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• **Supplementary File** •

On book thickness parameterized by the vertex cover number*

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Appendix A Preliminaries

We only consider undirected graphs. Given a graph $G = (V, E)$, we use uv to denote the edge $e \in E(G)$ with endpoints u and v. In particular, G is called an *empty graph* if $E(G) = \emptyset$. For $r \in \mathbb{N}$, we denote the set $\{1, \dots, r\}$ by $[1, r]$.

A *vertex cover* of a given graph $G = (V, E)$ is a subset V' of vertices in V such that each edge in E has at least one of its endpoints in V'. The vertex cover number of G, denoted by τ, is the size of a minimum vertex cover of G. We will use C to denote a minimum vertex cover of G. Given a graph $G = (V, E)$, a vertex cover C with size τ can be computed in time $\mathcal{O}(2^\tau + \tau \cdot |V|)$ [1]. We will use E_C to denote the set of all edges whose both endpoints lie in C. Given a subset W of C, a vertex in $V(G) \setminus C$ is of type W if its set of neighbors is equal to W [2].

A *k-page book embedding* $\langle \prec, \sigma \rangle$ of a graph G consists of a linear order $\prec$ of its vertices along a spine and an assignment σ that assigns each edge to one of k pages (half-planes sharing the spine) so that no two edges on the same page cross [3, 4]. The smallest $k > 0$ such that G admits a k-page book embedding is called the *book thickness* of G, denoted by $\mathrm{bt}(G)$. The BOOK THICKNESS problem asks, given a graph $G = (V, E)$ and a positive integer k, whether $\mathrm{bt}(G) \leqslant k$. This problem is specially called the FIXED-ORDER BOOK THICKNESS problem if the vertex order $\prec$ is given as part of the input. Correspondingly, the book thickness of G with respect to $\prec$ is specially called the *fixed-order book thickness* of G and denoted by $\mathrm{bt}(G, \prec)$ [2]. FIXED-ORDER BOOK THICKNESS parameterized by vertex cover number is abbreviated as FIXED-ORDER BTVC. Since the BTVC problem and the FIXED-ORDER BTVC problem can be solved in polynomial time when $k \geqslant \tau$ [2], we assume that $k < \tau$ in our algorithms.

Assume that (G, k, τ) is a yes-instance of the BTVC problem. Let $\langle \prec, \sigma \rangle$ be a k-page book embedding of G, let W be a subset of C, and let V_W be the set of vertices having type W. A vertex v in V_W is called an *uncloneable vertex* with respect to page p if v has two edges, say $w_i v$ and $w_j v$, assigned to page p simultaneously. Furthermore, a vertex v in V_W is called an *uncloneable vertex* with respect to $\langle \prec, \sigma \rangle$ if it is an uncloneable vertex with respect to at least one page in $\langle \prec, \sigma \rangle$. Otherwise, vertex v is called a *cloneable vertex* with respect to $\langle \prec, \sigma \rangle$ [5].

We also define a class of 1-page graphs extended from 1-page 2-degree embedding graphs. For a nonnegative b, a graph G is called a *b-crossings 1-page graph* if it meets the requirement (1) in the definition of 1-page 2-degree embedding graphs and has at most b edge-crossings. A b-crossings 1-page graph G' is called a *pure b-crossings 1-page graph* if each edge in $E(G')$ is involved in edge-crossing.

Appendix B A refined kernel for the BTVC problem

By Lemma 1, 2, and the analysis in [5], we immediately arrive at Theorem 1.

Appendix B.1 The proof of Lemma 1

Assume that the vertices in W are ordered as $c_1 \prec c_2 \prec \cdots \prec c_{|W|}$. We first show that $\mathcal{H}_W$ can be decomposed into two smaller parts by a side-vertex deletion and a middle-vertex deletion as follows.

We argue that there must exist one 2-degree vertex (denoted by u) in $V(\mathcal{H}_W) \setminus W$ on the left of c_1 or on the right of $c_{|W|}$ in $\prec$. Assume towards a contradiction that there exists no such vertex. Then we can introduce a new vertex u left next to c_1 (or right next to $c_{|W|}$) and add two edges uc_1 and $uc_{|W|}$ such that there exists no edge-crossing in the resulting graph, which contradicts the assumption that $|V(\mathcal{H}_W)|$ is maximum. Thus, we can always find such a vertex $u \in V(\mathcal{H}_W) \setminus W$. Without loss of generality, we assume that u lies on the left of c_1 and its neighbors are denoted by c_i and c_j respectively.

We then argue that there must exist another 2-degree vertex (denoted by v) in $V(\mathcal{H}_W) \setminus W$ such that $c_i \prec v \prec c_j$ and that both $c_i v$ and vc_j belong to $E(\mathcal{H}_W)$. Let S be the set of all 2-degree vertices that lie between c_i and c_j and belong to $V(\mathcal{H}_W) \setminus W$. We show that c_i connects at least one vertex in S. Assume towards a contradiction that c_i does not connect any vertex in S. Then we can extend $\prec$ by inserting a new vertex v right next to c_i, and adding edges $c_i v$ and vc_j, such that the resulting graph is still a 1-page 2-degree embedding graph, which contradicts the assumption that $|V(\mathcal{H}_W)|$ is maximum. Based on the same proof, c_j also connects at least one vertex in S. Without loss of generality, let $v_1 \in S$ be the leftmost vertex that connects c_i and let $v_2 \in S$ be the rightmost vertex that connects c_j. Next, we argue that v_1 and v_2 must be the same one. Assume towards a contradiction that v_1 and v_2 are two distinct vertices. By the assumption that $\mathcal{H}_W$ is a maximum 1-page 2-degree embedding graph, there must exist one vertex $c_z \in W$ on the right of v_1 such that $v_1 c_z \in E(\mathcal{H}_W)$. Then we can adjust $\mathcal{H}_W$ by inserting a new vertex v_h left next to u, deleting the edge uc_j, and adding edges uc_z, $v_h c_z$, and $v_h c_j$; see Figure B1 for an illustration. Since $\mathcal{H}_W$ is a 1-page 2-degree embedding graph with respect to W, the resulting graph is also a 1-page 2-degree embedding graph with respect to W. However, this resulting graph has $|V(\mathcal{H}_W)| + 1$ vertices, which contradicts the assumption that $|V(\mathcal{H}_W)|$ is maximum.

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Figure B1 A 1-page 2-degree embedding graph with respect to $\{c_i, c_j, c_z\}$ (left) and the resulting 1-page 2-degree embedding graph with respect to $\{c_i, c_j, c_z\}$ obtained by introducing a new vertex v_h (right).

Thus, a side-vertex deletion on u and a middle-vertex deletion on v can be successively executed, resulting in two subgraphs $\mathcal{H}_{W_1}$ and $\mathcal{H}_{W_2}$. Observe that $\{W_1, W_2\}$ is a partition of W. Observe further that $\mathcal{H}_W$ is a maximum 1-page 2-degree embedding graph with respect to W. Thus, $\mathcal{H}_{W_1}$ (resp. $\mathcal{H}_{W_2}$) is a maximum 1-page 2-degree embedding subgraph with respect to W_1 (resp. W_2).

Furthermore, $\mathcal{H}_{W_1}$ and $\mathcal{H}_{W_2}$ can be respectively decomposed into two smaller parts by a side-vertex deletion and a middle-vertex deletion in the same way as described above. This decomposition process can be recursively done until the resulting graph became an empty graph $\mathcal{H} = (W, \emptyset)$.

Appendix B.2 The proof of Lemma 2

Based on Lemma 1, $\mathcal{H}_W$ can be reduced to $\mathcal{H} = (W, \emptyset)$ by a sequence of 2-degree vertex deletions. Observe that this reduction is invertible. Hence, we can construct $\mathcal{H}_W$ from $\mathcal{H} = (W, \emptyset)$ by a sequence of 2-degree vertex additions in the inverse order.

Assume that W_{12} is a subset of W. We prove that $|V(\mathcal{H}_{W_{12}}) \setminus W_{12}| = 2(|W_{12}| - 1)$ for all $|W_{12}| \in [1, |W|]$ by the second principle of mathematical induction. In the basis step, it provides that $|W_{12}| = 1$. Since $\mathcal{H}_{W_{12}}$ can be seen as a simple graph with only one vertex, it obviously holds that $|V(H_{W_{12}}) \setminus W_{12}| = 0 = 2(|W_{12}| - 1)$. Let $q \in [2, |W|]$. In the inductive step, assume that the claim holds for all $|W_{12}| \in [1, q-1]$. We then consider the particular case that $|W_{12}| = q$. Let $\{W_1, W_2\}$ be a partition of W_{12} and let $\mathcal{H}_{W_1}$ (resp. $\mathcal{H}_{W_2}$) be a maximum 1-page 2-degree embedding subgraph with respect to W_1 (resp. W_2). The linear order of vertices in W_1 (resp. W_2) is denoted by $\prec_1$ (resp. $\prec_2$). We distinguish two cases based on the structural relationship among $\mathcal{H}_{W_{12}}$, $\mathcal{H}_{W_1}$ and $\mathcal{H}_{W_2}$.

Case 1: $\mathcal{H}_{W_{12}}$ is constructed by merging $\langle \mathcal{H}_{W_1}, \prec_1 \rangle$ with $\langle \mathcal{H}_{W_2}, \prec_2 \rangle$ parallelly. Without loss of generality, assume that $\langle \mathcal{H}_{W_1}, \prec_1 \rangle$ and $\langle \mathcal{H}_{W_2}, \prec_2 \rangle$ are placed in a half-plane with a left-to-right fashion. Observe that there exists a unique vertex, say c_1, in $\mathcal{H}_{W_1}$ and a unique vertex, say c_2, in $\mathcal{H}_{W_2}$ such that $\mathcal{H}_{W_1}$ and $\mathcal{H}_{W_2}$ can be merged into a larger 1-page 2-degree embedding graph by adding two 2-degree vertices connecting c_1 and c_2. Hence, we take two steps as follows. Step (1): introduce a vertex, say u_1, between the rightmost vertex in $\prec_1$ and the leftmost vertex in $\prec_2$; add edges u_1c_1 and u_1c_2. Step (2): introduce another vertex, say u_2, left next to the leftmost vertex in $\prec_1$ (or right next to the rightmost vertex in $\prec_2$); add edges u_2c_1 and u_2c_2. Since $|W_1| < q$ and $|W_2| < q$, by the inductive assumption, it follows that $|V(\mathcal{H}_{W_1}) \setminus W_1| = 2(|W_1| - 1)$ and that $|V(\mathcal{H}_{W_2}) \setminus W_2| = 2(|W_2| - 1)$. Thus, we obtain that $|V(\mathcal{H}_{W_{12}}) \setminus W_{12}| = 2(|W_1| - 1) + 2(|W_2| - 1) + 2 = 2(|W_1| + |W_2| - 1) = 2(|W_{12}| - 1)$.

Case 2: $\mathcal{H}_{W_{12}}$ is constructed by embedding $\langle \mathcal{H}_{W_1}, \prec_1 \rangle$ into $\langle \mathcal{H}_{W_2}, \prec_2 \rangle$. Let v_1 and v_2 be two arbitrary consecutive vertices in $\mathcal{H}_{W_2}$. Without loss of generality, assume that $\mathcal{H}_{W_1}$ is embedded between v_1 and v_2. If either of vertices v_1 and v_2 belongs to W_2 (we assume that $v_1 \in W_2$), then there exists a unique vertex c_1 in W_1 such that $\mathcal{H}_{W_1}$ and $\mathcal{H}_{W_2}$ can be merged by adding two 2-degree vertices connecting c_1 and v_1. Otherwise, $\mathcal{H}_{W_1}$ and $\mathcal{H}_{W_2}$ can be merged by adding two 2-degree vertices connecting c_1 and c_2, where c_2 is the common neighbor of vertices v_1 and v_2 in $\mathcal{H}_{W_2}$. The steps in merging process are the same as those described in case (1). Hence, we also obtain that $|V(\mathcal{H}_{W_{12}}) \setminus W_{12}| = 2(|W_1| - 1) + 2(|W_2| - 1) + 2 = 2(|W_1| + |W_2| - 1) = 2(|W_{12}| - 1)$. Note that if $\mathcal{H}_{W_{12}}$ is constructed by embedding $\langle \mathcal{H}_{W_2}, \prec_2 \rangle$ into $\langle \mathcal{H}_{W_1}, \prec_1 \rangle$, the proof is along the same line.

Altogether, it holds that $|V(\mathcal{H}_{W_{12}}) \setminus W_{12}| = 2(|W_{12}| - 1)$ for $|W_{12}| = q$.

Appendix C An improved time bound for the algorithm solving fixed-order BTVC

In the algorithm presented by Bhore et al. [2], the size of $\mathcal{R}_i(s)$ was bounded by $2^{\tau^3+\tau^2}$; see [2] for the explanations of related notations. Herein, we will prove an improved running time bound for this algorithm by reanalyzing the size of $\mathcal{R}_i(s)$.

The key technique we used is to adjust some edge on each page in $\alpha_i \cup s$ such that the endpoints of any edge in the resulting assignment $\alpha'_i \cup s$ fall into $C \cup \{u_1, u_{x_1}, u_{x_2}, \ldots, u_{x_z}\}$; see Figure C1 for an illustration. Theorem 2 can be immediately obtained from Lemma 3 and 4.

Appendix C.1 The proof of Lemma 3

Let $c_q \in C$ for $q \in [1, \tau]$ and let $u_x \in (V(G) \setminus C)$ for $x \geqslant i$. By the definition of the visibility matrix, it is sufficient to compare the visibility from c_q to u_x in $\alpha_i \cup s$ with that in $\alpha'_i \cup s$. By our adjusting rule, each edge in E_C remains unchanged in the resulting assignment $\alpha'_i \cup s$. Thus, we will not consider edges in E_C in the following proof.

Let (p, q) be an arbitrary entry in $M_i(x, \alpha_i, s)$, let $E(p) \subseteq (E_i \cup E_C)$ be the set of edges assigned to page p, and let $E(p, c_q) \subseteq E(p)$ be the set of edges incident to c_q. We distinguish two cases based on two possible values of (p, q).

Case 1: $(p, q) = 0$. Then there must exist some edge separating c_q from u_x on page p in $\alpha_i \cup s$. Let H_1 be the set of edges that embrace u_x but not embrace c_q and let H_2 be the set of edges that embrace c_q but not embrace u_x. Obviously, it holds that $(H_1 \cup H_2) \cap E(p, c_q) = \emptyset$. Assume that $H_1 \neq \emptyset$. Let e_1 be an arbitrary edge in H_1 and let e'_1 be the corresponding edge in $\alpha'_i \cup s$ resulted by adjusting e_1 (by the adjusting rule, there always exists such an edge). Since $x \geqslant i$, the right endpoint of e_1 must be in C. We analyze the position of e'_1 in two possible subcases. If $u_x \prec c_q$, then the right endpoint of e'_1 still lies on the right of u_x. Otherwise (i.e., $c_q \prec u_x$), the left endpoint of e'_1 is still on the right of c_q. In either case, e'_1 always separates c_q from u_x in $\alpha'_i \cup s$. Assume that $H_2 \neq \emptyset$. Let e_2 be an arbitrary edge in H_2 and let e'_2 be the corresponding edge in $\alpha'_i \cup s$ resulted by adjusting e_2. By the assumption that $x \geqslant i$, it follows that $c_q \prec u_x$. Moreover, both endpoints of e_2 lie on the left of u_x. By the rule of edge-adjusting, e'_2 still embraces c_q but not embrace u_x, which means that e'_2 separates c_q from u_x on page p in $\alpha'_i \cup s$. Since $H_1 \cup H_2 \neq \emptyset$, there is at least one edge separating c_q from u_x on page p in $\alpha'_i \cup s$, indicating that (p, q) is 0 in $M_i(x, \alpha'_i, s)$.

Figure C1 An original 2-page assignment of $E_5 \cup E_C$ (left) and the resulting 2-page assignment (right) obtained by executing 1-edge-adjusting(p) for p = 1,2.

Case 2: $(p, q) = 1$. If $E(p) \neq E(p, c_q)$, then all edges in $E(p) \setminus E(p, c_q)$ fall into two categories: (1) the edges that embrace neither c_q nor u_x and (2) the edges that embrace c_q and u_x simultaneously. We denote by H_1 the former edge set and by H_2 the latter. Assume that $H_1 \neq \emptyset$. Let e_1 be an arbitrary edge in H_1 and let e'_1 be the corresponding edge in $\alpha' \cup s$ resulted by adjusting e_1. Since $x \geqslant i$, both endpoints of e_1 lie on the left of u_x, which means that e'_1 will not embrace u_x in $\alpha' \cup s$. Moreover, both endpoints of e_1 either lie on the left of c_q or lie on the right of c_q. Thus, e'_1 will also not embrace c_q in $\alpha' \cup s$. Assume that $H_2 \neq \emptyset$. Let e_2 be an arbitrary edge in H_2 and let e'_2 be the corresponding edge in $\alpha' \cup s$ resulted by adjusting e_2. By the assumption that $x \geqslant i$, it follows that the right endpoint of e_2 must be in C. Thus, after adjusting e_2, the edge e'_2 still embraces c_q and u_x simultaneously. Altogether, no edge on page p in $\alpha'_i \cup s$ separates c_q from u_x, indicating that (p, q) is 1 in $M_i(x, \alpha'_i, s)$.

Appendix C.2 The proof of Lemma 4

Given a set S of $2\tau + 1$ vertices with a fixed linear order $\prec$, we first define a special class of 0-crossing 1-page graphs. A 0-crossing 1-page graph P is called a $((S, \prec), \tau)$ *0-crossing 1-page graph* if P meets requirements: (1) $V(P) = S$, (2) the order of vertices in $V(P)$ is $\prec$, and (3) the vertex cover number of P is equal to τ. Specifically, we set $S = V' \cup C$ and denote by $\mathcal{Q}$ the family of all $((S, \prec), \tau)$ 0-crossing 1-page graphs and estimate the cardinality of $\mathcal{Q}$ as follows. Let $P \in \mathcal{Q}$ be an arbitrary $((S, \prec), \tau)$ 0-crossing 1-page graph. Since P is a planar graph with $2\tau + 1$ vertices, it has at most $3 \times (2\tau + 1) - 6 = 6\tau - 3$ edges. Furthermore, all edges of P come from $\tau \times (2\tau + 1)$ possible vertex pairs since the vertex cover number of P is τ. Therefore, the cardinality of $\mathcal{Q}$ can be bounded by $|\mathcal{Q}| = \Sigma_{i=1}^{6\tau-3} \binom{(2\tau+1)\times\tau}{i} \leqslant (6\tau - 3) \cdot (2\tau^2 + \tau)^{6\tau-3}$.

Next, we denote by $\mathcal{D}$ the set $\{(P_1, P_2, \ldots, P_k) \mid P_i \in \mathcal{Q}, i \in [1, k]\}$. Then, the number of k-tuples in $\mathcal{D}$ can be bounded by $|\mathcal{D}| = (6\tau - 3)^k \cdot (2\tau^2 + \tau)^{(6\tau-3)k} = 2^{k \log(6\tau-3) + ((6\tau-3)k)(\log(2\tau^2+\tau))} = 2^{\mathcal{O}(k\tau \log \tau)}$.

Furthermore, we can bound the size of $\mathcal{R}_i(s)$ by the relationship between $\mathcal{R}_i(s)$ and $\mathcal{D}$. We denote by $\mathcal{P}_i(s)$ the set $\{\alpha_i \cup s \mid \alpha_i = \Lambda_i^s(\rho)$ and $\rho \in \mathcal{R}_i(s)\}$. Suppose that $\alpha'_i \cup s$ and $\alpha''_i \cup s$ are two distinct assignments in $\mathcal{P}_i(s)$. By the definition of $\mathcal{R}_i(s)$, there exists at least an integer $h \in \{i\} \cup X$ such that $M_i(h, \alpha'_i, s) \neq M_i(h, \alpha''_i, s)$. We argue that there exists an injective function f from $\mathcal{P}_i(s)$ to $\mathcal{D}$ as follows. (1). Assume that $\alpha_i \cup s$ is an arbitrary assignment in $\mathcal{P}_i(s)$. Since $\alpha_i \cup s$ is a k-page book embedding, it contains k pages. After executing 1-edge-adjusting(p) for $p = 1, 2, \ldots k$ in $\alpha_i \cup s$, each page is exactly transformed into a $((S, \prec), \tau)$ 0-crossing 1-page graph. Hence, there exists a unique tuple $(P_1, P_2, \ldots, P_k)$ in $\mathcal{D}$ such that $f(\alpha_i \cup s) = (P_1, P_2, \ldots, P_k)$. (2). Given two distinct assignments in $\mathcal{P}_i(s)$, say $\alpha'_i \cup s$ and $\alpha''_i \cup s$, we show that $f(\alpha'_i \cup s) \neq f(\alpha''_i \cup s)$ as follows. Assume towards a contradiction that $f(\alpha'_i \cup s) = f(\alpha''_i \cup s)$. Then, $M_i(x, f(\alpha'_i), s) = M_i(x, f(\alpha''_i), s)$ for each $x \in \{i\} \cup X$. By Lemma 3, it holds that $M_i(x, \alpha'_i, s) = M_i(x, \alpha''_i, s)$ for each $x \in \{i\} \cup X$, contradicting the fact that $\alpha'_i \cup s$ and $\alpha''_i \cup s$ are distinct assignments in $\mathcal{P}_i(s)$.

Consequently, $|\mathcal{R}_i(s)| = |\mathcal{P}_i(s)| \leqslant |\mathcal{D}|$. It was assumed that $k < \tau$ [2]. Therefore, $|\mathcal{R}_i(s)|$ can be bounded by $2^{\mathcal{O}(\tau^2 \log \tau)}$.

Appendix D Analysis on the algorithm solving an extended fixed-order BTVC

An extended FIXED-ORDER BOOK THICKNESS asks, given a tuple $(G, \prec)$ and two integers k, b, whether G admits a k-page book drawing $\langle \prec, \sigma \rangle$ with at most b crossings over all pages ? Its parameterized version with respect to the vertex cover number is abbreviated as E-F-BTVC. Bhore et al. [2] mentioned the techniques in their algorithm solving FIXED-ORDER BTVC can be extended to this problem, but they did not elaborate on them. To facilitate our analysis, we first briefly describe a specific algorithm for E-F-BTVC by extending the techniques in [2]. Then, we pay more attention to analyze its running time by our edge-adjusting technique.

Based on Lemma 5, we define an *extended record set* with the form $\mathcal{R}'_i(s) = \{(N_i(\alpha_i, s), M_i(i, \alpha_i, s), M_i(x_1, \alpha_i, s), M_i(x_2, \alpha_i, s), \ldots, M_i(x_z, \alpha_i, s)) \mid \exists$ valid partial page assignment $\alpha_i : E_i \to [1, k]\ \}$. The record corresponding to $\alpha_i \cup s$ in $\mathcal{R}'_i(s)$ is also denoted by $\mathcal{M}_i(\alpha_i, s)$. Along with $\mathcal{R}'_i(s)$, we also set a mapping $\Lambda_i'^s$ which maps $(N, M_0, \ldots, M_z) \in \mathcal{R}'_i(s)$ to some α_i such that $(N, M_0, \ldots, M_z) = \mathcal{M}_i(\alpha_i, s)$. Based on the framework of dynamic programming in [2], we can obtain an algorithm (denoted by ALGF) for the E-F-BTVC problem.

The basic strategy in ALGF is to dynamically construct page assignments containing at most b crossings over all pages from left to right along the linear order $\prec$. Assume the record set $\mathcal{R}'_{i-1}(s)$ has been computed. Each page assignment β of edges incident to the vertex u_{i-1} and each record $\rho \in \mathcal{R}'_{i-1}(s)$ are branched. For each such β and $\gamma = \Lambda'^s_{i-1}(\rho)$, the algorithm check whether $\gamma \cup s \cup \beta$ is valid. If $\gamma \cup \beta \cup s$ contains at most b crossings over all pages, then a new record $\mathcal{M}_i(\gamma \cup \beta, s)$ is computed and stored in $\mathcal{R}'_i(s)$, and the mapping Λ'^s_i is set to map this record to $\gamma \cup \beta$. Otherwise, the tuple (γ, s, β) is discarded.

Furthermore, we use $\text{bt}(G, \prec, b)$ to denote the smallest $k > 0$ such that $((G, \prec), k, \tau, b)$ is a yes-instance of the E-F-BTVC problem. It holds that $\text{bt}(G, \prec, b) < \tau$ since $\text{bt}(G, \prec, b) \leqslant \text{bt}(G, \prec)$ and $\text{bt}(G, \prec) < \tau$. By Lemma 5 and 6, we immediately arrive at Theorem 3.

Appendix D.1 The proof of Lemma 5

Let (p, q) be an arbitrary entry in $M_i(x_j, \alpha_i, s)$. Assume that $(p, q) = r$. In the following, we argue that the corresponding entry (p, q) in $M_i(h, \alpha_i, s)$ is also equal to r. We distinguish two cases based on whether $r = b + 1$ or not.

Case 1: $r \in [0, b]$. If $r = 0$, then there exists no edge separating c_q from u_{x_j} on page p in $\alpha_i \cup s$. By the assumption that $x_j \geqslant i$ and $h - x_j = \min\ \{|h - x||x \in \{i\} \cup X\}$, there exists no edge separating c_q from u_h on page p. Hence, (p, q) also equals 0 in $M_i(h, \alpha_i, s)$. Assume that $r \neq 0$. Since $r \leqslant b$, there are exactly r edges separating c_q from u_{x_j} on page p in $\alpha_i \cup s$. Moreover, each of these edges either embraces u_{x_j} but not embrace c_q or embraces c_q but not embrace u_{x_j}. We analyze two possible subcases as follows. Let e_1 be an arbitrary edge that embraces u_{x_j} but not embrace c_q. By the assumption that $x_j \geqslant i$, the right endpoint of

Figure D1 An original 2-page assignment of $E_6 \cup E_C$ (left) and the resulting 2-page assignment (right) obtained by executing 2-edge-adjusting(p) for $p = 1,2$, in which $b = 2$.

e_1 (denoted by c_{e_1}) must be in C. By the assumption that $h - x_j = \min\{|h-x||x \in \{i\} \cup X\}$, it follows that $u_{x_j} \prec u_h \prec c_{e_1} \prec c_q$. Thus, the edge e_1 also separates c_q from u_h. Similarly, let e_2 be an arbitrary edge that embraces c_q but not embrace u_{x_j}. Since $x_j \geqslant i$, it follows that $c_q \prec u_{x_j}$. Moreover, both endpoints of e_2 lie on the left of u_{x_j}. Thus, the edge e_2 also separates c_q from u_h. Altogether, there are r edges separating c_q from u_h on page p in $\alpha_i \cup s$, indicating that (p,q) is r in $M_i(h, \alpha_i, s)$.

Case 2: $r = b+1$. Then there must exist at least $b+1$ edges that separating c_q from u_{x_j} on page p in $\alpha_i \cup s$. By the proof of case 1, each of these edges will also separate c_q from u_h on page p. By the definition of crossing number matrix, the entry (p,q) is also set to $b+1$ in $M_i(h, \alpha_i, s)$.

Appendix D.2 The proof of Lemma 6

We first prove the correctness of algorithm ALGF. Suppose both α'_{i-1} and α''_{i-1} are two valid page assignments of E_{i-1}. Let β be a page assignment of edges incident to the vertex u_{i-1}.

Claim 1. If $\mathcal{M}_{i-1}(\alpha'_{i-1}, s) = \mathcal{M}_{i-1}(\alpha''_{i-1}, s)$, then $\mathcal{M}_i(\alpha'_{i-1} \cup \beta, s) = \mathcal{M}_i(\alpha''_{i-1} \cup \beta, s)$.

Proof of Claim 1. By the assumption that $M_{i-1}(i-1, \alpha'_{i-1}, s) = M_{i-1}(i-1, \alpha''_{i-1}, s)$, it follows that $\alpha'_{i-1} \cup \beta \cup s$ is a valid assignment on edges in $E_i \cup E_C \cup s$ if and only if $\alpha''_{i-1} \cup \beta \cup s$ is a valid assignment on edges in $E_i \cup E_C \cup s$.

We show that $N_i(\alpha'_{i-1} \cup \beta, s) = N_i(\alpha''_{i-1} \cup \beta, s)$ as follows. Let r_1 (resp. r_2) be the number of crossings resulted from adding some edge by β to $\alpha'_{i-1} \cup s$ (resp. $\alpha''_{i-1} \cup s$). Since $M_{i-1}(i-1, \alpha'_{i-1}, s) = M_{i-1}(i-1, \alpha''_{i-1}, s)$ and the assignment of edges by β to $\alpha'_{i-1} \cup s$ is identical to that to $\alpha''_{i-1} \cup s$, it follows that $r_1 = r_2$. By the assumption that $N_{i-1}(\alpha'_{i-1}, s) = N_{i-1}(\alpha''_{i-1}, s)$, it holds that $N_{i-1}(\alpha'_{i-1}, s) + r_1 = N_{i-1}(\alpha''_{i-1}, s) + r_2$, i.e., $N_i(\alpha'_{i-1} \cup \beta, s) = N_i(\alpha''_{i-1} \cup \beta, s)$.

We also show that $M_i(x, \alpha'_{i-1} \cup \beta, s) = M_i(x, \alpha''_{i-1} \cup \beta, s)$ for each $x \in \{i\} \cup X$ as follows. Let (p,q) be an arbitrary entry in $M_{i-1}(h, \alpha'_{i-1}, s)$ for $h \in \{i-1\} \cup X$. By the assumption that $\mathcal{M}_{i-1}(\alpha'_{i-1}, s) = \mathcal{M}_{i-1}(\alpha''_{i-1}, s)$, it follows that the entry (p,q) in $M_{i-1}(h, \alpha'_{i-1}, s)$ equals that in $M_{i-1}(h, \alpha''_{i-1}, s)$. At the same time, in the assignment β, the edges assigned to page p in α'_{i-1} are identical to those in α''_{i-1}. Hence, the number of edges in β separating c_q from u_x on page p in $\alpha'_{i-1} \cup \beta$ is equal to that in $\alpha''_{i-1} \cup \beta$, in which $x \in \{i\} \cup X$. Thus, if the entry (p,q) in $M_i(x, \alpha'_{i-1} \cup \beta, s)$ varies, then its increments is the same as that for the entry (p,q) in $M_i(x, \alpha''_{i-1} \cup \beta, s)$.

Based on Claim 1, we conclude that if $((G, \prec), k, \tau, b)$ is a yes-instance of the E-F-BTVC problem, then the algorithm ALGF $((G, \prec), k, \tau, b)$ outputs a valid page assignment.

Next, we apply the edge-adjusting technique to estimate the size of $\mathcal{R}'_i(s)$ along similar lines in Appendix C.

We define another unit operation 2-ADJUST(u_j, u_h) for adjusting edges from $E(p, u_j)$ to $E(p, u_h)$. By replacing the unit operation 1-ADJUST(u_j, u_{x_y}) with 2-ADJUST(u_j, u_{x_y}) in procedure 1-edge-adjusting(p), we can obtain another procedure named as 2-edge-adjusting(p). Given a valid partial assignment $\alpha_i \cup s$, we execute the procedure 2-edge-adjusting(p) on each page p for $p = 1, 2, \ldots, k$. After adjusting all possible edges, we obtain a resulting assignment $\alpha'_i \cup s$ such that the subgraph induced by the edges on each page has at most $2\tau+1$ vertices; see Figure D1 for an illustration. Note that there may be multiple edges between two vertices on some page in $\alpha'_i \cup s$.

Claim 2. For each $x \in \{i\} \cup X$, $M_i(x, \alpha'_i, s) = M_i(x, \alpha_i, s)$.

Proof of Claim 2. Let (p,q) be an arbitrary entry in $M_i(x, \alpha_i, s)$. Assume that (p,q) is r. In the following, we argue that (p,q) is also equal to r in $M_i(x, \alpha'_i, s)$. We distinguish two cases based on whether $r = b+1$ or not.

Case 1: $r \in [0, b]$. If $r = 0$, then there exists no edge separating c_q from u_x on page p in $\alpha_i \cup s$. By the proof of Lemma 3, there exists no edge separating c_q from u_x on page p in $\alpha'_i \cup s$. Hence, (p,q) also equals 0 in $M_i(x, \alpha'_i, s)$. Assume that $r \neq 0$. Since $r \leqslant b$, there must exist r edges separating c_q from u_x on page p in $\alpha_i \cup s$. Along the same lines as that in Lemma 3, we can show that there are exactly r edges separating c_q from u_x on page p in $\alpha'_i \cup s$, indicating that (p,q) is r in $M_i(x, \alpha'_i, s)$.

Case 2: $r = b+1$. Then there must be h ($h \geqslant b+1$) edges (denoted by H) that separating c_q from u_x on page p in $\alpha_i \cup s$. By the proof of case 1, after adjusting some edge, each of the resulting edges still separates c_q from u_x on page p in $\alpha'_i \cup s$. Next, we analyze the value of (p,q) in $M_i(x, \alpha'_i, s)$. Let uc be an arbitrary edge in H. It holds that $c \in C$ and u lies on one of the $\tau+1$ intervals formed by the vertices in C. The edges in H fall into at most $\tau(\tau+1)$ bunches. We further distinguish two subcases based on the number of edges in each bunch. Subcase 2.1: there exists at least one bunch, say B, such that $|B| > b$. After adjusting edges in B, we keep one multiple edge with multiplicities $b+1$ corresponding to B. This multiple edge can be seen as a set containing $b+1$ edges. Thus the number of edges separating c_q from u_x can be seen as at least $b+1$. Subcase 2.2: none of bunches contains $b+1$ edges. Let B' denote an arbitrary bunch. During edge-adjusting, we keep one multiple edge with multiplicities $|B'|$. By the assumption that $h \geqslant b+1$, the total number of edges separating c_q from u_x can be also seen as at least $b+1$. Therefore, the entry (p,q) is also set to $b+1$ in $M_i(x, \alpha'_i, s)$.

Given a nonnegative b and a set S of $2\tau+1$ vertices with a fixed linear order $\prec$, we further define two special classes of 1-page graphs. A 0-crossing (resp. pure b-crossing) 1-page graph P is called a $((S, \prec), \tau, b+1)$ 0-crossing (resp. pure b-crossing) 1-page graph if P meets requirements: (1) $V(P) = S$, (2) the order of vertices in $V(P)$ is $\prec$,(3) the vertex cover number of P is equal to τ, and (4) the multiplicity of each edge in $E(P)$ is at most $b+1$. Specifically, we set $S = V' \cup C$ and denote by $\mathcal{P}_1$ (resp. $\mathcal{P}_2$) the family of all $((S, \prec), \tau, b+1)$ 0-crossing (resp. pure b-crossing) 1-page graphs.

The size of $\mathcal{P}_1$ and $\mathcal{P}_2$ can be respectively estimated as follows. Since each $((S, \prec), \tau, b+1)$ 0-crossing graph in $\mathcal{P}_1$ is a planar multigraph, it follows that $|\mathcal{P}_1| = \Sigma_{i=1}^{6\tau-3}\binom{(2\tau+1)\tau}{i} \cdot (b+1)^i \leqslant (6\tau-3) \cdot ((2\tau+1)\tau)^{6\tau-3} \cdot (b+1)^{6\tau-3}$, in which the factor $(b+1)$ denotes the maximum multiplicity of each edge. Similarly, since each $((S, \prec), \tau, b+1)$ pure b-crossing 1-page graph in $\mathcal{P}_2$ contains at most b crossings produced by at most $2b$ edges, it follows that $|\mathcal{P}_2| = \Sigma_{i=1}^{2b}\binom{(2\tau+1)\tau}{i} \cdot (b+1)^i \leqslant 2b \cdot ((2\tau+1)\tau)^{2b} \cdot (b+1)^{2b}$. We assume that $\mathcal{Q} = \mathcal{P}_1 \times \mathcal{P}_2$. Then, $|\mathcal{Q}| \leqslant (6\tau-3) \cdot ((2\tau+1)\tau)^{6\tau-3} \cdot (b+1)^{6\tau-3} \cdot 2b \cdot ((2\tau+1)\tau)^{2b} \cdot (b+1)^{2b} = 2b \cdot (6\tau-3) \cdot ((2\tau+1)(b+1)\tau)^{6\tau+2b-3}$.

Let N be an integer variable ranging from 0 to b. We denote by $\mathcal{D}'$ the set $\{(N, B_1, B_2, \ldots, B_k) \mid B_i \in \mathcal{Q}$ for $i \in [1, k]\}$. Then, $|\mathcal{D}'| = (b+1) \cdot (2b)^k \cdot (6\tau - 3)^k \cdot ((2\tau + 1)(b+1)\tau)^{(6\tau+2b-3)k} = 2^{\mathcal{O}((\tau+b)k \log \tau(b+1))}$.

Denote by $\mathcal{P}'_i(s)$ the set $\{\alpha_i \cup s \mid \alpha_i = \Lambda'^s_i(\rho)$ and $\rho \in \mathcal{R}'_i(s)\}$. By Claim 2, we can show that there exists an injective function from $\mathcal{P}'_i(s)$ to $\mathcal{D}'$ along the same lines in the proof of Lemma 4. Hence, it holds that $|\mathcal{R}'_i(s)| \leqslant |\mathcal{D}'|$. In other words, $|\mathcal{R}'_i(s)|$ can be bounded by $2^{\mathcal{O}((\tau+b)k \log \tau(b+1))}$.

By the fact $|S| < \tau^{\tau^2}$ and Lemma 6, we further conclude that the algorithm ALGF$((G, \prec), k, \tau, b)$ for the E-F-BTVC problem runs in time $2^{\mathcal{O}((\tau+b)k \log \tau(b+1))+\tau^2 \log \tau} \cdot |V|$.

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