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• **Supplementary File** •

Observer-based boundary control for an asymmetric output-constrained flexible robotic manipulator

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Appendix A Definition of $\mathcal{A}$

The manipulator system model can be transformed as

$$\begin{cases} \rho\ddot{z} + EIz'''' - Tz'' + c\dot{z} = 0 \\ I_h\ddot{\theta} = EIz''(0,t) + T(z(l,t) - lz'(0,t)) - k_1\dot{\theta} - e(k_2 + \frac{J(e)}{b_2^2 - e^2} + \frac{1-J(e)}{b_1^2 - e^2}) + \tilde{d}_1 \\ m\ddot{z}(l,t) = EIz'''(l,t) - T(z'(l,t) - z'(0,t)) - k_3\dot{z}(l,t) - z_e(l,t)(k_4 + \frac{J(z_e(l,t))}{a_2^2 - z_e^2(l,t)} + \frac{1-J(z_e(l,t))}{a_1^2 - z_e^2(l,t)}) + \tilde{d}_2 \\ z(0,t) = z''(l,t) = 0 \\ z'(0,t) = \theta \end{cases} \tag{A1}$$

The linear operator $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ is defined as

$$\mathcal{A}\begin{bmatrix} g \\ v \\ w \\ T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} w \\ w \\ \frac{Tg''}{\rho} - \frac{EIg''''}{\rho} - \frac{cw}{\rho} \\ -\frac{k_3T_1}{m} + \varpi_1 \\ -\frac{k_1T_2}{I_h} + \varpi_2 \end{bmatrix} \tag{A2}$$

with the domain

$$D(\mathcal{A}) = \{[g, v, w, T_1, T_2]^T \in H | T_1 = w(l,t), T_2 = w'(0,t)\} \tag{A3}$$

where

$$\begin{cases} \varpi_1 = \frac{EIg'''(l,t)}{m} - \frac{T(g'(l,t) - g'(0,t))}{m} - \frac{v(l,t)}{m}(k_4 + \frac{J(v(l,t))}{a_2^2 - v^2(l,t)} + \frac{1-J(v(l,t))}{a_1^2 - v^2(l,t)}) \\ \varpi_2 = \frac{EIg''(0,t)}{I_h} + \frac{T(g(l,t) - lg'(0,t))}{I_h} - \frac{v'(0,t)}{m}(k_2 + \frac{J(v'(0,t))}{b_2^2 - (v'(0,t))^2} + \frac{1-J(v'(0,t))}{b_1^2 - (v'(0,t))^2}) \end{cases} \tag{A4}$$

From (8), for any $Y \in H$, we have

$$< Y, \mathcal{A}Y >_H \leqslant -\mathcal{M}\{(v'(0,t))^2 + v^2(l,t) + T_1^2 + T_2^2 + \int_0^l ((g'')^2 + w^2 + (g' - g'(0,t))^2)dx\} \leqslant 0 \tag{A5}$$

where $\mathcal{M}$ is a positive constant, and the operator $\mathcal{A}$ is dissipative in H.

We next prove that $\mathcal{A}^{-1}$ is compact in H. For any $Q = [q_1, q_2, q_3, q_4, q_5]^T \in H$, consider the equation $\mathcal{A}Y = Q$, $Y = [g, v, w, T_1, T_2] \in D(\mathcal{A})$. From (A2), we can obtain the following equations

$$\begin{cases} w = q_1 \\ w = q_2 \\ \frac{T}{\rho}g'' - \frac{EI}{\rho}g'''' - \frac{c}{\rho}w = q_3 \\ -\frac{k_3}{m}T_1 + \varpi_1 = q_4 \\ -\frac{k_1}{I_h}T_2 + \varpi_2 = q_5 \end{cases} \tag{A6}$$

with the corresponding boundary conditions

$$g(0,t) = g''(l,t) = 0,\ T_1 = w(l,t),\ T_2 = w'(0,t) \tag{A7}$$

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Solving (A6) results in

$$\begin{cases} g = \mathcal{M}_1 + \mathcal{M}_2 x + \mathcal{M}_3 e^{\nu x} + \mathcal{M}_4 e^{-\nu x} + \frac{e^{-\nu x}}{EI} \int_0^x e^{2\nu \varrho} \int_0^{\varrho} e^{-\nu \varsigma} \int_0^{\varsigma} \int_0^{\xi} (\rho q_3 + c q_1) d\varrho d\varsigma d\xi dx \\ v = g - x\theta_d \\ w = q_2 \\ T_1 = \frac{m\varpi_1 - m q_4}{k_3} \\ T_2 = \frac{I_h \varpi_2 - I_h q_5}{k_1} \end{cases} \tag{A8}$$

where $\nu = \sqrt{\frac{T}{EI}}$, $\mathcal{M}_i$ $(i = 1, 2, 3, 4)$ are the constants which are exclusively determined by the boundary conditions (A7), ϖ_1 and ϖ_2 can be calculated by $g(l,t), g'(0,t), g''(0,t), g'''(0)$ and $g'''(l,t)$. By using the inverse-operator theorem and Sobolev embedding theorem, it can be concluded that $\mathcal{A}^{-1}$ is a compact operator.

Appendix B Proof of Theorem 1

Lemma 1. [1] From Poincaré inequalities, we can get the following result

$$\int_0^l \pi^2 dx \leqslant 8l^3(\pi'(0,t))^2 + 2l\pi^2(0,t) + 16l^4 \int_0^l (\pi'')^2 dx \tag{B1}$$

Lemma 2. [2] For any $C > 0$ and $\hbar \in \mathbb{R}$ in the interval $|\hbar| < C$, the following inequality holds

$$\ln \frac{C^2}{C^2 - \hbar^2} \leqslant \frac{\hbar^2}{C^2 - \hbar^2} \tag{B2}$$

Lemma 3. [3] Define $\mathcal{V} = \{\zeta \in \mathbb{R} | -c_1 < \zeta < c_2\}$ and $\mathcal{N} = \mathbb{R}^m \times \mathcal{V}$ as two open sets, where c_1, $c_2 \in \mathbb{R}^+$. Consider the following system

$$\eta = \dot{h}(t, \eta) \tag{B3}$$

where $\eta = [w, \zeta]^T \in \mathcal{N}$ $(w \in \mathbb{R}^m)$ and $h : \mathbb{R}^+ \times \mathcal{N} \to \mathbb{R}^{m+1}$ is piecewise continuous and locally Lipschitz in η, uniformly in t, on $\mathbb{R}^+ \times \mathcal{N}$. Assume there are two positive definite and continuously differentiable functions $U_1 : \mathbb{R}^m \times \mathbb{R}^+$ and $U_2 : \mathcal{V} \to \mathbb{R}^+$ such that

$$U_2(\zeta) \to \infty \quad as \quad \zeta \to -c_1 \quad or \quad \zeta \to c_2, \kappa_1(||w||) \leqslant U_1(w,t) \leqslant \kappa_2(||w||) \tag{B4}$$

where κ_1, κ_2 are class k_∞ functions. Construct $\mathbb{U}(\eta) = U_2(\zeta, t) + U_1(w)$ with $\zeta(0) \in \mathcal{V} \subset (-c_1, c_2)$. If there exist positive constants λ and $\aleph$ satisfying the inequality

$$\dot{\mathbb{U}} = \frac{\partial \mathbb{U}}{\partial \eta} h \leqslant -\lambda \mathbb{U} + \aleph \tag{B5}$$

then ζ remains in the open set $\zeta \in (-c_1, c_2), \forall t \in [0, \infty]$.

From (12), we can conclude that $\mathbb{V}_a$ is continuous and differentiable in two intervals, $z_e(l,t) \in (-a_1, 0)$ and $z_e(l,t) \in (0, a_2)$. Taking the left and right limits at point $z_e(l,t) = 0$, we can obtain

$$\lim_{z_e(l,t)\to 0^-} \mathbb{V}_a = \lim_{z_e(l,t)\to 0^-} \frac{\beta}{2} \ln \frac{a_1^2}{a_1^2 - z_e^2(l,t)} = 0 \tag{B6}$$

$$\lim_{z_e(l,t)\to 0^+} \mathbb{V}_a = \lim_{z_e(l,t)\to 0^+} \frac{\beta}{2} \ln \frac{a_2^2}{a_2^2 - z_e^2(l,t)} = 0 \tag{B7}$$

$$\lim_{z_e(l,t)\to 0^-} \mathbb{V}_a = \lim_{z_e(l,t)\to 0^+} \mathbb{V}_a = \mathbb{V}_a(0) \tag{B8}$$

When $z_e(l,t) = 0$, from $\mathbb{V}_a$, we can obtain

$$\lim_{z_e(l,t)\to 0^-} \frac{\partial \mathbb{V}_a}{\partial z_e(l,t)} = \lim_{z_e(l,t)\to 0^-} \frac{\frac{\beta}{2} \ln \frac{a_1^2}{a_1^2 - z_e^2(l,t)} - 0}{z_e(l,t)} = 0 \tag{B9}$$

$$\lim_{z_e(l,t)\to 0^+} \frac{\partial \mathbb{V}_a}{\partial z_e(l,t)} = \lim_{z_e(l,t)\to 0^+} \frac{\frac{\beta}{2} \ln \frac{a_2^2}{a_2^2 - z_e^2(l,t)} - 0}{z_e(l,t)} = 0 \tag{B10}$$

Therefore, we can conclude that $\mathbb{V}_a$ is continuous and differentiable in the interval $z_e(l,t) \in (-a_1, a_2)$. Similarly, we can get the continuity and differentiability of $\mathbb{V}_b$ in the interval $e \in (-b_1, b_2)$. Furthermore, $\mathbb{V}_a$ and $\mathbb{V}_b$ are continuous and differentiable.

It is clearly that $\mathbb{V}_a + \mathbb{V}_b + \mathbb{V}_m + \mathbb{V}_u + \mathbb{V}_d \geqslant 0$, using Young's inequality we can get

$$|\mathbb{V}_s| \leqslant \sigma\rho\delta_2 \int_0^l z_e^2 dx + \frac{\sigma\rho}{\delta_2} \int_0^l \dot{z}^2 dx + \frac{\sigma I_h}{\delta_1} \dot{\theta}^2 + \frac{\sigma m}{\delta_3} \dot{z}^2(l,t) + \sigma I_h \delta_1 e^2 + \sigma m \delta_3 z_e^2(l,t) \leqslant \lambda_0 [\mathbb{V}_m + \mathbb{V}_u] \tag{B11}$$

where δ_1, δ_2, $\delta_3 > 0$ and $1 \geqslant \lambda_0 = 2\max\{\frac{\sigma I_h \delta_1}{\sigma k_1 + \beta k_2}, \frac{\sigma}{\beta\delta_1}, \frac{\rho\delta_2}{c}, \frac{\sigma}{\beta\delta_2}, \frac{\sigma m \delta_3}{\sigma k_3 + \beta k_4}\} > 0$. Further we have

$$0 \leqslant (1 - \lambda_0)[\mathbb{V}_a + \mathbb{V}_b + \mathbb{V}_m + \mathbb{V}_u + \mathbb{V}_d] \leqslant \mathbb{V} \leqslant (1 + \lambda_0)[\mathbb{V}_a + \mathbb{V}_b + \mathbb{V}_m + \mathbb{V}_u + \mathbb{V}_d] \tag{B12}$$

Differentiating $\mathbb{V}$, we have

$$\dot{\mathbb{V}} = \dot{\mathbb{V}}_s + \dot{\mathbb{V}}_a + \dot{\mathbb{V}}_b + \dot{\mathbb{V}}_m + \dot{\mathbb{V}}_u + \dot{\mathbb{V}}_d \tag{B13}$$

where

$$\begin{aligned}\dot{\mathbb{V}}_s \leqslant & \sigma\rho\int_0^l \dot{z}dx - \sigma c\int_0^l z_e\dot{z}dx - \sigma EI\int_0^l (y'')^2dx - \sigma(k_2-\frac{1}{2})e^2 - \sigma T\int_0^l (y')^2dx \\ & - \sigma(k_4-\frac{1}{2})z_e^2(l,t) - \sigma[1-J(z_e(l,t))]\ln\frac{a_1^2}{a_1^2-z_e^2(l,t)} - \sigma J(z_e(l,t))\ln\frac{a_2^2}{a_2^2-z_e^2(l,t)} \\ & - \sigma k_1 e\dot{\theta} - \sigma[1-J(e)]\ln\frac{b_1^2}{b_1^2-e^2} + \sigma m\dot{z}^2(l,t) - \sigma J(e)\ln\frac{b_2^2}{b_2^2-e^2} + \frac{\sigma}{2}\tilde{d}_2^2 + \frac{\sigma}{2}\tilde{d}_1^2 \\ & - \sigma k_3 z_e(l,t)\dot{z}(l,t) + \sigma I_h\dot{\theta}^2\end{aligned} \tag{B14}$$

$$\dot{\mathbb{V}}_a = \beta z_e(l,t)\dot{z}(l,t)[\frac{J(z_e(l,t))}{a_2^2-z_e^2(l,t)} + \frac{1-J(z_e(l,t))}{a_1^2-z_e^2(l,t)}] \tag{B15}$$

$$\dot{\mathbb{V}}_b = \beta e\dot{\theta}[\frac{J(e)}{b_2^2-e^2} + \frac{1-J(e)}{b_1^2-e^2}] \tag{B16}$$

$$\dot{\mathbb{V}}_m = \sigma c\int_0^l z_e\dot{z}dx - \beta c\int_0^l \dot{z}^2dx - \beta\dot{\theta}[EIy''(0,t) + Ty(l,t)] - \beta\dot{z}(l,t)[EIy'''(l,t) - Ty'(l,t)] \tag{B17}$$

$$\begin{aligned}\dot{\mathbb{V}}_u =& (\sigma k_1 + \beta k_2)e\dot{\theta} + \beta\dot{\theta}[Ty(l,t) + EIy''(0,t) + \tau + d_1] + (\sigma k_3 + \beta k_4)mz_e(l,t)\dot{z}(l,t) \\ & + \beta\dot{z}(l,t)[EIy'''(l,t) - Ty'(l,t) + u + d_2]\end{aligned} \tag{B18}$$

Substituting (12), (B14)-(B18) into (B13), combining (5), (B12) and applying Lemmas 1-2, result in

$$\begin{aligned}\dot{\mathbb{V}} \leqslant & -\iota_1\dot{\theta}^2 - \iota_2 e^2 - \iota_3\dot{z}^2(l,t) - \iota_4 z_e^2(l,t) - \iota_5\int_0^l (y'')^2dx - \iota_6\int_0^l \dot{z}^2dx - \tau_4\int_0^l z_e^2dx - \sigma T\int_0^l (y')^2dx \\ & + (\tau_2 + \frac{1}{\tau_3})\ddot{D}^T\ddot{D} - \tilde{D}^T\begin{bmatrix}\iota_7, 0\\ 0, \iota_8\end{bmatrix}\tilde{D} - \frac{1}{2}[\tilde{D} - \tilde{\dot{D}}]^T[\tilde{D} - \tilde{\dot{D}}] - \sigma J(z_e(l,t))\ln\frac{a_2^2}{a_2^2-z_e^2(l,t)} - \sigma J(e)\ln\frac{b_2^2}{b_2^2-e^2} \\ & - \iota_9\tilde{\dot{D}}^T\tilde{\dot{D}} - \sigma[1-J(z_e(l,t))]\ln\frac{a_1^2}{a_1^2-z_e^2(l,t)} - \sigma[1-J(e)]\ln\frac{b_1^2}{b_1^2-e^2} \\ \leqslant & -\lambda_1[\mathbb{V}_m + \mathbb{V}_u + \mathbb{V}_b + \mathbb{V}_d] + \epsilon\end{aligned} \tag{B19}$$

where $\tau_4 > 0$, $\epsilon = 2(\tau_2 + \frac{1}{\tau_3})\mathcal{D}^2 \geqslant 0$, $\iota_1 = \beta k_1 - \frac{\beta}{2} - \sigma I_h > 0$, $\iota_2 = \sigma k_2 - \frac{\sigma}{2} - 8l^3 > 0$, $\iota_3 = \beta k_3 - \frac{\beta}{2} - \sigma m > 0$, $\iota_4 = \sigma k_4 - \frac{\sigma}{2} > 0$, $\iota_5 = \sigma EI - 16l^4\tau_4 > 0$, $\iota_6 = \beta c - \sigma\rho > 0$, $\iota_7 = \gamma_1 - \frac{1+\beta+\sigma}{2} - \tau_1 - \tau_3 > 0$, $\iota_8 = \gamma_2 - \frac{1+\beta+\sigma}{2} - \tau_1 - \tau_3 > 0$ and $\iota_9 = \frac{1}{2} - \frac{1}{\tau_1} - \frac{1}{\tau_2}$. Combining (B11) and (B19), the parameter values of $\gamma_1 \sim \gamma_2$, $\tau_1 \sim \tau_4$, $\delta_1 \sim \delta_3$, $k_1 \sim k_4$, β and σ are properly chosen to satisfy the following conditions

$$\iota_j > 0,\ j = 1 \sim 8 \tag{B20}$$

$$\iota_9 \geqslant 0 \tag{B21}$$

$$1 \geqslant \lambda_0 = 2\max\{\frac{\sigma I_h\delta_1}{\sigma k_1 + \beta k_2}, \frac{\sigma}{\beta\delta_1}, \frac{\rho\delta_2}{c}, \frac{\sigma}{\beta\delta_2}, \frac{\sigma m\delta_3}{\sigma k_3 + \beta k_4}\} > 0 \tag{B22}$$

$$\lambda_1 = 2\min\{\frac{\iota_1}{\beta I_h}, \frac{\iota_2}{\sigma k_1 + \beta k_2}, \frac{\iota_3}{\beta m}, \frac{\iota_4}{\sigma k_3 + \beta k_4}, \frac{\iota_5}{\beta EI}, \frac{\iota_6}{\beta\rho}, \frac{\iota_7}{\beta}, \frac{\iota_8}{\beta}, \frac{\tau_4}{\sigma c}\} > 0 \tag{B23}$$

Then we can obtain

$$\dot{\mathbb{V}} \leqslant -\lambda_2\mathbb{V} + \epsilon \tag{B24}$$

where $\lambda_2 = \frac{\lambda_1}{1+\lambda_0}$.

Multiplying (B24) by $e^{\lambda_2 t}$ and integrating the resulting equation, results in

$$\mathbb{V} \leqslant [\mathbb{V}(0) - \frac{\epsilon}{\lambda_2}]e^{-\lambda_2 t} + \frac{\epsilon}{\lambda_2} \leqslant \mathbb{V}(0)e^{-\lambda_2 t} + \frac{\epsilon}{\lambda_2} \tag{B25}$$

Using Lemma 1, combining (12), (B12) and the boundary conditions (2), we can obtain

$$\frac{\beta EI}{2l^3}y^2 \leqslant \frac{\beta EI}{2}[y''] \leqslant \frac{\mathbb{V}}{1-\lambda_0} \tag{B26}$$

$$\frac{\sigma k_1 + \beta k_2}{2}e^2 \leqslant \mathbb{V}_u \leqslant \frac{\mathbb{V}}{1-\lambda_0} \tag{B27}$$

$$\frac{\sigma k_3 + \beta k_4}{2}z_e^2(l,t) \leqslant \mathbb{V}_u \leqslant \frac{\mathbb{V}}{1-\lambda_0} \tag{B28}$$

Form (B26)-(B28), we have

$$|y| \leqslant \sqrt{\frac{2l^3}{2\beta EI(1-\lambda_0)}[\mathbb{V}(0)e^{-\lambda_2 t} + \frac{\epsilon}{\lambda_2}]} \tag{B29}$$

$$|e| \leqslant \sqrt{\frac{2}{(\sigma k_1 + \beta k_2)(1-\lambda_0)}[\mathbb{V}(0)e^{-\lambda_2 t} + \frac{\epsilon}{\lambda_2}]} \tag{B30}$$

$$|z_e(l,t)| \leqslant \sqrt{\frac{2}{(\sigma k_3+\beta k_4)(1-\lambda_0)}[\mathbb{V}(0)e^{-\lambda_2 t}+\frac{\epsilon}{\lambda_2}]} \tag{B31}$$

When $t\to\infty$, we can obtain

$$\lim_{t\to\infty}|y| \leqslant \sqrt{\frac{2l^3\epsilon}{2\beta EI\lambda_2(1-\lambda_0)}} \tag{B32}$$

$$\lim_{t\to\infty}|e| \leqslant \sqrt{\frac{2\epsilon}{\lambda_2(\sigma k_1+\beta k_2)(1-\lambda_0)}} \tag{B33}$$

$$\lim_{t\to\infty}|z_e(l,t)| \leqslant \sqrt{\frac{2\epsilon}{\lambda_2(\sigma k_3+\beta k_4)(1-\lambda_0)}} \tag{B34}$$

Therefore, for the flexible robotic manipulator system under the developed boundary control, the uniform ultimate boundedness is verified. It is noted that $\mathbb{V}_a$ and $\mathbb{V}_b$ are bounded indicating by (B12). Hence $\mathbb{V}_a\to\infty$ as $z_e(l,t)\to -a_1$ or $z_e(l,t)\to a_2$, and $\mathbb{V}_b\to\infty$ as $e\to -b_1$ or $e\to b_2$. Given $z_e(l,0)\in(-a_1,a_2)$, $e(0)\in(-b_1,b_2)$, we can conclude that $z_e(l,t)\in(-a_1,a_2)$ and $e\in(-b_1,b_2)$ by applying Lemma 3.

Appendix C Simulation settings

In this paper, the space step size and time step size are given as $\Delta x = 0.02$ and $\Delta t = 0.0001$ respectively. The system parameters are listed as $EI = 1\text{Nm}^2$, $T = 10\text{N}$, $I_h = 0.1\text{kgm}^2$, $\rho = 1\text{kg/m}$, $m = 5\text{kg}$, $l = 1\text{m}$, $c = 0.01\text{Ns/m}$ and $\theta_d = 0.2\text{rad}$. Besides, boundary disturbance d_1 is described as $d_1 = 1+\sin(t)+\sin(0.5t)$ and d_2 is given as $d_2 = 1+\cos(t)+\cos(0.5t)$. Define the manipulator system initial conditions as $y(x,0) = \dot{y}(x,0) = \theta(0) = \dot{\theta}(0) = 0$. The end-point displacement constraints are given as $a_1 = 0.25$ and $a_2 = 0.05$, and the angle constraints are $b_1 = 0.25$ and $b_2 = 0.05$.

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