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• LETTER •

Special Focus on Theory and Applications of Models of Computation

On book thickness parameterized by the vertex cover number

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Dear editor,

Book thickness for graphs forms a major theme in graph theory and has a broad application in the fields of sorting permutations, fault tolerant VLSI design, parallel computing, and others. The BOOK THICKNESS problem, even if the vertex order is taken as part of the input, was shown to be NP-complete in general [1, 2]. Just because of this, parameterized algorithms have been proposed to deal with it [3]. In particular, the parameterized BOOK THICKNESS problem with respect to the vertex cover number τ of the input graph, abbreviated as BTVC, can be described as follows [3].

Input: an undirected graph G and a positive integer k.

Parameter: the vertex cover number τ.

Task: decide whether G admits a k-page book embedding.

The BTVC problem was firstly studied by Bhore et al. [3] and has attracted much attention from several aspects. First of all, an algorithm of running time $\mathcal{O}(2^{2^{\mathcal{O}(\tau)}} + \tau \cdot |V|)$ was developed by Liu et al. [4], improving that of running time $\mathcal{O}(2^{\tau^{\mathcal{O}(\tau)}} + \tau \cdot |V|)$ in [3]. Secondly, two sister problems (note that the BOOK THICKNESS problem is also named the STACK LAYOUT problem), including the QUEUE LAYOUT problem and the ARCH LAYOUT problem, have been studied by using an approach originally applied to BTVC [4, 5]. Thirdly, in recent studies, several problems including MIXED s-STACK q-QUEUE LAYOUT [4], UPWARD STACK LAYOUT on directed acyclic graphs [4], and BOOK DRAWING with a limited number of crossings per edge [6], can be seen as extensions of the BTVC problem from different angles.

In this study, we first present a refined kernel for the BTVC problem. By introducing a class of 1-page 2-degree embedding graphs, we derive an upper bound of $\mathcal{O}(\tau^2)$ for the number of uncloneable vertices with the same type, improving the previous bound of $\mathcal{O}(\tau^3)$ in [4] (the explanations of related notations are provided in Appendix A).

Assume that (G, k, τ) is an instance of the BTVC problem. Let C be a minimum vertex cover of G and let W be a subset of C. A graph $\mathcal{H}_W$ is called a 1-page 2-degree embedding graph with respect to W if it meets the following requirements: (1) all its vertices lie on a horizontal axis l and all its edges are drawn inside the upper half-plane bounded by l; (2) each of vertices in $V(\mathcal{H}_W) \setminus W$ has exactly degree 2; and (3) no two edges in $E(\mathcal{H}_W)$ cross. A 1-page 2-degree embedding graph $\mathcal{H}_W$ with respect to W is maximum if there exists no other 1-page 2-degree embedding graph $\mathcal{H}'_W$ with respect to W such that $|V(\mathcal{H}_W)| < |V(\mathcal{H}'_W)|$. See Figure 1 for an example, where $W = \{c_1, c_2, c_3, c_4\}$.

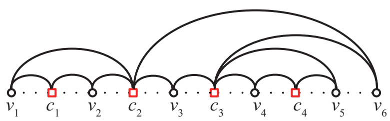

Figure 1 (Color online) A maximum 1-page 2-degree embedding graph with respect to the vertex set $\{c_1, c_2, c_3, c_4\}$.

Let $\mathcal{H}_W$ be a maximum 1-page 2-degree embedding graph with respect to W and let $\prec$ be the specified order of vertices in $V(\mathcal{H}_W)$. To estimate the size of $V(\mathcal{H}_W) \setminus W$, we adopt a strategy in which $\mathcal{H}_W$ is decomposed into smaller parts recursively by two types of vertex deletions. Let $v \in V(\mathcal{H}_W) \setminus W$ and let vc_i and vc_j be edges incident to v. Assuming that $c_i \prec c_j$, two types of deletions on v are classified by the ordering of vertices v, c_i, and c_j. The deletion is called a middle-vertex deletion if $c_i \prec v \prec c_j$. Otherwise, it is called a side-vertex deletion.

Lemma 1. Let $\mathcal{H}_W$ be an arbitrary maximum 1-page 2-degree embedding graph with respect to W. Then $\mathcal{H}_W$ can be reduced to an empty graph $\mathcal{H} = (W, \emptyset)$ by a sequence of side-vertex deletions alternated with middle-vertex deletions.

Based on Lemma 1, the number of vertices in $V(\mathcal{H}_W) \setminus W$ can be expressed by a function of $|W|$.

Lemma 2. Let $\mathcal{H}_W$ be an arbitrary maximum 1-page 2-degree embedding graph with respect to W. Then $|V(\mathcal{H}_W) \setminus W| = 2(|W| - 1)$.

From Lemma 2, the number of uncloneable vertices having the same type can be bounded by $\mathcal{O}(\tau^2)$. Following the

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analysis in [4], we arrive at our first result.

Theorem 1. The BTVC problem admits a kernel with at most $2^{\tau}\cdot 2\tau^2+\tau$ vertices.

Next, we prove a time bound of $2^{\mathcal{O}(\tau^2\log\tau)}\cdot|V|$ for the algorithm solving the FIXED-ORDER BTVC problem in [3], improving the previous bound of $2^{\mathcal{O}(\tau^3)}\cdot|V|$. Our result depends heavily on a new technique named edge-adjusting, which is described as follows.

Suppose that $\alpha_i\cup s$ is a valid page assignment of $E_i\cup E_C$ and let $p\in[1,k]$ be an arbitrary page in $\alpha_i\cup s$. Let $E(p,u_j)=\{u_jc^1_{u_j},u_jc^2_{u_j},\ldots,u_jc^d_{u_j}\}$ (for $u_j\in V(G)\setminus C$ and $j<i$) be the subset of edges incident to u_j and assigned to page p. Without loss of generality, assume that $c^1_{u_j}\prec c^2_{u_j}\prec\cdots\prec c^d_{u_j}$. A unit operation 1-ADJUST(u_j,u_h) (for $u_h\prec u_j$) is defined as follows for adjusting edges from $E(p,u_j)$ to $E(p,u_h)$.

For $r=1$ **to** d **do**:
(1) remove the edge $u_jc^r_{u_j}$;
(2) if $E(p,u_h)$ does not contain the edge $u_hc^r_{u_j}$, then insert a new edge $u_hc^r_{u_j}$ on page p.

Assume that the vertices in $V(G)\setminus C$ are ordered as $u_1\prec\cdots\prec u_{n-\tau}$ and that $V'=\{u_x\mid x\in X\}\cup\{u_1\}$ (refer to [3] for the definition on set X). Let $x_1,x_2,\ldots,x_z$ denote the integers in X. Based on the unit operation above, we develop a procedure 1-edge-adjusting(p), whose main steps are described as follows.

For $j=2$ **to** $i-1$ **do**
If $u_j\notin V'$ and $E(p,u_j)\neq\emptyset$ **then**
(1) locate the vertex u_{x_y} in V' such that u_{x_y} is the nearest vertex to the left of u_j;
(2) execute the operation 1-ADJUST(u_j,u_{x_y}).

After executing procedure 1-edge-adjusting(p), we obtain a simplified page p' in which the endpoints of any edge fall into $C\cup V'$. Let $\alpha'_i\cup s$ be the assignment obtained by executing 1-edge-adjusting(p) on each page p for $p=1,2,\ldots,k$ in $\alpha_i\cup s$.

Lemma 3. The equation $M_i(x,\alpha'_i,s)=M_i(x,\alpha_i,s)$ holds for each x in $\{i\}\cup X$.

From Lemma 3, the size of $\mathcal{R}_i(s)$ can be estimated by an improved upper bound.

Lemma 4. $|\mathcal{R}_i(s)|\leqslant 2^{\mathcal{O}(\tau^2\log\tau)}$.

Based on the fact $|S|<\tau^{\tau^2}$ and Lemma 4, the following result holds for FIXED-ORDER BTVC.

Theorem 2. There is an algorithm that, given a graph $G=(V,E)$ with a vertex order $\prec$, runs in time $2^{\mathcal{O}(\tau^2\log\tau)}\cdot|V|$ and returns a bt$(G,\prec)$-page book embedding of G, where τ is the vertex cover number of G and bt$(G,\prec)$ denotes the book thickness of G with respect to $\prec$.

Finally, we employ the edge-adjusting technique to analyze the algorithm for an extended FIXED-ORDER BTVC with the relaxation that at most b crossings over all pages are allowed. To facilitate our analysis, we give a specific description for that algorithm by extending the techniques used in [3], whose feasibility was mentioned by Bhore et al. [3]. A page assignment $\alpha_i\cup s$ that maps edges in $E_i\cup E_C$ to pages $[1,k]$ with at most b crossings over all pages is called a valid page assignment of $E_i\cup E_C$. Let $\alpha_i\cup s$ be a valid assignment of $E_i\cup E_C$. We set a variable $N_i(\alpha_i,s)$ to store the number of crossings that already exist over all pages. Given a vertex $u_a\in(V(G)\setminus C)$, we also design a $k\times\tau$ matrix $M_i(a,\alpha_i,s)$ to record the number of "potential" crossings. More precisely, the entry (p,q) in $M_i(a,\alpha_i,s)$ stores the number of edges separating c_q from u_a on page p. It only stores the integer $b+1$ if this number is any number greater than $b+1$.

Lemma 5. Let $\alpha_i\cup s$ be a valid page assignment of $E_i\cup E_C$. Assuming that $u_h\in V(G)\setminus C$ with $h>i$ and $h\notin X$, let $x_j\in\{i\}\cup X$ be such that $x_j\geqslant i$ and $h-x_j=\min\{|h-x|\mid x\in\{i\}\cup X\}$. Then $M_i(h,\alpha_i,s)=M_i(x_j,\alpha_i,s)$.

In algorithm analysis, we define another unit operation 2-ADJUST(u_j,u_h) for adjusting edges from $E(p,u_j)$ to $E(p,u_h)$ as follows.

For r=1 **to** d **do**:
(1) remove the edge $u_jc^r_{u_j}$;
(2) if the multiplicity of $u_hc^r_{u_j}$ in $E(p,u_h)$ is no more than b, then insert an edge between u_h and $c^r_{u_j}$ on page p.

By introducing a procedure 2-edge-adjusting(p), the extended record set $\mathcal{R}'_i(s)$ can be estimated as follows.

Lemma 6. $|\mathcal{R}'_i(s)|\leqslant 2^{\mathcal{O}((\tau^2+b\tau)\log\tau(b+1))}$.

Based on Lemma 6, we arrive at the following result.

Theorem 3. There exists an algorithm that, given a graph $G=(V,E)$ with a vertex order $\prec$, and a nonnegative integer b, runs in time $2^{\mathcal{O}((\tau^2+b\tau)\log\tau(b+1))}\cdot|V|$ and returns a bt$(G,\prec,b)$-page book drawing of G, where τ is the vertex cover number of G and bt$(G,\prec,b)$ denotes the smallest $k>0$ such that $((G,\prec),k,\tau,b)$ is a yes-instance of the extended FIXED-ORDER BTVC problem.

Remark. The edge-adjusting technique is the key ingredient in our analysis. Recently, one variant of this technique was applied to the problem FIXED-ORDER BOOK THICKNESS parameterized by the pathwidth of the vertex ordering [7].

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Supporting information Appendixes A–D. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

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