, the gradient information is not always available, or even intractable in some cases. Next, we first introduce stochastic perturbation to the local objective functions $f^i(\cdot),\ i \in \mathcal{N}$, construct randomized differences to replace the gradient measurements, and then introduce the distributed gradient-free algorithm.

2.2 Randomized differences

For each $i \in \mathcal{N}$, let $\{\Delta_k^i = [\Delta_k^{i1}, \Delta_k^{i2},\ldots,\Delta_k^{ip}]^{\mathrm{T}} \in \mathbb{R}^p\}_{k\geqslant 1}$ be a sequence of mutually independent and identically distributed random vectors, which we call the perturbation signal of agent i at time k. Furthermore, we choose a positive sequence $\{\beta_k\}_{k\geqslant 0}$ tending to zero, which will be specified later. Assuming that at iteration $k+1$, we have two observations of each local objective function $f^i(\cdot)$ at $x_k^i + \beta_k\Delta_{k+1}^i$ and $x_k^i - \beta_k\Delta_{k+1}^i$, denoted by $[y_{k+1}^i]^+ = f^i(x_k^i + \beta_k\Delta_{k+1}^i)$ and $[y_{k+1}^i]^- = f^i(x_k^i - \beta_k\Delta_{k+1}^i)$, respectively; where $[\cdot]^+$ and $[\cdot]^-$ represent positive and negative perturbation directions, respectively.

Let us consider $[\Delta_k^i]^{-1} \triangleq [\frac{1}{\Delta_k^{i1}}, \frac{1}{\Delta_k^{i2}}, \ldots, \frac{1}{\Delta_k^{ip}}]^{\mathrm{T}}$. The randomized differences adopted in this paper are given as follows:

$$d_{k+1}^i \triangleq \frac{([y_{k+1}^i]^+ - [y_{k+1}^i]^-)[\Delta_{k+1}^i]^{-1}}{2\beta_k}. \tag{2}$$

From the definition of the randomized differences, we find that d_{k+1}^i only uses the measurements of the function values $f^i(x_k^i + \beta_k \Delta_{k+1}^i)$ and $f^i(x_k^i - \beta_k \Delta_{k+1}^i)$, but can be regarded as an approximation or estimate of the true gradient of $f^i(\cdot)$ at x_k^i.

Remark 1. Another type of randomized difference, which is usually called a random gradient-free oracle, can be expressed as ([18, 26])

$$g_\mu(x_k) = \frac{f(x_k + \mu u_k) - f(x_k - \mu u_k)}{2\mu} u_k, \tag{3}$$

where $\mu > 0$ is a constant, called the smoothing parameter, and $u_k \in \mathbb{R}^p$ is uniformly generated over a unit ball. For the centralized/distributed stochastic optimization algorithms with gradients replaced by the abovementioned randomized differences, it can be proved that the estimates converge to a neighborhood of the optimal value, whose size is characterized by the smoothing parameter μ. The problem considered in this paper is not only to design a gradient-free algorithm but also to ensure that the estimates converge to the optimal value almost certainly. Thus, the randomized difference provided by (3) cannot be applied here, and correspondingly, the mathematical proofs presented in this paper differ from those presented in [18, 26].

2.3 Distributed resource allocation algorithm

The basic concept of the algorithm is motivated by [31–33]; that is, first transform the resource allocation problem into saddle-point-seeking of Lagrangian duality, and then use the primal-dual gradient dynamics (PDGD) to solve the problem. However, for PDGD, the gradients of the objective functions are applied. In this paper we use the above-defined randomized differences to replace the gradients in PDGD. In addition, by considering the network uncertainties, we assume that each agent i can only access its local resource r^i with noise observations and the information communicated among agents is corrupted by noises. As mentioned in Section 3, the randomized differences adopted in the algorithm and the network uncertainties make the problem nontrivial and the convergence analysis of the distributed algorithm more involved. For simplicity, the concept of the algorithm design is defined as being based on a connected and time-invariant network. The detailed assumptions made on the network are presented in Section 3.

First, we define the Lagrangian function $\mathcal{L} : \mathbb{R}^p \times \cdots \times \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$, with the multiplier $\lambda \in \mathbb{R}^p$ associated with problem (1), as

$$\mathcal{L}(x^1, \ldots, x^n, \lambda) \triangleq \sum_{i=1}^n f^i(x^i) - \lambda^{\mathrm{T}} \left(\sum_{i=1}^n x^i - \sum_{i=1}^n r^i \right), \tag{4}$$

where $\mathrm{dom}\mathcal{L} = \Omega^1 \times \cdots \times \Omega^n \times \mathbb{R}^p$. Then, the Lagrangian dual problem is presented as

$$\max_{\lambda \in \mathbb{R}^p} g(\lambda) \tag{5}$$

with

$$g(\lambda) = \sum_{i=1}^n g^i(\lambda) = \sum_{i=1}^n \inf_{x^i \in \Omega^i} \{f^i(x^i) - \lambda^{\mathrm{T}} x^i + \lambda^{\mathrm{T}} r^i\}. \tag{6}$$

Let L denote the Laplacian matrix of the network, and set $\Lambda = \mathrm{col}\{\lambda^1, \ldots, \lambda^n\} \in \mathbb{R}^{np}$ and $\Gamma = \mathrm{col}\{\gamma^1, \ldots, \gamma^n\} \in \mathbb{R}^{np}$. Then, the Lagrangian duality problem (5) can be equivalently formulated as

$$\begin{cases} \max\limits_{\Lambda} \ G(\Lambda) \\ \text{s.t. } (L \otimes I_p)\Lambda = \mathbf{0} \in \mathbb{R}^{np} \end{cases} \tag{7}$$

with

$$G(\Lambda) = \sum_{i=1}^{n} g^i(\lambda^i) \tag{8}$$

and the augmented Lagrangian duality is given by

$$\min_{\Gamma} \max_{\Lambda} \ G(\Lambda, \Gamma) \tag{9}$$

with

$$G(\Lambda, \Gamma) = \sum_{i=1}^{n} g^i(\lambda^i) - \Gamma^{\mathrm{T}}(L \otimes I_p)\Lambda - \frac{1}{2}\Lambda^{\mathrm{T}}(L \otimes I_p)\Lambda. \tag{10}$$

For the resource allocation problem (1), the Karush-Kuhn-Tucker (KKT) condition can be formulated as

$$\begin{aligned} & -\nabla f^i(x^{i*}) + \lambda^* \in N_{\Omega^i}(x^{i*}), \\ & \sum_{i=1}^{n} x^{i*} = \sum_{i=1}^{n} r^i, \quad x^{i*} \in \Omega^i, \end{aligned} \tag{11}$$

where $X^* = [x^{1*\mathrm{T}} \dots x^{n*\mathrm{T}}]^{\mathrm{T}}$ and λ^* denotes one of the solutions of KKT.

Lemma 2.38 in [32] shows that, under certain moderate conditions, X^* is the optimal solution of the original problem (1) if and only if (X^*, λ^*) is a saddle point of the Lagrangian (4). Furthermore, $\Lambda^* = [\lambda^{*\mathrm{T}} \dots \lambda^{*\mathrm{T}}]^{\mathrm{T}}$ is the optimal solution of the Lagrangian duality (7) if and only if (Λ^*, Γ^*) is a saddle point of the augmented Lagrangian duality (10). Thus, the optimal solution of resource allocation (1) is equivalent to seeking the saddle points of the Lagrangian (4) and the augmented Lagrangian duality (10).

A widely used algorithm for the seeking of the saddle points of the Lagrangian (4) and the augmented Lagrangian duality (10) is called PDGD or the saddle point dynamics [34, 35], for which the distributed continuous-time algorithm can be formulated as, e.g., [33],

$$\begin{cases} \dot{x}^i = \mathcal{P}_{\Omega^i}\{x^i - \nabla f^i(x^i) + \lambda^i\} - x^i, \\ \dot{\lambda}^i = r^i - x^i - \sum_{j=1}^{n} a^{ij}(\lambda^i - \lambda^j) - \sum_{j=1}^{n} a^{ij}(\gamma^i - \gamma^j), \\ \dot{\gamma}^i = \sum_{j=1}^{n} a^{ij}(\lambda^i - \lambda^j), \end{cases} \tag{12}$$

for all $i \in \mathcal{N}$, where the matrix $[a^{ij}]_{i,j=1}^{n}$ represents the link structure of the network. However, the above algorithm cannot be directly applied to problem (1), as in this paper, we consider the observations in discrete time and the time-varying network; furthermore, there is no gradient information available for the objective functions in (1).

Motivated by PDGD, based on the randomized differences d_{k+1}^i in (2) in this paper, we introduce the gradient-free algorithm for distributed resource allocation, see Algorithm 1.

Algorithm 1 Gradient-free algorithm for distributed resource allocation

For each agent $i \in \mathcal{N}$, initialize $x_0^i, \lambda_0^i, \gamma_0^i$,
for $k = 0, 1, 2, \dots$ **do**

$$\begin{cases} x_{k+1}^i = \mathcal{P}_{\Omega^i}\{x_k^i + \alpha_k(-d_{k+1}^i + \lambda_k^i)\}, \\ \lambda_{k+1}^i = \lambda_k^i + \alpha_k\{(r^i + \delta_{k+1}^i) - x_k^i - \sum_{j \in \mathcal{N}_k^i} a_k^{ij}(\lambda_k^i - (\lambda_k^j + \xi_{k+1}^{ij})) - \sum_{j \in \mathcal{N}_k^i} a_k^{ij}(\gamma_k^i - (\gamma_k^j + \eta_{k+1}^{ij}))\}, \\ \gamma_{k+1}^i = \gamma_k^i + \alpha_k \sum_{j \in \mathcal{N}_k^i} a_k^{ij}(\lambda_k^i - (\lambda_k^j + \xi_{k+1}^{ij})), \end{cases} \tag{13}$$

end for

In Algorithm 1, k is the time index; $\{\alpha_k\}_{k \geqslant 0}$ is a positive sequence tending to zero, which we call the step size in this paper; $[a_k^{ij}]_{i,j=1}^{n}$ indicates the adjacent matrices of the network; x_k^i represents the resource allocated to agent i at time k; $([x_k^{1\mathrm{T}}, \dots, x_k^{n\mathrm{T}}]^{\mathrm{T}}, \lambda_k^i), k \geqslant 0$ are the estimates for the saddle points of the Lagrangian (4), while $([\lambda_k^{1\mathrm{T}}, \dots, \lambda_k^{n\mathrm{T}}]^{\mathrm{T}}, [\gamma_k^{1\mathrm{T}}, \dots, \gamma_k^{n\mathrm{T}}]^{\mathrm{T}}), k \geqslant 0$ serve as the estimates for the saddle points of the augmented Lagrangian duality (10). From Algorithm 1, we find that the gradient information of local objective functions is not applied, and for each agent i, it only uses its local information and that information exchanged with its adjacent neighbors. Thus, Algorithm 1 is gradient-free and fully distributed.

Considering the uncertainties in the communication between the agents, in Algorithm 1, ξ^{ij}_{k+1} and η^{ij}_{k+1} are random noises occurring when λ^j_k and γ^j_k are sent from agent j to agent i, and δ^i_{k+1} denotes the random noise when agent i gets access to its local resource r^i.

In Section 3, we will introduce the network assumptions, and then establish a high-probability convergence—and in fact, convergence with probability one—of the proposed algorithm.

3 Assumptions and convergence of algorithm

We make the following network assumptions.

Assumption 1. Problem (1) is feasible and has a finite optimal solution. For each $i \in \mathcal{N}$, $f^i(\cdot)$ is strictly convex and differentiable in $\mathrm{dom}(f^i)$, and has a Lipschitzian continuous gradient, i.e., $\|\nabla f^i(x) - \nabla f^i(y)\|_2 \leqslant l_c\|x-y\|_2$ for some $l_c > 0$ and $\forall x, y \in \mathrm{dom}(f^i)$. Furthermore, $f^i(\cdot)$ has a bounded gradient, i.e., $\|\nabla f^i(x)\|_2 \leqslant L$, for some $L > 0$ and $\forall x \in \mathrm{dom}(f^i)$.

Assumption 2. For each $i \in \mathcal{N}$, Ω^i is convex and closed and the set of its interior points is nonempty. Let us define $I^i(x) \triangleq \{j : h^{ij}(x) = 0\}$. For each x with nonempty $I^i(x)$, $\{\nabla_x h^{ij}(x), j \in I^i(x)\}$ are linearly independent.

Remark 2. In what follows, we will apply the ODE method (e.g., [30]) for convergence analysis of the distributed algorithm. The primary concept of ODE is based on transforming the convergence of the discrete-time algorithm into the stability of the equilibrium of an associated ODE. Assumption 2, called the Slater condition in the optimization literature, aims to guarantee that the equilibrium of the associated ODE is precisely the solution of the KKT condition (11), and then, the convergence of the distributed algorithm can be established.

Assumption 3. Let L_k denote the Laplacian matrix of the network at time k. Let us assume that $\{L_k\}_{k\geqslant 0}$ is an i.i.d. sequence[1)]. Set $\overline{L} \triangleq \mathrm{E}(L_k)$. $\overline{L}$ is symmetric with a positive second-smallest eigenvalue.

Remark 3. Assumption 3 does not require the connectivity of the communication graph at any time instance k. In contrast, it requires the mean graph of the network to be connected. This assumption is often applied in the distributed optimization/estimation (e.g., [36, 37]).

Next, we introduce the conditions on the stochastic perturbation and the perturbation step size adopted in the randomized differences.

Assumption 4. For each $i \in \mathcal{N}$ and $j \in \{1, 2, \ldots, p\}$, we choose $\{\Delta^{ij}_k\}_{k\geqslant 0}$ as an i.i.d. sequence with

$$|\Delta^{ij}_k| < a, \quad \left|\frac{1}{\Delta^{ij}_k}\right| < b, \quad \mathrm{E}\left[\frac{1}{\Delta^{ij}_k}\right] = 0, \quad \forall k \geqslant 0 \tag{14}$$

for some $a > 0$ and $b > 0$, and $\{\Delta^{ij}_k\}_{k\geqslant 0}$ and $\{\Delta^{i'j'}_k\}_{k\geqslant 0}$ are mutually independent if $i \neq i'$ or $j \neq j'$.

Assumption 5. Let us assume that $\{\alpha_k\}_{k\geqslant 0}$ and $\{\beta_k\}_{k\geqslant 0}$ are two positive sequences that satisfy the following:

(a) $\alpha_k > 0$, $\sum_{k=0}^{\infty} \alpha_k = \infty$, $\sum_{k=0}^{\infty} \alpha_k^2 < \infty$;

(b) $\beta_k > 0$, $\beta_k \to 0$, $\sum_{k=0}^{\infty} \alpha_k\beta_k < \infty$.

Remark 4. We select

$$\alpha_k = \frac{1}{k^\theta} \quad \text{and} \quad \beta_k = \frac{1}{k^\nu}$$

with $\theta \in (\frac{1}{2}, 1]$ and $\nu \in (0, \frac{1}{2}]$, to directly verify that such choices on α_k and β_k meet the requirements in Assumption 5.

Let $\{\mathcal{F}_k\}_{k\geqslant 0}$ and $\{\mathcal{F}'_k\}_{k\geqslant 0}$, which are two sequences of σ-algebras, by

$$\mathcal{F}_k \triangleq \sigma\left\{\xi^{ij}_t, \eta^{ij}_t, \delta^i_t, \Delta^i_t, L_t,\ 0 \leqslant t \leqslant k, i \in \mathcal{N}, j \in \mathcal{N}, X(0), \Lambda(0), \Gamma(0)\right\} \tag{15}$$

and $\mathcal{F}'_k \triangleq \sigma\{\mathcal{F}_k, L_{k+1}\}$, respectively.

Now, we introduce the conditions on communication noises.

1) Independent and identically distributed.

Assumption 6. (a) Assume that for each $i \in \mathcal{N}$, $\{\delta_k^i\}_{k\geqslant 0}$ is an i.i.d. sequence with $\mathrm{E}(\delta_k^i) = 0$ and $\mathrm{E}(\|\delta_k^i\|_2^2) \leqslant \sigma_1^2 < \infty$, where σ_1 is a positive constant and $\{\delta_k^i\}_{k\geqslant 0}$ is independent of $\{L_k\}_{k\geqslant 0}$, $\{\xi_k^{i'j'}\}_{k\geqslant 0}$, and $\{\eta_k^{i'j'}\}_{k\geqslant 0}$ for any $i', j' \in \mathcal{N}$.

(b) For each $i, j \in \mathcal{N}$, $\{\xi_{k+1}^{ij}, \mathcal{F}_k'\}_{k\geqslant 0}$ and $\{\eta_{k+1}^{ij}, \mathcal{F}_k'\}_{k\geqslant 0}$ are martingale difference sequences (m.d.s.) and there exists a positive constant σ_2 such that $\mathrm{E}(\|\xi_{k+1}^{ij}\|_2^2|\mathcal{F}_k') \leqslant \sigma_2^2$ and $\mathrm{E}(\|\eta_{k+1}^{ij}\|_2^2|\mathcal{F}_k') \leqslant \sigma_2^2$ for all $k \geqslant 0$.

(c) Furthermore, the perturbation signals $\{\Delta_k^{ij}, i \in \mathcal{N}, j \in \{1, \ldots, p\}\}_{k\geqslant 0}$ are independent of $\{\delta_k^i, i \in \mathcal{N}\}_{k\geqslant 0}$, $\{\xi_k^{ij}, i \in \mathcal{N}, j \in \mathcal{N}\}_{k\geqslant 0}$ and $\{\eta_k^{ij}, i \in \mathcal{N}, j \in \mathcal{N}\}_{k\geqslant 0}$.

Remark 5. Assumptions 6(a) and (b) are made on the uncertainties of the observations/communications over random networks (e.g., [14, 36, 38, 39]). Assumption 6(a) is made on the uncertainties of the local resource observation. Assumption 6(b) aims to describe the noises when the information is exchanged over random networks with imperfect communications, which includes independent and identically distributed random noises, such as Gaussian distributed variables and uniformly distributed variables, as special cases. Assumption 6(c) requires that the perturbation signals adopted in the algorithm are independent of the communication and resource observation noises.

Before proving the strong consistency of the distributed resource allocation algorithm, we first introduce the notations used throughout this section.

Let $R = \mathrm{col}\{r^1, r^2, \ldots, r^n\}$,

$$\begin{aligned}
&\xi_k^i = \sum_{j=1}^n a_k^{ij}\xi_k^{ij}, \quad \eta_k^i = \sum_{j=1}^n a_k^{ij}\eta_k^{ij},\\
&X_k = \mathrm{col}\{x_k^1, x_k^2, \ldots, x_k^n\}, \quad \Lambda_k = \mathrm{col}\{\lambda_k^1, \lambda_k^2, \ldots, \lambda_k^n\}, \quad \Gamma_k = \mathrm{col}\{\gamma_k^1, \gamma_k^2, \ldots, \gamma_k^n\},\\
&\delta_k = \mathrm{col}\{\delta_k^1, \delta_k^2, \ldots, \delta_k^n\}, \quad \xi_k = \mathrm{col}\{\xi_k^1, \xi_k^2, \ldots, \xi_k^n\}, \quad \eta_k = \mathrm{col}\{\eta_k^1, \eta_k^2, \ldots, \eta_k^n\},\\
&D_k = \mathrm{col}\{d_k^1, d_k^2, \ldots, d_k^n\},
\end{aligned}$$

and

$$\nabla f(X_k) = \mathrm{col}\{\nabla f^1(x_k^1), \nabla f^2(x_k^2), \ldots, \nabla f^n(x_k^n)\}.$$

With the above notations, the algorithm (13) can then be rewritten in a compact form as follows:

$$\begin{cases}
X_{k+1} = \mathcal{P}_\Omega\{X_k + \alpha_k(-D_{k+1} + \Lambda_k)\},\\
\Lambda_{k+1} = \Lambda_k + \alpha_k\{R + \delta_{k+1} - X_k - (L_k \otimes I_p)(\Lambda_k + \Gamma_k) + \xi_{k+1} + \eta_{k+1}\},\\
\Gamma_{k+1} = \Gamma_k + \alpha_k\{(L_k \otimes I_p)\Lambda_k - \xi_{k+1}\},
\end{cases} \tag{16}$$

where $\Omega = \prod_{i=1}^n \Omega^i$ is the Cartesian product of $\Omega^i, i = 1, \ldots, n$.

Let

$$\begin{aligned}
e_{k+1}^1 &= ((\bar{L} - L_k) \otimes I_p)(\Lambda_k + \Gamma_k),\\
e_{k+1}^2 &= \delta_{k+1} + \xi_{k+1} + \eta_{k+1},\\
e_{k+1}^3 &= ((L_k - \bar{L}) \otimes I_p)\Lambda_k - \xi_{k+1}.
\end{aligned}$$

Then, Eq. (16) can be formulated as

$$\begin{cases}
X_{k+1} = \mathcal{P}_\Omega\{X_k + \alpha_k(-D_{k+1} + \Lambda_k)\},\\
\Lambda_{k+1} = \Lambda_k + \alpha_k\{R - X_k - (\bar{L} \otimes I_p)(\Lambda_k + \Gamma_k) + e_{k+1}^1 + e_{k+1}^2\},\\
\Gamma_{k+1} = \Gamma_k + \alpha_k\{(\bar{L} \otimes I_p)\Lambda_k + e_{k+1}^3\},
\end{cases} \tag{17}$$

or equivalently,

$$S_{k+1} = \mathcal{P}_\Phi\{S_k + \alpha_k(J(S_k) + E_{k+1})\}, \tag{18}$$

where

$$S_k = \begin{pmatrix} X_k \\ \Lambda_k \\ \Gamma_k \end{pmatrix}, \quad J(S) = \begin{pmatrix} -\nabla f(X) + \Lambda \\ R - X - (\bar{L} \otimes I_p)(\Lambda + \Gamma) \\ (\bar{L} \otimes I_p)\Lambda \end{pmatrix}, \quad E_{k+1} = \begin{pmatrix} \nabla f(X_k) - D_{k+1} \\ e_{k+1}^1 + e_{k+1}^2 \\ e_{k+1}^3 \end{pmatrix} \tag{19}$$

and $\Phi = \Omega \times \mathbb{R}^{np} \times \mathbb{R}^{np}$.

Remark 6. Let $Z_{k+1} = \frac{1}{\alpha_k}\big(\mathcal{P}_\Phi\{S_k + \alpha_k(J(S_k) + E_{k+1})\} - S_k - \alpha_k(J(S_k) + E_{k+1})\big)$. Then Eq. (18) can further be written as

$$S_{k+1} = S_k + \alpha_k(J(S_k) + E_{k+1} + Z_{k+1}), \tag{20}$$

which is in a standard SA algorithm form [30,40]. According to [30], $Z_{k+1} \in -N_\Phi(S_{k+1})$ and Z_{k+1} can be regarded as the minimum norm force required to take $S_k + \alpha_k(J(S_k) + E_{k+1})$ back to the constraint set Φ. In what follows, we will apply the ODE method (e.g., [30]) for the convergence analysis of (18). The basic concept of ODE lies in first interpolating the discrete-time estimates $\{S_k\}_{k\geqslant 0}$ into a sequence of uniformly bounded and equicontinuous functions, then using Arzelà-Ascoli Theorem [30,40], determining an associated ODE, and finally proving the convergence of estimation sequence to the ODE equilibrium.

Let us define the following continuous-time projected dynamics as the underlying ODE of (13):

$$\dot{S} = J(S) + z, \quad S_0 = \mathrm{col}\{X_0, \Lambda_0, \Gamma_0\} \in \Phi \tag{21}$$

with $z \in -N_\Phi(S)$ and $J(S)$ defined by (19).

We first present two technical lemmas, which will be used in our analysis. Their proofs are omitted here as they can be found in [14].

Lemma 1 ([14])**.** Suppose that Assumptions 1–3 hold. Then, the ODE (21) has at least one equilibrium point. Denote any one equilibrium point of (21) by $S^* = \mathrm{col}\{X^*, \Lambda^*, \Gamma^*\}$. Then, $X^* = \mathrm{col}\{x^{1*} \cdots x^{n*}\}$ is the optimal solution to problem (1), and the following formulas hold:

$$\begin{cases} -\nabla f^i(x^{i*}) + \lambda^* \in N_{\Omega^i}(x^{i*}), \ x^{i*} \in \Omega^i, \ i \in \mathcal{N}, \\ (\bar{L} \otimes I_p)(\Lambda^* + \Gamma^*) + X^* - R = 0, \\ (\bar{L} \otimes I_p)\Lambda^* = 0. \end{cases} \tag{22}$$

Moreover, the trajectories of (21) converge to one of its equilibrium points if the initial $S_0 \in \Phi$.

For ease of reading, we list some notations to be used later. Set

$$C_1 = 3(2n^3\sigma_2^2 + n\sigma_1^2), \tag{23}$$
$$C_2 = \mathrm{E}(\|(\bar{L} - L_k) \otimes I_p\|_2^2), \tag{24}$$
$$C_3 = \|(\bar{L} \otimes I_p)\|_2^2, \tag{25}$$
$$c_1 = pa\sqrt{(p-1)a^2b^2 + 1}, \tag{26}$$
$$c_2 = \|n_\Omega(X^*)\|_2^2 = \|\nabla f(X^*) - \Lambda^*\|_2^2, \tag{27}$$

where X^* and Λ^* are guaranteed by Lemma 1 and $n_\Omega(X^*) = \mathrm{col}\{n_{\Omega^1}(x^{1*}), \ldots, n_{\Omega^n}(x^{n*})\}$ with $n_{\Omega^i}(x^{i*}) \in N_{\Omega^i}(x^{i*})$ such that $-\nabla f^i(x^{i*}) + \lambda^* = n_{\Omega^i}(x^{i*})$.

Lemma 2 ([14])**.** Suppose that Assumptions 3 and 6 hold. Then,

$$\mathrm{E}(e_{k+1}^1 + e_{k+1}^2|\mathcal{F}_k) = 0, \quad \mathrm{E}(e_{k+1}^3|\mathcal{F}_k) = 0, \tag{28}$$
$$\mathrm{E}(\|e_{k+1}^1 + e_{k+1}^2\|_2^2|\mathcal{F}_k) \leqslant C_2\|\Lambda_k + \Gamma_k\|_2^2 + C_1, \tag{29}$$
$$\mathrm{E}(\|e_{k+1}^3\|_2^2|\mathcal{F}_k) \leqslant C_2\|\Lambda_k\|_2^2 + n^3\sigma_2^2, \tag{30}$$

where C_1 and C_2 are defined by (23) and (24), respectively.

We have the following lemma.

Lemma 3. Let $\{S_k\}_{k\geqslant 0}$ be generated by algorithm (17) and S^* be any of the equilibrium points of (21). If Assumptions 1 and 4 hold, then

$$(S_k - S^*)^{\mathrm{T}}\mathrm{E}(E_{k+1}|\mathcal{F}_k) \leqslant c_1 l_c \beta_k \|X_k - X^*\|_2^2 + nc_1 l_c \beta_k \tag{31}$$

and

$$J(S_k)^{\mathrm{T}}\mathrm{E}(E_{k+1}|\mathcal{F}_k) \leqslant c_1 l_c (l_c^2 + 1)\beta_k(\|X_k - X^*\|_2^2 + \|\Lambda_k - \Lambda^*\|_2^2) + c_1 c_2 l_c \beta_k + 3nc_1 l_c \beta_k, \tag{32}$$

where c_1 and c_2 are positive constants defined by (26) and (27), respectively, and E_{k+1} and $J(\cdot)$ are defined in (19).

Proof. We first prove (31). According to the definitions of S_k and E_{k+1},

$$(S_k - S^*)^{\mathrm{T}}\mathrm{E}(E_{k+1}|\mathcal{F}_k) = (X_k - X^*)^{\mathrm{T}}\mathrm{E}(\nabla f(X_k) - D_{k+1}|\mathcal{F}_k) + (\Lambda_k - \Lambda^*)^{\mathrm{T}}\mathrm{E}(e^1_{k+1} + e^2_{k+1}|\mathcal{F}_k) + (\Gamma_k - \Gamma^*)^{\mathrm{T}}\mathrm{E}(e^3_{k+1}|\mathcal{F}_k).$$

According to (28), for (31), it suffices to show that

$$(X_k - X^*)^{\mathrm{T}}\mathrm{E}(\nabla f(X_k) - D_{k+1}|\mathcal{F}_k) \leqslant c_1 l_c \beta_k \|X_k - X^*\|_2^2 + nc_1 l_c \beta_k. \tag{33}$$

For $i \in \mathcal{N}$ and $k \geqslant 0$, by the definition of the randomized difference d^i_{k+1}, we have

$$\begin{aligned}\mathrm{E}\left(\nabla f^i(x^i_k) - d^i_{k+1}\middle|\mathcal{F}_k\right) &= \mathrm{E}\left(\nabla f^i(x^i_k) - \frac{f^i(x^i_k + \beta_k\Delta^i_{k+1}) - f^i(x^i_k - \beta_k\Delta^i_{k+1})}{2\beta_k}[\Delta^i_{k+1}]^{-1}\middle|\mathcal{F}_k\right)\\ &= \mathrm{E}\left(\nabla f^i(x^i_k) - \frac{\langle\nabla f^i(x^i_k + \mu^i_k\beta_k\Delta^i_{k+1}), 2\beta_k\Delta^i_{k+1}\rangle}{2\beta_k}[\Delta^i_{k+1}]^{-1}\middle|\mathcal{F}_k\right),\end{aligned}$$

where $\mu^i_k \in [-1, 1]$ and the mean value theorem is applied in the second equality.

By noting that $\nabla f^i(x^i_k)$ is $\mathcal{F}_k$-measurable and Δ^i_{k+1} and $\mathcal{F}_k$ are mutually independent, we have

$$\begin{aligned}\mathrm{E}\left(\nabla f^i(x^i_k) - d^i_{k+1}\middle|\mathcal{F}_k\right) &= \mathrm{E}\left(\nabla f^i(x^i_k) - [\Delta^i_{k+1}]^{-1}(\Delta^i_{k+1})^{\mathrm{T}}\nabla f^i(x^i_k + \mu^i_k\beta_k\Delta^i_{k+1})\middle|\mathcal{F}_k\right)\\ &= \mathrm{E}\left([\Delta^i_{k+1}]^{-1}(\Delta^i_{k+1})^{\mathrm{T}}(\nabla f^i(x^i_k) - \nabla f^i(x^i_k + \mu^i_k\beta_k\Delta^i_{k+1}))\middle|\mathcal{F}_k\right)\\ &\quad + \mathrm{E}\left((I - [\Delta^i_{k+1}]^{-1}(\Delta^i_{k+1})^{\mathrm{T}})\nabla f^i(x^i_k)\middle|\mathcal{F}_k\right)\\ &= \mathrm{E}\left([\Delta^i_{k+1}]^{-1}(\Delta^i_{k+1})^{\mathrm{T}}(\nabla f^i(x^i_k) - \nabla f^i(x^i_k + \mu^i_k\beta_k\Delta^i_{k+1}))\middle|\mathcal{F}_k\right)\\ &\quad + \mathrm{E}\left(I - [\Delta^i_{k+1}]^{-1}(\Delta^i_{k+1})^{\mathrm{T}}\right)\nabla f^i(x^i_k).\end{aligned} \tag{34}$$

Assumption 4 indicates that $\mathrm{E}\left(I - [\Delta^i_{k+1}]^{-1}(\Delta^i_{k+1})^{\mathrm{T}}\right) = 0$. Then, from (34), we obtain

$$\mathrm{E}\left(\nabla f^i(x^i_k) - d^i_{k+1}\middle|\mathcal{F}_k\right) = \mathrm{E}\left([\Delta^i_{k+1}]^{-1}(\Delta^i_{k+1})^{\mathrm{T}}(\nabla f^i(x^i_k) - \nabla f^i(x^i_k + \mu^i_k\beta_k\Delta^i_{k+1}))\middle|\mathcal{F}_k\right), \tag{35}$$

from which, by noting $x^i_k - x^{i*} \in \mathcal{F}_k$, we have

$$\begin{aligned}&(x^i_k - x^{i*})^{\mathrm{T}}\mathrm{E}\left(\nabla f^i(x^i_k) - d^i_{k+1}\middle|\mathcal{F}_k\right)\\ &= (x^i_k - x^{i*})^{\mathrm{T}}\mathrm{E}\left([\Delta^i_{k+1}]^{-1}(\Delta^i_{k+1})^{\mathrm{T}}(\nabla f^i(x^i_k) - \nabla f^i(x^i_k + \mu^i_k\beta_k\Delta^i_{k+1}))\middle|\mathcal{F}_k\right)\\ &\leqslant \|x^i_k - x^{i*}\|_2\mathrm{E}\left(\|[\Delta^i_{k+1}]^{-1}(\Delta^i_{k+1})^{\mathrm{T}}(\nabla f^i(x^i_k) - \nabla f^i(x^i_k + \mu^i_k\beta_k\Delta^i_{k+1}))\|_2\middle|\mathcal{F}_k\right)\\ &\leqslant \|x^i_k - x^{i*}\|_2\mathrm{E}\left(\|[\Delta^i_{k+1}]^{-1}(\Delta^i_{k+1})^{\mathrm{T}}\|_2\|\nabla f^i(x^i_k) - \nabla f^i(x^i_k + \mu^i_k\beta_k\Delta^i_{k+1})\|_2\middle|\mathcal{F}_k\right),\end{aligned} \tag{36}$$

where for the first inequality, the Cauchy-Schwarz inequality and the Jensen inequality are applied.

Assumption 4 indicates that

$$\|[\Delta^i_{k+1}]^{-1}(\Delta^i_{k+1})^{\mathrm{T}}\|_2 \leqslant \sqrt{p(p-1)a^2b^2 + p}. \tag{37}$$

From (36) and (37) and under the Lipschitzian condition of the gradient (Assumption 1), we further obtain

$$\begin{aligned}(x^i_k - x^{i*})^{\mathrm{T}}\mathrm{E}\left(\nabla f^i(x^i_k) - d^i_{k+1}\middle|\mathcal{F}_k\right) &\leqslant \sqrt{p(p-1)a^2b^2 + p}\,l_c\beta_k\|x^i_k - x^{i*}\|_2\mathrm{E}\left(\|\Delta^i_{k+1}\|_2\middle|\mathcal{F}_k\right)\\ &\leqslant \sqrt{p(p-1)a^2b^2 + p}\,l_c\beta_k\sqrt{p}a\|x^i_k - x^{i*}\|_2 = c_1 l_c\beta_k\|x^i_k - x^{i*}\|_2\end{aligned} \tag{38}$$

with c_1 defined by (26), and hence,

$$(x^i_k - x^{i*})^{\mathrm{T}}\mathrm{E}\left(\nabla f^i(x^i_k) - d^i_{k+1}\middle|\mathcal{F}_k\right) \leqslant c_1 l_c\beta_k(\|x^i_k - x^{i*}\|_2^2 + 1) = c_1 l_c\beta_k\|x^i_k - x^{i*}\|_2^2 + c_1 l_c\beta_k. \tag{39}$$

Noting that $X^* = \mathrm{col}\{x^{1*}, x^{2*}, \ldots, x^{n*}\}$, from (39), we derive that

$$(X_k - X^*)^{\mathrm{T}}\mathrm{E}\left(\nabla f(X_k) - D_{k+1}\middle|\mathcal{F}_k\right) \leqslant c_1 l_c\beta_k\|X_k - X^*\|_2^2 + nc_1 l_c\beta_k.$$

Thus, Eq. (31) is proved. Next, we will prove (32).

According to the definitions of $J(\cdot)$ and E_{k+1}, we have

$$\begin{aligned} J(S_k)^{\mathrm{T}}\mathrm{E}(E_{k+1}|\mathcal{F}_k) =&(-\nabla f(X_k)+\Lambda_k)^{\mathrm{T}}\mathrm{E}(\nabla f(X_k)-D_{k+1}|\mathcal{F}_k) \\ &+(R-X_k-(\bar{L}\otimes I_p)(\Lambda_k+\Gamma_k))^{\mathrm{T}}\mathrm{E}(e^1_{k+1}+e^2_{k+1}|\mathcal{F}_k)+((\bar{L}\otimes I_p)\Lambda_k)^{\mathrm{T}}\mathrm{E}(e^3_{k+1}|\mathcal{F}_k) \\ =&(-\nabla f(X_k)+\Lambda_k)^{\mathrm{T}}\mathrm{E}\left(\nabla f(X_k)-D_{k+1}\middle|\mathcal{F}_k\right). \end{aligned}$$

Thus, for (32), we only need to prove

$$\begin{aligned} &(-\nabla f(X_k)+\Lambda_k)^{\mathrm{T}}\mathrm{E}\left(\nabla f(X_k)-D_{k+1}\middle|\mathcal{F}_k\right) \\ &\quad\leqslant c_1 l_c(l_c^2+1)\beta_k\left(\|X_k-X^*\|_2^2+\|\Lambda_k-\Lambda^*\|_2^2\right)+c_1c_2l_c\beta_k+3nc_1l_c\beta_k. \end{aligned} \tag{40}$$

Noting that $n_{\Omega^i}(x^{i*})\in N_{\Omega^i}(x^{i*})$ and $-\nabla f^i(x^{i*})+\lambda^*=n_{\Omega^i}(x^{i*})$, by (35) and conducting a similar analysis as that for (36)–(38), we can obtain

$$\begin{aligned} &(-\nabla f^i(x^i_k)+\lambda^i_k)^{\mathrm{T}}\mathrm{E}\left(\nabla f^i(x^i_k)-d^i_{k+1}\middle|\mathcal{F}_k\right) \\ &=(-\nabla f^i(x^i_k)+\lambda^i_k+\nabla f^i(x^{i*})-\lambda^*+n_{\Omega^i}(x^{i*}))^{\mathrm{T}} \\ &\quad\cdot\mathrm{E}\left([\Delta^i_{k+1}]^{-1}(\Delta^i_{k+1})^{\mathrm{T}}(\nabla f^i(x^i_k)-\nabla f^i(x^i_k+\mu^i_k\beta_k\Delta^i_{k+1}))\middle|\mathcal{F}_k\right) \\ &\leqslant(\|\nabla f^i(x^i_k)-\nabla f^i(x^{i*})\|_2+\|\lambda^i_k-\lambda^*\|_2+\|n_{\Omega^i}(x^{i*})\|_2) \\ &\quad\cdot\mathrm{E}\left(\|[\Delta^i_{k+1}]^{-1}(\Delta^i_{k+1})^{\mathrm{T}}\|_2\|(\nabla f^i(x^i_k)-\nabla f^i(x^i_k+\mu^i_k\beta_k\Delta^i_{k+1}))\|_2\middle|\mathcal{F}_k\right) \\ &\leqslant c_1l_c\beta_k(\|\nabla f^i(x^i_k)-\nabla f^i(x^{i*})\|_2+\|\lambda^i_k-\lambda^*\|_2+\|n_{\Omega^i}(x^{i*})\|_2) \\ &\leqslant c_1l_c\beta_k(\|\nabla f^i(x^i_k)-\nabla f^i(x^{i*})\|_2^2+\|\lambda^i_k-\lambda^*\|_2^2+\|n_{\Omega^i}(x^{i*})\|_2^2+3) \\ &\leqslant c_1l_c\beta_k(\|\nabla f^i(x^i_k)-\nabla f^i(x^{i*})\|_2^2+\|\lambda^i_k-\lambda^*\|_2^2+\|n_{\Omega^i}(x^{i*})\|_2^2)+3c_1l_c\beta_k, \end{aligned} \tag{41}$$

and then

$$\begin{aligned} &(-\nabla f(X_k)+\Lambda_k)^{\mathrm{T}}\mathrm{E}\left(\nabla f(X_k)-D_{k+1}\middle|\mathcal{F}_k\right) \\ &\leqslant c_1l_c\beta_k(\|\nabla f(X_k)-\nabla f(X^*)\|_2^2+\|\Lambda_k-\Lambda^*\|_2^2+\|n_\Omega(X^*)\|_2^2)+3nc_1l_c\beta_k \\ &\leqslant c_1l_c\beta_k(l_c^2\|X_k-X^*\|_2^2+\|\Lambda_k-\Lambda^*\|_2^2+\|n_\Omega(X^*)\|_2^2)+3nc_1l_c\beta_k \\ &\leqslant c_1l_c\beta_k(l_c^2+1)(\|X_k-X^*\|_2^2+\|\Lambda_k-\Lambda^*\|_2^2)+c_1c_2l_c\beta_k+3nc_1l_c\beta_k, \end{aligned} \tag{42}$$

with c_1 and c_2 defined by (26) and (27), respectively.

Thus, Eq. (40) is established, and the proof is complete.

Lemma 4. If Assumptions 1 and 4 hold, then

$$\mathrm{E}\left(\|E_{k+1}\|_2^2\middle|\mathcal{F}_k\right)\leqslant 2nc_1^2l_c^2\beta_k^2+6C_2(\|S_k-S^*\|_2^2+\|S^*\|_2^2)+c_3, \tag{43}$$

where c_1 and C_2 are defined by (26) and (24), respectively, and $c_3\triangleq 2np(p-1)a^2b^2L^2+C_1+n^3\sigma_2^2$.

Proof. According to the definitions of d^i_{k+1} and (34),

$$\begin{aligned} \mathrm{E}\left(\|\nabla f^i(x^i_k)-d^i_{k+1}\|_2^2\middle|\mathcal{F}_k\right)&=\mathrm{E}\left(\|\nabla f^i(x^i_k)-[\Delta^i_{k+1}]^{-1}(\Delta^i_{k+1})^{\mathrm{T}}\nabla f^i(x^i_k+\mu^i_k\beta_k\Delta^i_{k+1})\|_2^2\middle|\mathcal{F}_k\right) \\ &\leqslant 2\mathrm{E}\left(\|[\Delta^i_{k+1}]^{-1}(\Delta^i_{k+1})^{\mathrm{T}}(\nabla f^i(x^i_k)-\nabla f^i(x^i_k+\mu^i_k\beta_k\Delta^i_{k+1}))\|_2^2\middle|\mathcal{F}_k\right) \\ &\quad+2\mathrm{E}\left(\|(I-[\Delta^i_{k+1}]^{-1}(\Delta^i_{k+1})^{\mathrm{T}})\nabla f^i(x^i_k)\|_2^2\middle|\mathcal{F}_k\right), \end{aligned} \tag{44}$$

from which, by Assumptions 1 and 4 and the fact that $\nabla f^i(x^i_k)$ is $\mathcal{F}_k$-measurable and Δ^i_{k+1} is independent of $\mathcal{F}_k$,

$$\mathrm{E}\left(\|\nabla f^i(x^i_k)-d^i_{k+1}\|_2^2\middle|\mathcal{F}_k\right)\leqslant 2c_1^2l_c^2\beta_k^2+2p(p-1)a^2b^2L^2. \tag{45}$$

By combining (45) with (29) and (30), we have

$$\mathrm{E}\left(\|E_{k+1}\|_2^2\middle|\mathcal{F}_k\right)=\mathrm{E}\left(\|\nabla f(X_k)-D_{k+1}\|_2^2\middle|\mathcal{F}_k\right)+\mathrm{E}\left(\|e^1_{k+1}+e^2_{k+1}\|_2^2\middle|\mathcal{F}_k\right)+\mathrm{E}\left(\|e^3_{k+1}\|_2^2\middle|\mathcal{F}_k\right)$$

$$
\begin{aligned}
&\leqslant \sum_{i=1}^{n} \mathrm{E}(\|\nabla f^i(x_k^i) - d_{k+1}^i\|_2^2|\mathcal{F}_k) + C_2 \|\Lambda_k + \Gamma_k\|_2^2 + C_1 + C_2\|\Lambda_k\|_2^2 + n^3\sigma_2^2 \\
&\leqslant 2nc_1^2 l_c^2 \beta_k^2 + 2np(p-1)a^2b^2L^2 + 3C_2\|S_k\|_2^2 + C_1 + n^3\sigma_2^2 \\
&\leqslant 2nc_1^2 l_c^2 \beta_k^2 + 2np(p-1)a^2b^2L^2 + 6C_2(\|S_k - S^*\|_2^2 + \|S^*\|_2^2) + C_1 + n^3\sigma_2^2 \\
&= 2nc_1^2 l_c^2 \beta_k^2 + 6C_2(\|S_k - S^*\|_2^2 + \|S^*\|_2^2) + c_3. \qquad (46)
\end{aligned}
$$

Thus, we can now prove the strong consistency of the proposed algorithm. The mathematical proofs can be formulated into two steps: (1) establish the boundedness of the estimates generated from the algorithm and (2) prove that the estimates converge almost certainly to the optimal solution of the resource allocation problem. The main results are given as follows.

Theorem 1. Let us suppose that Assumptions 1–6 hold. Then, there exists an ω-set Ω_0 with $\mathbb{P}\{\Omega_0\} = 1$ such that for any $\omega \in \Omega_0$, $\{S_k(\omega)\}_{k\geqslant 0}$ generated by (17) is a bounded sequence with any initial S_0.

Proof. Let us suppose S^* is an equilibrium point of (21). According to Assumption 1 and (22), S^* is finite. Defining $V(S) \triangleq \|S - S^*\|_2^2$, we have

$$
\begin{aligned}
V(S_{k+1}) = \|S_{k+1} - S^*\|_2^2 &\leqslant \|S_k + \alpha_k(J(S_k) + E_{k+1}) - S^*\|_2^2 \\
&\leqslant \|S_k - S^*\|_2^2 + \alpha_k^2\|J(S_k) + E_{k+1}\|_2^2 + 2\alpha_k(S_k - S^*)^{\mathrm{T}}(J(S_k) + E_{k+1}), \quad (47)
\end{aligned}
$$

where for the first inequality, the nonexpansive property of the projection operator (see Proposition A1 in Appendix A) is applied. Note that S_k is $\mathcal{F}_k$-measurable. By taking the conditional expectation on both sides of (47), we derive

$$
\begin{aligned}
\mathrm{E}\left(V(S_{k+1})|\,\mathcal{F}_k\right) &\leqslant \mathrm{E}\left\{\|S_k - S^*\|_2^2 + \alpha_k^2\|J(S_k) + E_{k+1}\|_2^2 + 2\alpha_k(S_k - S^*)^{\mathrm{T}}(J(S_k) + E_{k+1})\big|\,\mathcal{F}_k\right\} \\
&= V(S_k) + \alpha_k^2\|J(S_k)\|_2^2 + \alpha_k^2\mathrm{E}\left(\|E_{k+1}\|_2^2 + 2E_{k+1}^{\mathrm{T}}J(S_k)\big|\,\mathcal{F}_k\right) + 2\alpha_k(S_k - S^*)^{\mathrm{T}}J(S_k) \\
&\quad + 2\alpha_k(S_k - S^*)^{\mathrm{T}}\mathrm{E}\left(E_{k+1}|\,\mathcal{F}_k\right). \qquad (48)
\end{aligned}
$$

From the definitions of S_k and $J(\cdot)$ and by noting (22), we have

$$
\begin{aligned}
(S_k - S^*)^{\mathrm{T}}J(S_k) &= (X_k - X^*)^{\mathrm{T}}\left(-\nabla f(X_k) + \Lambda_k + \nabla f(X^*) - \Lambda^* + n_\Omega(X^*)\right) \\
&\quad + (\Lambda_k - \Lambda^*)^{\mathrm{T}}\left(-(\bar{L}\otimes I_p)(\Lambda_k + \Gamma_k) + R - X_k + (\bar{L}\otimes I_p)(\Lambda^* + \Gamma^*) - R + X^*\right) \\
&\quad + (\Gamma_k - \Gamma^*)^{\mathrm{T}}\left((\bar{L}\otimes I_p)\Lambda_k - (\bar{L}\otimes I_p)\Lambda^*\right) \\
&= -(X_k - X^*)^{\mathrm{T}}\left(\nabla f(X_k) - \nabla f(X^*)\right) + (X_k - X^*)^{\mathrm{T}} n_\Omega(X^*) \\
&\quad - (\Lambda_k - \Lambda^*)^{\mathrm{T}}(\bar{L}\otimes I_p)(\Lambda_k - \Lambda^*).
\end{aligned}
$$

Through the convexity of $f^i(\cdot), i \in \mathcal{N}$, it follows that $-(X_k - X^*)^{\mathrm{T}}\left(\nabla f(X_k) - \nabla f(X^*)\right) \leqslant 0$. From the definition of cone $N_\Omega(X^*)$, $(X_k - X^*)^{\mathrm{T}} n_\Omega(X^*) \leqslant 0$. By noting that $\bar{L}$ is non-negative definite, we find that $-(\Lambda_k - \Lambda^*)^{\mathrm{T}}(\bar{L}\otimes I_p)(\Lambda_k - \Lambda^*) \leqslant 0$. Hence, from the above equalities we obtain

$$
(S_k - S^*)^{\mathrm{T}}J(S_k) \leqslant 0. \qquad (49)
$$

From (48) and (49), we derive

$$
\begin{aligned}
\mathrm{E}\left(V(S_{k+1})|\,\mathcal{F}_k\right) &\leqslant V(S_k) + \alpha_k^2\|J(S_k)\|_2^2 + \alpha_k^2\mathrm{E}\left(\|E_{k+1}\|_2^2\big|\,\mathcal{F}_k\right) \\
&\quad + 2\alpha_k^2 J(S_k)^{\mathrm{T}}\mathrm{E}\left(E_{k+1}|\,\mathcal{F}_k\right) + 2\alpha_k(S_k - S^*)^{\mathrm{T}}\mathrm{E}\left(E_{k+1}|\,\mathcal{F}_k\right). \qquad (50)
\end{aligned}
$$

Again, by using (22) and the definition of $J(\cdot)$, we have the following equality and inequalities:

$$
\begin{aligned}
\|J(S_k)\|_2^2 &= \|-\nabla f(X_k) + \Lambda_k + \nabla f(X^*) - \Lambda^* + n_\Omega(X^*)\|_2^2 \\
&\quad + \|(\bar{L}\otimes I_p)(\Lambda_k - \Lambda^*) + (\bar{L}\otimes I_p)(\Gamma_k - \Gamma^*) + X_k - X^*\|_2^2 + \|(\bar{L}\otimes I_p)(\Lambda_k - \Lambda^*)\|_2^2 \\
&\leqslant 3\left(\|\nabla f(X_k) - \nabla f(X^*)\|_2^2 + \|\Lambda_k - \Lambda^*\|_2^2 + \|n_\Omega(X^*)\|_2^2\right) \\
&\quad + 3\left(\|(\bar{L}\otimes I_p)(\Lambda_k - \Lambda^*)\|_2^2 + \|(\bar{L}\otimes I_p)(\Gamma_k - \Gamma^*)\|_2^2 + \|X_k - X^*\|_2^2\right) + \|(\bar{L}\otimes I_p)(\Lambda_k - \Lambda^*)\|_2^2 \\
&\leqslant (3l_c^2 + 3)\|X_k - X^*\|_2^2 + (4C_3 + 3)\|(\Lambda_k - \Lambda^*)\|_2^2 + 3C_3\|(\Gamma_k - \Gamma^*)\|_2^2 + 3c_2
\end{aligned}
$$

$$\leqslant (3l_c^2+3+4C_3)\|S_k-S^*\|_2^2+3c_2 \triangleq c_4V(S_k)+3c_2, \tag{51}$$

where C_3 and c_2 are defined by (25) and (27), respectively.

By noting (31), (32), (43), and (51), from (50), we have

$$\begin{aligned}
&\mathrm{E}\left(V(S_{k+1})\middle|\mathcal{F}_k\right)\\
&\leqslant V(S_k)+c_4\alpha_k^2V(S_k)+3c_2\alpha_k^2+2nc_1^2l_c^2\alpha_k^2\beta_k^2+6C_2\alpha_k^2(\|S_k-S^*\|_2^2+\|S^*\|_2^2)+c_3\alpha_k^2\\
&\quad+2c_1l_c(l_c^2+1)\alpha_k^2\beta_k(\|X_k-X^*\|_2^2+\|\Lambda_k-\Lambda^*\|_2^2)+2c_1c_2l_c\alpha_k^2\beta_k+6nc_1l_c\alpha_k^2\beta_k\\
&\quad+2c_1l_c\alpha_k\beta_k\|X_k-X^*\|_2^2+2nc_1l_c\alpha_k\beta_k\\
&\leqslant V(S_k)+(c_4+6C_2)\alpha_k^2V(S_k)+2c_1l_c(l_c^2+1)\alpha_k^2\beta_kV(S_k)+2c_1l_c\alpha_k\beta_kV(S_k)\\
&\quad+(3c_2+6C_2\|S^*\|_2^2+c_3)\alpha_k^2+2nc_1^2l_c^2\alpha_k^2\beta_k^2+(2c_1c_2l_c+6nc_1l_c)\alpha_k^2\beta_k+2nc_1l_c\alpha_k\beta_k\\
&\leqslant \left(1+(c_4+6C_2)\alpha_k^2+2c_1l_c(l_c^2+1)\alpha_k^2\beta_k+2c_1l_c\alpha_k\beta_k\right)V(S_k)\\
&\quad+(3c_2+6C_2\|S^*\|_2^2+c_3)\alpha_k^2+2nc_1^2l_c^2\alpha_k^2\beta_k^2+(2c_1c_2l_c+6nc_1l_c)\alpha_k^2\beta_k+2nc_1l_c\alpha_k\beta_k.
\end{aligned}\tag{52}$$

Assumption 5 indicates that $\sum_{k=1}^{\infty}\alpha_k^2\beta_k<\infty$ and $\sum_{k=1}^{\infty}\alpha_k^2\beta_k^2<\infty$. Then by Proposition A2 in Appendix A, $V(S_k)$ converges to a finite non-negative random variable as $k\to\infty$ almost certainly. Hence, $\{S_k\}_{k\geqslant 0}$ is bounded with probability one.

Theorem 2. Let us suppose that Assumptions 1–6 hold. Then, there exists an ω-set Ω_0 with $\mathbb{P}\{\Omega_0\}=1$ such that, for any $\omega\in\Omega_0$, $\{S_k(\omega)\}_{k\geqslant 0}$ generated by (17) converges to $S^*=(X^*,\Lambda^*,\Gamma^*)$, which is one of the equilibrium points of (21). Hence, $\{X_k(\omega)\}_{k\geqslant 0}$ converges to X^*, which is the optimal solution to problem (1).

Proof. Note that S_{k+1}, $J(S_k)+E_{k+1}$, and $\Phi=\Omega\times\mathbb{R}^{np}\times\mathbb{R}^{np}$ correspond to χ_{k+1}, Y_{k+1}, and Φ, respectively, in Proposition A4 in Appendix A. Thus, to prove the convergence of $\{S_k\}_{k\geqslant 0}$ and $\{X_k\}_{k\geqslant 0}$, we only need to verify conditions (C1)–(C5) of Proposition A4.

We first verify that condition (C1) holds for algorithm (17). Noting that S_k is $\mathcal{F}_k$-measurable, we have

$$\begin{aligned}
\mathrm{E}(\|Y_{k+1}\|_2^2)&=\mathrm{E}\left(\mathrm{E}\left(\|J(S_k)+E_{k+1}\|_2^2\middle|\mathcal{F}_k\right)\right)\\
&=\mathrm{E}\left(\|J(S_k)\|_2^2+\mathrm{E}(\|E_{k+1}\|_2^2|\mathcal{F}_k)+2J(S_k)^{\mathrm{T}}\mathrm{E}\left(E_{k+1}\middle|\mathcal{F}_k\right)\right).
\end{aligned}\tag{53}$$

Let

$$\begin{aligned}
&g_1(\alpha_k,\beta_k)=(c_4+6C_2)\alpha_k^2+2c_1l_c(l_c^2+1)\alpha_k^2\beta_k+2c_1l_c\alpha_k\beta_k,\\
&g_2(\alpha_k,\beta_k)=(3c_2+6C_2\|S^*\|_2^2+c_3)\alpha_k^2+2nc_1^2l_c^2\alpha_k^2\beta_k^2+(2c_1c_2l_c+6nc_1l_c)\alpha_k^2\beta_k+2nc_1l_c\alpha_k\beta_k.
\end{aligned}$$

Then, considering the expectation on both sides of (52), we obtain

$$\mathrm{E}(V(S_{k+1}))\leqslant(1+g_1(\alpha_k,\beta_k))\mathrm{E}(V(S_k))+g_2(\alpha_k,\beta_k).$$

Noting that $\sum_{k=0}^{\infty}g_1(\alpha_k,\beta_k)<\infty$ and $\sum_{k=0}^{\infty}g_2(\alpha_k,\beta_k)<\infty$, by Proposition A3 in Appendix A, $\{\mathrm{E}(V(S_k))\}$ is considered to be bounded.

Therefore, according to (32), (43), (51), and (53), we have $\mathrm{E}(\|Y_{k+1}\|_2^2)<\infty$, and hence, condition (C1) holds for algorithm (17).

We now verify that conditions (C2) and (C3) hold for algorithm (17). It directly follows that

$$\mathrm{E}\left[Y_{k+1}\middle|\chi_0,Y_i,i\leqslant k\right]=\mathrm{E}\left[J(S_k)+E_{k+1}\middle|\mathcal{F}_k\right]=J(S_k)+\mathrm{E}\left(E_{k+1}\middle|\mathcal{F}_k\right)=J(S_k)+\zeta_{k+1},$$

and by (28)

$$\zeta_{k+1}=\mathrm{E}\left(E_{k+1}\middle|\mathcal{F}_k\right)=\mathrm{col}\{\mathrm{E}\left(\nabla f(X_k)-D_{k+1}\middle|\mathcal{F}_k\right),0,0\},$$

with $0\in\mathbb{R}^{np}$. Thus, the measurable function $h(\cdot)$ and random variable ζ_{k+1} required by (C2) in Proposition A4 can be chosen as $h(\cdot)=J(\cdot)$ and $\zeta_{k+1}=\mathrm{E}\left(E_{k+1}\middle|\mathcal{F}_k\right)$, respectively, and hence, conditions (C2) and (C3) hold for algorithm (17).

We now proceed to consider (C4). Let $\zeta_{k+1,i}=\mathrm{E}\left(\nabla f^i(x_k^i)-d_{k+1}^i\middle|\mathcal{F}_k\right)$. By (35), we derive

$$\begin{aligned}
\|\zeta_{k+1,i}\|_1&=\left\|\mathrm{E}\left([\Delta_{k+1}^i]^{-1}(\Delta_{k+1}^i)^{\mathrm{T}}(\nabla f^i(x_k^i)-\nabla f^i(x_k^i+\mu_k^i\beta_k\Delta_{k+1}^i))\middle|\mathcal{F}_k\right)\right\|_1\\
&\leqslant c_5\left\|\mathrm{E}\left([\Delta_{k+1}^i]^{-1}(\Delta_{k+1}^i)^{\mathrm{T}}(\nabla f^i(x_k^i)-\nabla f^i(x_k^i+\mu_k^i\beta_k\Delta_{k+1}^i))\middle|\mathcal{F}_k\right)\right\|_2\\
&\leqslant c_5\mathrm{E}\left(\|[\Delta_{k+1}^i]^{-1}(\Delta_{k+1}^i)^{\mathrm{T}}\|_2\|\nabla f^i(x_k^i)-\nabla f^i(x_k^i+\mu_k^i\beta_k\Delta_{k+1}^i)\|_2\middle|\mathcal{F}_k\right)\\
&\leqslant c_5c_1l_c\beta_k,
\end{aligned}$$

where the first inequality holds by equivalence of vector norms in Euclidean space, and similarly, the last two inequalities follow as (36)–(38). Therefore,

$$\sum_{k=0}^{\infty}\alpha_k\|\zeta_{k+1}\|_1=\sum_{k=0}^{\infty}\alpha_k\sum_{i=1}^{n}\|\zeta_{k+1,i}\|_1\leqslant\sum_{k=0}^{\infty}\alpha_k\sum_{i=1}^{n}c_5c_1l_c\beta_k=nc_5c_1l_c\sum_{k=0}^{\infty}\alpha_k\beta_k,$$

which is finite, as by Assumption 5, $\sum_{k=0}^{\infty}\alpha_k\beta_k<\infty$. Hence condition (C4) holds for algorithm (17).

Finally, Theorem 1 indicates that $\{S_k\}_{k\geqslant 0}$ is bounded almost certainly, and thus, condition (C5) holds for algorithm (17).

Thus, all conditions required in Proposition A4 hold for algorithm (17). By Proposition A4, we know that $\{S_k\}_{k\geqslant 0}$ converges to a limit set of (21). By Lemma 1, the trajectories of (21) converge to one of its equilibria S^*. Therefore, $\{S_k\}_{k\geqslant 0}$ converges to S^* and $\{X_k\}_{k\geqslant 0}$ converges to the optimal solution of problem (1) as $k\to\infty$.

4 Numerical example

Example 1. Consider a network with five agents. Each agent i ($i=1,\ldots,5$) has its local objective function $f^i(x^i)=\frac{1}{i}((x^i)^2+x^i)$ and the resource over the network is $\sum_{i=1}^{5}r^i=100$. The constraint set for each agent is set as $0\leqslant x^i\leqslant 100,\ i=1,\ldots,5$. The resource allocation problem is given as follows:

$$\begin{aligned}&\min_{x^i}\sum_{i=1}^{5}\frac{1}{i}((x^i)^2+x^i),\\ &\text{s.t.}\sum_{i=1}^{5}x^i=\sum_{i=1}^{5}r^i=100,\quad 0\leqslant x^i\leqslant 100,\ i=1,\ldots,5.\end{aligned}$$

A simulation is performed over an Erdős-Rényi random graph. Let us consider a graph set $\mathcal{G}$ containing 50 graphs, each of which is generated according to the E-R graph $G(5,p)$, where the probability p is selected independently and uniformly over $[0.2,0.3]$. At each iteration, a graph is randomly selected from the graph set $\mathcal{G}$.

The perturbation signals $\{\Delta_k^i\}_{k\geqslant 0}$ are chosen as independently and uniformly distributed random variables over $[-1,-0.5]\cup[0.5,1]$. The resource observation noises $\{\delta_k^i\}_{k\geqslant 0}$ and communication noises $\{\xi_k^{ij}\}_{k\geqslant 0}$ and $\{\eta_k^{ij}\}_{k\geqslant 0}$ are assumed to be i.i.d. random variables with Gaussian distribution $N(0,1)$. We choose the step sizes $\alpha_k=\frac{1}{k}$ for algorithm (13) and $\beta_k=\frac{1}{k^{0.1}}$ for the randomized differences (2).

The simulation results are shown in Figures 1–4. Figure 1 shows how the agents in the network cooperatively determine the optimal resource allocation solution. Specifically, in Figure 1, the solid lines denote the trajectories generated by the proposed gradient-free algorithm; the colored dashed lines denote the estimates generated by the algorithms proposed in [14], which are gradient-based; and the black dashed lines denote true values of the optimal resource allocation. Figure 2 compares the algorithm in this paper with that proposed in [14] in terms of the error between the estimates and the optimal resource allocation. Figure 3 shows the changes in the value of the global objective function as iteration k increases, and Figure 4 shows those in the global resource after being allocated over the network. In Figures 3 and 4, the solid lines denote the trajectories generated by the proposed gradient-free algorithm, whereas the colored dashed lines denote the estimates generated by the algorithms proposed in [14].

Example 2. Consider a three-period demand response management problem over a time-varying network with 10 agents, which can be formulated as

$$\begin{aligned}&\min_{P_i^d\in\mathbb{R}^3,i\in\mathcal{N}}\sum_{i\in\mathcal{N}}\mathrm{E}_{\Psi_i,\theta_i}\left({P_i^d}^{\mathrm{T}}(Q_i+\Psi_i)P_i^d+(c_i+\theta_i)^{\mathrm{T}}P_i^d\right),\\ &\text{s.t.}\sum_{i\in\mathcal{N}}P_i^d=\sum_{i\in\mathcal{N}}P_i^g,\quad P_i^d\in\Omega^i,\end{aligned}$$

where P_i^d indicates the load demand decision variables to be optimized later, P_i^g indicates the randomly generated scheduling variables, Q_i and c_i are the randomly generated positive definite matrix and vector,

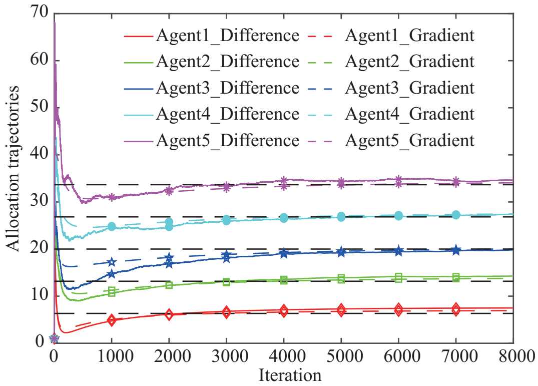

Figure 1 (Color online) Changes in agents' allocated resources.

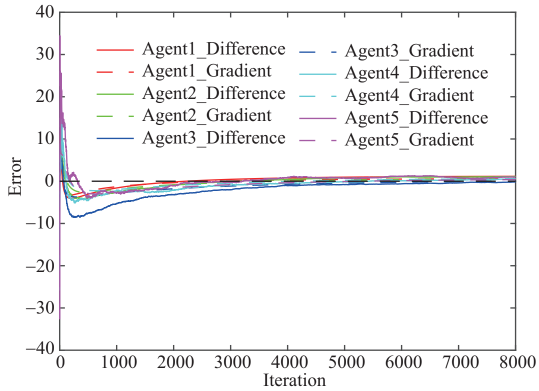

Figure 2 (Color online) Changes in errors between agents' estimates and the optimal solution of resource allocation.

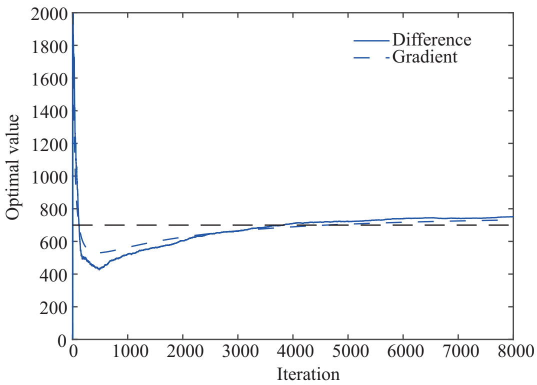

Figure 3 (Color online) Changes in the values of the global objective function.

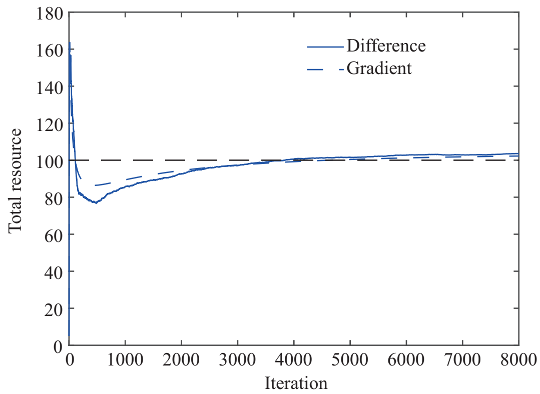

Figure 4 (Color online) Changes in the allocated global resources.

respectively, and the entries of Ψ_i and θ_i are random variables with standard Gaussian distribution. Set $\Omega_i = \{x|Rx \leqslant l_i\}$ with l_i being randomly generated and

$$R = \begin{pmatrix} -1 & 1 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}^{\mathrm{T}}.$$

It is directly verified that such a choice of Ω_i meets the requirement in Assumption 2.

The simulation is still conducted over the Erdős-Rényi random graph. Consider a graph set $\mathcal{G}$ containing 50 graphs, each of which is generated according to the E-R graph $G(10, p)$, where the probability p is selected independently and uniformly over $[0.2, 0.3]$. At each iteration, a graph is randomly selected from the graph set $\mathcal{G}$.

Assume that the resource observation noise $\{\delta_k^i\}_{k\geqslant 0}$ and communication noises $\{\xi_k^{ij}\}_{k\geqslant 0}$ and $\{\eta_k^{ij}\}_{k\geqslant 0}$ are i.i.d. random variables with Gaussian distribution $N(0, 2)$. We choose the perturbation signals $\{\Delta_k^i\}_{k\geqslant 0}$ as random vectors, with the entries being mutually independent and uniformly distributed over $[-1, -0.5] \cup [0.5, 1]$. Set $\alpha_k = \frac{1}{k^{0.6}}$ and $\beta_k = \frac{1}{k^{0.2}}$.

In this simulation, we conduct five experiments, and Figure 5 shows the average trajectories of the allocated resource of all agents during these experiments, where the colored solid lines denote the trajectories generated by the proposed gradient-free algorithm and the black dashed lines denote the estimates generated by the gradient-based algorithms present in [14]. In Figure 6, we show the performance $\|\bar{L}\Lambda\|_2$ generated by the gradient-free algorithm and the gradient-based algorithms, respectively, which indicate that the Lagrange multiplier of each agent achieves consensus.

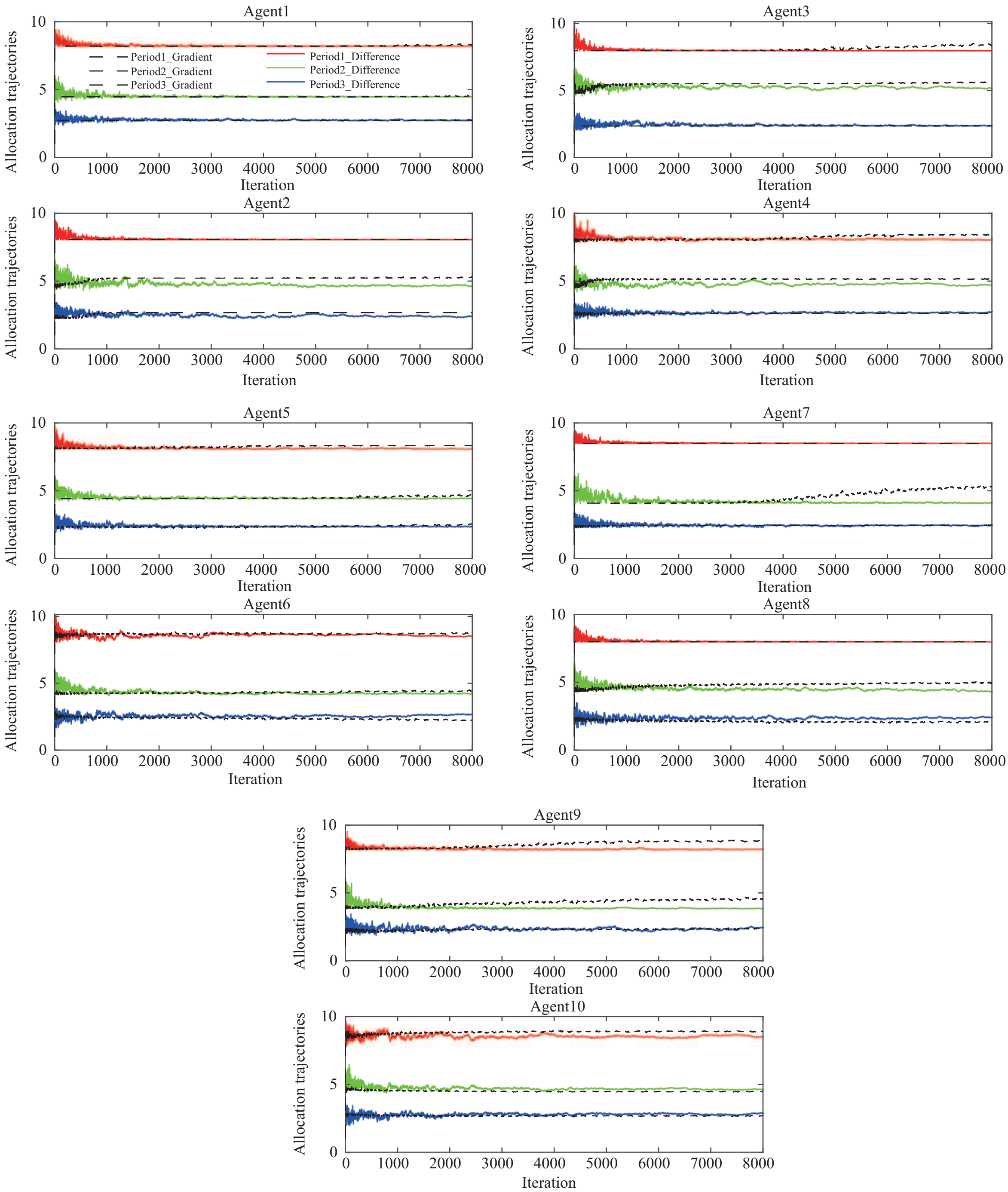

Figure 5 (Color online) Changes in the allocated resources of Agent1–Agent10.

From the above two simulations, we find that the performance of the proposed gradient-free algorithm is comparable to the gradient-based algorithm proposed in [14].

5 Conclusion

This paper proposes a gradient-free distributed algorithm for resource allocation based on a randomized difference technique. We established an almost certain convergence of the algorithm by applying the ODE method and tested its performance through numerical examples. For further research, it will be

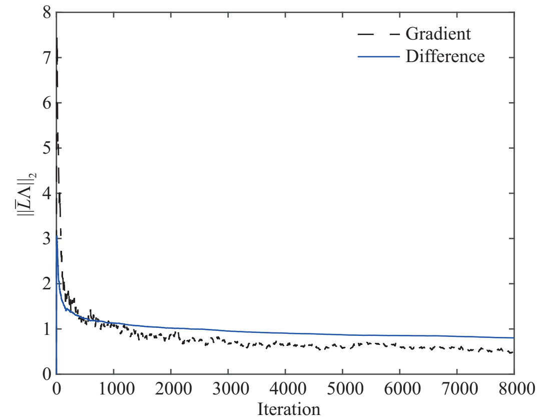

Figure 6 (Color online) Changes in performance index $\|\bar{L}\Lambda\|_2$ of gradient-based and gradient-free algorithms.

interesting to improve the convergence rate of the proposed algorithm and consider the time-varying objective functions for distributed resource allocation.

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Appendix A

Proposition A1 ([25]). Let $\mathcal{P}_\Omega(\cdot)$ be the projection operator onto a closed convex set Ω in $\mathbb{R}^p$. Then, for all $x, y \in \mathbb{R}^p$, the nonexpansive inequality holds as follows:

$$\|\mathcal{P}_\Omega(x) - \mathcal{P}_\Omega(y)\|_2 \leqslant \|x - y\|_2.$$

Proposition A2 ([25]). Let $(\Omega, \mathcal{F}, P)$ denote the probability space. Let $\{V_k\}_{k\geqslant 0}$, $\{U_k\}_{k\geqslant 0}$, and $\{W_k\}_{k\geqslant 0}$ be non-negative random variables, and $\mathcal{F}_k = \sigma\{V_1, \dots, V_k, U_1, \dots, U_k, W_1, \dots, W_k\}$. Suppose that

$$\mathrm{E}(V_{k+1}|\mathcal{F}_k) \leqslant (1+\omega_k)V_k - U_k + W_k \tag{A1}$$

and $\sum_{k=0}^{\infty} W_k < \infty$ hold with probability one, and $\{\omega_k\}_{k\geqslant 0}$ is a deterministic non-negative scalar sequence with $\sum_{k=0}^{\infty}\omega_k < \infty$. Then, $\{V_k\}_{k\geqslant 0}$ converges to a finite non-negative random variable and $\sum_{k=0}^{\infty} U_k < \infty$ with probability one.

Proposition A3 [2]**.** Let $u_k \geqslant 0$ and

$$u_{k+1} \leqslant (1+a_k)u_k + b_k$$

with $a_k \geqslant 0$, $b_k \geqslant 0$, and $\sum_{k=0}^{\infty} a_k < \infty$, $\sum_{k=0}^{\infty} b_k < \infty$. Then $u_k \to u \geqslant 0$.

Proposition A4 ([30]). Let us assume that the following conditions hold for the algorithm $\chi_{k+1} = \mathcal{P}_\Phi\{\chi_k + \alpha_k Y_{k+1}\}$, where Φ is a compatible dimension convex constraint set:

(C1) $\sup_k \mathrm{E}(\|Y_{k+1}\|_2^2) < \infty$;

(C2) There is a measurable function $h(\cdot)$ of χ and random variables ζ_{k+1} such that $\mathrm{E}\,[\,Y_{k+1}|\,\chi_0, Y_i, i \leqslant k] = h(\chi_k) + \zeta_{k+1}$;

(C3) $h(\cdot)$ is continuous;

(C4) $\sum_{k=0}^{\infty} \alpha_k|\zeta_{k+1}| < \infty$ with probability one;

(C5) χ_k is bounded with probability one.

Suppose that Φ has the same form as Ω^i, and Assumption 2 holds for Φ. Then, the sequence $\{\chi_k\}$ is associated with the projected ODE:

$$\dot{\chi} = h(\chi) + z, \tag{A2}$$

where $z \in -N_\Phi(\chi)$ is the projection term, which indicates the minimum force required to keep $\chi(\cdot)$ in Φ. In addition, $\{\chi_k\}$ converges to some limit set of ODE (A2) in Φ.

2) Polyak B T. Introduction to Optimization. New York: Optimization Software Inc., 1987.