SCIENCE CHINA
Information Sciences

• LETTER •

March 2022, Vol. 65 139202:1–139202:3
https://doi.org/10.1007/s11432-019-2843-9

pth moment $\mathcal{D}$-stability/stabilization of linear discrete-time stochastic systems

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Received 20 November 2019/Revised 16 January 2020/Accepted 29 February 2020/Published online 26 February 2021

Citation Zhang H S, Xia J W, Zhang Y N, et al. pth moment $\mathcal{D}$-stability/stabilization of linear discrete-time stochastic systems. Sci China Inf Sci, 2022, 65(3): 139202, https://doi.org/10.1007/s11432-019-2843-9

Dear editor,

Stability is a minimum requirement for systems, which includes stability in probability, stability in distribution, almost sure stability, and moment stability. Recently, pth moment stability and stabilization have attracted the attention of some researchers and many qualitative theories were obtained.

Regional pole placement is a good method to guarantee satisfactory transients, which places the closed-loop poles in a suitable region of the complex plane. Regional pole assignment of the linear deterministic system was investigated in various aspects, such as robust pole placement in linear matrix inequality (LMI) regions [1, 2], and H_2 or H_∞ performance with pole placement constraints [3]. Meanwhile, a technique named the $\mathcal{H}$-representation was proposed explicitly in [2], which can transform the stability in a mean square sense of Itô stochastic systems to deterministic systems.

Motivated by the regional pole placement and the $\mathcal{H}$-representation method, we generalize the state matrix of a linear discrete-time stochastic system to the extended state matrix by the generalized $\mathcal{H}$-representation. Meanwhile, the quadratic terms of state vectors in the $\mathcal{H}$-representation can be extended to pth terms using the lexicographical order of monomials. After this generalization, we can transform the pth stability of linear discrete-time stochastic systems to deterministic discrete-time systems with the extended state matrix and vector.

Definitions and preliminaries. Let us first introduce the definition of the LMI region and some common examples.

Definition 1 ([1]). A subset $\mathcal{D}$ of the complex plane is an LMI region if there is a symmetric matrix M and a matrix N such that

$$\mathcal{D} = \{s \in \mathcal{C} : f_{\mathcal{D}}(s) = M + sN + \overline{s}N^{\mathrm{T}} < 0\}. \tag{1}$$

In practical projects, a good controller should deliver sufficiently fast, well-damped time responses and shorten adjustment time. The transient response and the damping ratio of a linear discrete-time stochastic system are related to the location of its poles. For instance, an interesting region of linear discrete-time stochastic systems for control purposes is the set $R((q,0),r,a,b)$. If an eigenvalue $\lambda = \sigma \pm \xi \cdot i$ lies in $R((q,0),r,a,b)$, then

$$\frac{-\sigma}{\sqrt{\sigma^2+\xi^2}} \geqslant \cos\theta := \frac{|a|}{(a^2+r^2-(a-q)^2)^{1/2}},$$

which implies that the damping ratio lies in $[\cos\theta \ 1]$ and $a \leqslant \sigma \leqslant (b^2+r^2-(b-q)^2)^{1/2}$, i.e., the damped natural frequency lies in $[a \ (b^2+r^2-(b-q)^2)^{1/2}]$. Another interesting region for control purposes is the set $R((q_1,0),r_1,(q_2,0),r_2)$, which has similar properties to $R((q_1,0),r_1,(q_2,0),r_2)$.

Lemma 1 ([3]). A matrix A is $\mathcal{D}$-stable, i.e., $\sigma(A) \subset \mathcal{D}$, if and only if there exists a symmetric matrix $P > 0$, such that

$$M_{\mathcal{D}}(A,P) := M \otimes P + N \otimes (PA) + N^{\mathrm{T}} \otimes (A^{\mathrm{T}}P) < 0.$$

Let us consider the following discrete-time linear stochastic system:

$$\begin{cases} x(t+1) = Ax(t) + Bx(t)\omega(t), \\ x(0) = x_0, \end{cases} \tag{2}$$

where A, B are real constant matrices, $t \geqslant 0$, $t \in \mathcal{Z}_+$, $x(t) \in \mathcal{R}^n$, and $\omega(t)$ is an independent random variable standard 1-D Wiener process defined on the filtered probability space $(\Omega,\mathcal{F},\mathcal{P};\mathcal{F}_t)$. We assume the probability space $(\Omega,\mathcal{F},\mathcal{P})$ is complete and the filtration satisfies the usual conditions.

Definition 2 ([4]). The linear stochastic system (2) is asymptotically stable in the pth moment, if for any $x_0 \in \mathcal{R}^n$, we have

$$\lim_{t\to\infty} \mathcal{E}\|x(t;t_0,x_0)\|^p = 0.$$

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Definition 3 ([5]). Power vector $x^{\{m\}}$ is a vector of form

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}^{\{m\}} = \begin{cases} \begin{pmatrix} x_1(x_1,\ldots,x_n)^{\mathrm{T}\{m-1\}} \\ x_2(x_2,\ldots,x_n)^{\mathrm{T}\{m-1\}} \\ \vdots \\ x_{n-1}(x_{n-1},x_n)^{\mathrm{T}\{m-1\}} \\ x_n(x_n)^{\mathrm{T}\{m-1\}} \end{pmatrix}, & m>0, \\ 1, & \text{otherwise}, \end{cases}$$

where $x=(x_1,x_2,\ldots,x_n)$.

Definition 4 ([6]). Assuming $A \in \mathcal{R}^{m\times n}$, the Moore-Penrose generalized inverse matrix is a unique matrix $A^\dagger \in \mathcal{R}^{n\times m}$ satisfying

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger,$$
$$(AA^\dagger)^{\mathrm{T}} = AA^\dagger, \quad (A^\dagger A)^{\mathrm{T}} = A^\dagger A.$$

Now let us investigate the pth moment $\mathcal{D}$-stability and $\mathcal{D}$-stabilization.

Theorem 1. The linear stochastic system (2) is asymptotically stable in pth ($p>1$) moment if and only if the following deterministic system

$$\begin{cases} X^{\{p\}}(t+1) = \hat{A}_{n\cdot p}X^{\{p\}}(t), \\ X^{\{p\}}(0) = X_0^{\{p\}} \end{cases} \tag{3}$$

is asymptotically stable, where

$$\hat{B}_{n\cdot p-1} = H^\dagger_{n\cdot p-1}(\hat{A}_{n\cdot p-2}\otimes B + \hat{B}_{n\cdot p-2}\otimes A)H_{n\cdot p-1},$$
$$\hat{A}_{n\cdot p} = H^\dagger_{n\cdot p}(\hat{A}_{n\cdot p-1}\otimes A + \hat{B}_{n\cdot p-1}\otimes B)H_{n\cdot p}.$$

In other words, the sufficient and necessary condition for asymptotical stability in the pth moment of system (2) is

$$\hat{A}_{n\cdot p}P\hat{A}^{\mathrm{T}}_{n\cdot p} - P < 0, \tag{4}$$

or

$$\hat{A}_{n\cdot p}P\hat{A}^{\mathrm{T}}_{n\cdot p} - P = -Q, \tag{5}$$

or

$$\sigma(\hat{A}_{n\cdot p}) \subset (0,1), \tag{6}$$

where $P>0$, $Q>0$.

Proof. See Appendix A.

Remark 1. Based on the power vector and the generalized $\mathcal{H}$-representation method, we transformed the asymptotical stability in the pth moment of stochastic systems into the asymptotical stability of the deterministic systems, which can be dealt with the Lyapunov direct method and Hurwitz or eigenvalue criterion. Theorem 1 generalized the mean square stability of the discrete-time stochastic system through the $\mathcal{H}$-representation method to asymptotic stability in the pth moment of the discrete-time stochastic system.

As a special case of the pth moment asymptotic stability, the sufficient and necessary conditions of the mean square stability of the discrete-time stochastic system can be presented by Corollary 1.

Corollary 1. The discrete linear stochastic system (2) is asymptotically stable in the mean square if and only if the following deterministic system

$$\begin{cases} X^{\{2\}}(t+1) = \hat{A}_{n\cdot 2}X^{\{2\}}(t), \\ X^{\{2\}}(0) = X_0^{\{2\}} \end{cases} \tag{7}$$

is asymptotically stable, where

$$\hat{A}_{n\cdot 2} = H^\dagger_{n\cdot 2}(A\otimes A + B\otimes B)H_{n\cdot 2}. \tag{8}$$

In other words, the sufficient and necessary condition for asymptotical stability in the pth moment of system (2) is

$$\hat{A}_{n\cdot 2}P\hat{A}^{\mathrm{T}}_{n\cdot 2} - P < 0, \tag{9}$$

or

$$\hat{A}_{n\cdot 2}P\hat{A}^{\mathrm{T}}_{n\cdot 2} - P = -Q, \tag{10}$$

or

$$\sigma(\hat{A}_{n\cdot 2}) \subset (0,1), \tag{11}$$

where $P>0$, $Q>0$.

Definition 5. For an LMI region $\mathcal{D}$ of the complex plane, the discrete-time linear stochastic system (2) is $\mathcal{D}$-stable in the pth moment if the eigenvalues of $\hat{A}_{n\cdot p}$ lie in $\mathcal{D}$, i.e., $\sigma(\hat{A}_{n\cdot p}) \subset \mathcal{D}$, and $\sigma(\hat{A}_{n\cdot p})$ is called the pth moment pole.

In the following, we investigate the regional stability in the pth moment of a linear stochastic system. Based on Theorem 1 and Lemma 1, it can be easily generalized to the regional stability in the pth moment. We omit the proof and only list the result.

Theorem 2. The discrete-time linear stochastic system (2) is pth $\mathcal{D}$-stable, if and only if there exists a symmetric matrix $P>0$, such that

$$M\otimes P + N\otimes(\hat{A}_{n\cdot p}P) + N^{\mathrm{T}}\otimes(\hat{A}_{n\cdot p}P)^{\mathrm{T}} < 0,$$

where $\hat{A}_{n\cdot p}$ is definited as (4).

Appendix B shows the corollaries of Theorem 2.

The following shows how to use our main pth $\mathcal{D}$-stable results for pth $\mathcal{D}$-stabilization purposes.

Definition 6. The linear stochastic system (2) is a generalized pth $\mathcal{D}$-stabilization for the given sub-collection $\mathcal{D}$ of $\mathcal{C}$, if there exists a constant state feedback $u=Kx$, such that

$$\begin{cases} \mathrm{d}x(t) = (A+CK)x(t)\mathrm{d}t + Bx(t)\mathrm{d}w(t), \\ x(0) = x_0 \in \mathcal{R}^n, \end{cases} \tag{12}$$

is pth $\mathcal{D}$-stable.

Theorem 3. The linear stochastic system (2) is generalized as pth $\mathcal{D}$-stabilizable, with a constant matrix K and symmetry matrix $P>0$, such that

$$\begin{bmatrix} -r\hat{P} & \begin{matrix} q\hat{P}+\Gamma+H^\dagger_{n\cdot 2} \\ \times(B\otimes B)H_{n\cdot 2}\hat{P} \end{matrix} \\ \begin{matrix} (q\hat{P}+\Gamma^{\mathrm{T}}+\hat{P}H^\dagger_{n\cdot 2} \\ \times(B\otimes B)H_{n\cdot 2})^{\mathrm{T}} \end{matrix} & -r\hat{P} \end{bmatrix} < 0, \tag{13}$$

$$\Gamma+\Gamma^{\mathrm{T}}+H^\dagger_{n\cdot 2}(B\otimes B)H_{n\cdot 2}\hat{P} + (\hat{P}H^\dagger_{n\cdot 2}(B\otimes B)H_{n\cdot 2})^{\mathrm{T}} - 2a\hat{P} > 0, \tag{14}$$

and

$$\Gamma+\Gamma^{\mathrm{T}}+H^\dagger_{n\cdot 2}(B\otimes B)H_{n\cdot 2}\hat{P} + (\hat{P}H^\dagger_{n\cdot 2}(B\otimes B)H_{n\cdot 2})^{\mathrm{T}} - 2b\hat{P} < 0, \tag{15}$$

where

$$\Gamma = H_{n\cdot 2}^{\dagger}\Bigg(A\otimes A+(A\otimes C)\left[\overrightarrow{\eta^{-1}(\hat{p}_1)K^{\rm T}}\cdots\overrightarrow{\eta^{-1}(\hat{p}_{n(n+1)/2})K^{\rm T}}\right]+(C\otimes A)\left[\overrightarrow{K\eta^{-1}(\hat{p}_1)}\,\overrightarrow{K\eta^{-1}(\hat{p}_2)}\cdots\overrightarrow{K\eta^{-1}(\hat{p}_{n(n+1)/2})}\right]+(C\otimes C)\left[\overrightarrow{K\eta^{-1}(\hat{p}_1)K^{\rm T}}\cdots\overrightarrow{K\eta(\hat{p}_{n(n+1)/2})K^{\rm T}}\right]\Bigg).$$

Proof. See Appendix A.

Now let us introduce a numerical example.

Example 1. Consider the system (2) with matrices

$$A=\begin{bmatrix}1 & -4\\ -3 & -2\end{bmatrix},\quad B=\begin{bmatrix}0.9 & 0.1\\ -1.5 & 0.6\end{bmatrix}.$$

According to Theorem 1, we can get the eigenvalues of $\hat{A}_{2\cdot 2}$ as $\sigma(\hat{A}_{2\cdot 2})=\{-14.75, 12.92, 18.39\}$. Obviously, the system is unstable in the mean square.

Letting $C=\left[\begin{smallmatrix}-2 & -1\\ -0.3 & 3\end{smallmatrix}\right]$, we give the region $R((q,0),r,a,b)=R((0.5,0),0.3,0.4,0.7)$ to get better system properties. The controller $K_1=\left[\begin{smallmatrix}0 & -2.22\\ 1 & 0.44\end{smallmatrix}\right]$ is obtained in light of Theorem 3. After adding the controller K_2, the eigenvalues of the system matrix is

$$\sigma(\hat{A}_{K_2})=\{0.43+0.53i, 0.43-0.53i, 0.69\},$$

which is asymptotically stable in the mean square sense. Meanwhile, we can get other eigenvalues of the system matrix as $\sigma(\hat{A}_{K_2})=\{0.62+0.58i, 0.62-0.58i, 0.99\}$ under the controller $K_2=\left[\begin{smallmatrix}0.09 & -2.11\\ 1.01 & 0.22\end{smallmatrix}\right]$ without any regional constrain. The system converges faster with controller K_1 than with controller K_2.

See Appendix C for more detailed examples.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant No. 61973148), Natural Science Foundation of Shandong Province (Grant Nos. ZR2018MF028, ZR2016JL025).

Supporting information Appendixes A–C. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

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