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• **Supplementary File** •

pth moment $\mathcal{D}$-stability/stabilization of linear discrete-time stochastic systems

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Appendix A Proof of Theorem 1 and Theorem 3

Proof of Theorem 1 Basis step: We start with the statement $p = 2$ and find that

$$\begin{aligned} x^{[2]}(t+1) &= [Ax(t) + Bx(t)\omega(k)] \otimes [Ax(t) + Bx(t)\omega(k)] \\ &= [A^{[2]} + B^{[2]}]x^{[2]}(t) + [A \otimes B + B \otimes A]x^{[2]}(t)\omega(k) \\ &=: A_{n,2}x^{[2]}(t) + B_{n,2}x^{[2]}(t)\omega(k). \end{aligned} \quad \text{(A1)}$$

It's not an state equation of a system, because it has repeating elements in $x^{[2]}(t+1)$ and $x^{[2]}(t)$. According to the generalized $\mathcal{H}$-representation, we have

$$x^{[2]}(t+1) = H_{n,2}x^{\{2\}}(t+1)$$

and

$$x^{[2]}(t) = H_{n,2}x^{\{2\}}(t).$$

Such that, the formula (A1) can be rewritten as

$$H_{n,2}x^{\{2\}}(t+1) = [A^{[2]} + B^{[2]}]H_{n,2}x^{\{2\}}(t) + [A \otimes B + B \otimes A]H_{n,2}x^{\{2\}}(t)\omega(k). \quad \text{(A2)}$$

Noting that $H_{n,2}$ is column full rank and pre-multiplying $(H_{n,2}^T H_{n,2})^{-1}H_{n,2}^T$ both sides of above formula, the following is derived.

$$x^{\{2\}}(t+1) = H_{n,2}^{\dagger}[A^{[2]} + B^{[2]}]H_{n,2}x^{\{2\}}(t) + H_{n,2})^{\dagger}[A \otimes B + B \otimes A]H_{n,2}x^{\{2\}}(t)\omega(k).$$

Letting

$$X^{[2]}(t+1) = \mathcal{E}[x^{[2]}(t+1)],$$

we have

$$\begin{aligned} X^{[2]}(t+1) &= H_{n,2}^{\dagger}[A^{[2]} + B^{[2]}]H_{n,2}x^{\{2\}}(t) \\ &:= \hat{A}_{n,2}x^{\{2\}}(t). \end{aligned} \quad \text{(A3)}$$

Inductive step: Now we assume

$$x^{[p-1]}(t+1) = A_{n,p-1}x^{[p-1]}(t) + B_{n,p-1}x^{[p-1]}(t)\omega(k)$$

and

$$x^{\{p-1\}}(t+1) = \hat{A}_{n,p-1}x^{\{p-1\}}(t) + B_{n,p-1}x^{\{p-1\}}(t)\omega(k)$$

are true statement. From this assumption and the properties of the Kronecker product, we can deduce that

$$\begin{aligned} x^{[p]}(t+1) &= [A_{n,p-1}x^{[p-1]}(t) + B_{n,p-1}x^{[p-1]}(t)\omega(k)] \otimes [Ax(t)dt + Bx(t)\omega(t)] \\ &= [A_{n,p-1} \otimes A + B_{n,p-1} \otimes B]x^{[p]}(t) + [A_{n,p-1} \otimes B + B_{n,p-1} \otimes A]x^{[p]}(t)\omega(t). \end{aligned} \quad \text{(A4)}$$

Substituting

$$x^{[p]}(t+1) = H_{n,p}x^{\{p\}}(t+1)$$

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and

$$x^{[p]}(t) = H_{n,p}x^{\{p\}}(t)$$

into (A4), we can get the following equation

$$\begin{aligned} x^{\{p\}}(t+1) &= H_{n,p}^{\dagger}[A_{n,p-1}\otimes A + B_{n,p-1}\otimes B]H_{n,p}x^{\{p\}}(t) + H_{n,p}^{\dagger}[A_{n,p-1}\otimes B + B_{n,p-1}\otimes A]H_{n,p}x^{\{p\}}(t)\omega(k) \\ &=: \hat{A}_{n,p}x^{\{p\}}(t) + \hat{B}_{n,p}x^{\{p\}}(t)\omega(k). \end{aligned} \tag{A5}$$

Letting

$$X^{[p]}(t+1) = \mathcal{E}[x^{[p]}(t+1)],$$

we have

$$X^{[p]}(t+1) = \hat{A}_{n,p}X^{\{p\}}(t). \tag{A6}$$

Observe that

$$\lim_{t\to\infty}\mathcal{E}\|x(t)\|^p = 0 \Leftrightarrow \lim_{t\to\infty}[X^{[p]}(t)] = 0 \Leftrightarrow \lim_{t\to\infty}[X^{\{p\}}(t)] = 0.$$

Such that, stochastic system (2) is asymptotically stable in $pth(p > 1)$ moment if and only if deterministic system (3) is asymptotically stable. According to Lyapunov stability theorem, we have (4) and (5) in Theorem 1. By the Hurwitz stability criterion, the system (2) is asymptotically stable if and only if

$$\sigma(\hat{A}_{n\cdot p}) \subset (0,1).$$

The proof is complete.

Proof of Theorem 3 According to Definition 6 and Theorem 2, system (2) is generalized $\mathcal{D}$-stabilizable if and only if there are constant matrix K and symmetric matrix $P > 0$ such that

$$\begin{bmatrix} -r\hat{P} & \begin{array}{c} q\hat{P} + H_{n\cdot 2}^{\dagger}((A+CK)\otimes(A+CK) \\ +B\otimes B)H_{n\cdot 2}\hat{P} \end{array} \\ \begin{array}{c} q\hat{P} + \hat{P}(H_{n\cdot 2}^{\dagger}((A+CK)\otimes(A+CK) \\ +B\otimes B)H_{n\cdot 2})^T \end{array} & -r\hat{P} \end{bmatrix} < 0, \tag{A7}$$

$$\begin{aligned} &(H_{n\cdot 2}^{\dagger}((A+CK)\otimes(A+CK)+B\otimes B)H_{n\cdot 2}\hat{P} \\ &+\hat{P}(H_{n\cdot 2}^{\dagger}((A+CK)\otimes(A+CK)+B\otimes B)H_{n\cdot 2})^T - 2a\hat{P} > 0 \end{aligned} \tag{A8}$$

and

$$\begin{aligned} &(H_{n\cdot 2}^{\dagger}((A+CK)\otimes(A+CK)+B\otimes B)H_{n\cdot 2}\hat{P} \\ &+\hat{P}(H_{n\cdot 2}^{\dagger}((A+CK)\otimes(A+CK)+B\otimes B)H_{n\cdot 2})^T - 2b\hat{P} < 0. \end{aligned} \tag{A9}$$

In the following, we deal with the formula

$$H_{n\cdot 2}^{\dagger}((A+CK)\otimes(A+CK))H_{n\cdot 2}\hat{P}.$$

According to the properties of Kronecker power, the above formula can be rewritten as

$$H_{n\cdot 2}^{\dagger}(A\otimes A + (A\otimes C)(I\otimes K) + (C\otimes A)(K\otimes I) + (C\otimes C)(K\otimes K))H_{n\cdot 2}\hat{P}. \tag{A10}$$

By Lemma 4 and the definition of $H_{n\cdot 2}$, the formula (A10) is equivalent to

$$\begin{aligned} &H_{n\cdot 2}^{\dagger}(A\otimes A + (A\otimes C)[\overrightarrow{\eta^{-1}(\hat{p}_1)K^T}\ \cdots\ \overrightarrow{\eta^{-1}(\hat{p}_{n(n+1)/2})K^T}] \\ &+(C\otimes A)[\overrightarrow{K\eta^{-1}(\hat{p}_1)}\ \overrightarrow{K\eta^{-1}(\hat{p}_2)}\ \cdots\ \overrightarrow{K\eta^{-1}(\hat{p}_{n(n+1)/2})}] \\ &+(C\otimes C)[\overrightarrow{K\eta^{-1}(\hat{p}_1)K^T}\ \cdots\ \overrightarrow{K\eta(\hat{p}_{n(n+1)/2})K^T}]) \\ &:= H_{n\cdot 2}^{\dagger}(A\otimes A + (A\otimes C)[\overrightarrow{Y_1},\ \cdots,\ \overrightarrow{Y_{n(n+1)/2}}] \\ &+(C\otimes A)[\overrightarrow{Y_1^T},\ \overrightarrow{Y_2^T},\ \cdots,\ \overrightarrow{Y_{n(n+1)/2}^T}] \\ &+(C\otimes C)[\overrightarrow{Z_1},\ \cdots,\ \overrightarrow{Z_{n(n+1)/2}}] \\ &:= \Gamma(H_{n\cdot 2},\ A,\ C,\ Y,\ Z), \end{aligned} \tag{A11}$$

where $\hat{p}_i$ is the ith column of matrix $\hat{P}$ and $\eta : P = (p_{ij}) \in \mathcal{S}_n \mapsto (p_{11}, \cdots, p_{1n}, p_{22}, \cdots, p_{2n}, \cdots, p_{n-1,n})^T$. Such that we can get inequations (13) of Theorem 3. As the similar procedures, we can get inequations (14) and (15) of Theorem 3. The proof is end.

Appendix B Corollaries of Theorem 2

As the special cases of Theorem 2, below are a few corollaries. Because the disk region, vertical/horizontal strip region and their intersection region have very important practical significance in pole assignment, the *pth* moment stability conditions of the disk region, vertical/horizontal strip region and their intersection region are given below. The following corollaries can be presented when take the characteristic function of $\mathcal{D}$ as a disk, i.e., $M = \begin{bmatrix} -r & c \\ c & -r \end{bmatrix}, N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and a conic sector, i.e., $M = 0, N = \begin{bmatrix} sin\theta & cos\theta \\ -cos\theta & sin\theta \end{bmatrix}$, respectively.

Corollary 1. The discrete-time linear stochastic system (2) is *pth* moment stable in the disk centered at $(q, 0)$ with radius r $(|q| + |r| < 1)$, i.e., all eigenvalues of $\hat{A}_{n \cdot p}$ lie in the disk, if and only if the linear matrix inequality

$$\begin{bmatrix} -r\hat{P} & q\hat{P} + \hat{A}_{n\cdot p}\hat{P} \\ q\hat{P} + \hat{P}\hat{A}_{n\cdot p}^T & -r\hat{P} \end{bmatrix} < 0. \tag{B1}$$

admits a solution $\hat{P} > 0$.

In particular, we can get the sufficient and necessary conditions for the *pth* moment stability of the discrete-time linear stochastic system, when $M = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Corollary 2. System (2) is *pth* moment stable, i.e., all eigenvalues of $\hat{A}_{n \cdot p}$ lie in the unit disk, if and only if there exists a symmetric matrix $\hat{P} > 0$, such that the linear matrix inequalities

$$\begin{bmatrix} \hat{P} & \hat{A}_{n\cdot p}\hat{P} \\ \hat{P}\hat{A}_{n\cdot p}^T & -\hat{P} \end{bmatrix} < 0 \tag{B2}$$

or

$$\begin{aligned} \hat{A}_{n\cdot p}\hat{P}\hat{A}_{n\cdot p}^T - \hat{P} < 0, \\ \hat{P} > 0 \end{aligned} \tag{B3}$$

hold.

Remark 1. When $p = 2$, it becomes the sufficient and necessary conditions for the stability in the mean square sense of the discrete-time linear stochastic systems, which has been shown in Corollary 3.

Another important conclusion is about the rate of convergence for the discrete-time linear stochastic systems. Based on Corollary 2, let $M = \begin{bmatrix} -r & 0 \\ 0 & -r \end{bmatrix} (r < 1), N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, we have the following corollary.

Corollary 3. System (2) is *pth* moment stable in interval $(0, r)(r < 1)$, i.e., all eigenvalues of $\hat{A}_{n \cdot p}$ are assigned in the specified disk with the center at the origin and the radius r, if and only if there is a symmetric matrix $\hat{P} > 0$, such that the linear matrix inequalities

$$\begin{bmatrix} r\hat{P} & \hat{A}_{n\cdot p}\hat{P} \\ \hat{P}\hat{A}_{n\cdot p}^T & -r\hat{P} \end{bmatrix} < 0 \tag{B4}$$

or

$$\begin{aligned} \hat{A}_{n\cdot p}\hat{P}\hat{A}_{n\cdot p}^T - r^2\hat{P} < 0, \\ \hat{P} > 0 \end{aligned} \tag{B5}$$

hold.

Remark 2. In corollary 4, the smaller the radius r, the faster convergence in *pth* moment of the discrete-time linear stochastic systems.

In most practical situations, we usually face the intersection of LMI regions in order to guarantee satisfactory some design objectives. Fortunately, the intersection of LMI regions is also an LMI region with characteristic functions as a combination by Lemma 2. A frequently-used intersection of regions for control purpose is the set

$$R((q, 0), r, a, b) = \{x + yj \in \mathcal{C} : -b < x < -a, -r < y < r,\ tan\theta = \frac{(r^2 - a^2)^{1/2}}{|a|}\} \tag{B6}$$

. Confining the closed-loop poles to this region ensures that the damping ratio lies in $[cos\theta\ 1]$, the decay rate lies in $[a\ (b^2 + (r^2 - (b - q)^2)^{1/2}]$. The region (B6) can be characterized as the following corollary.

Corollary 4. The discrete-time linear stochastic system (2) is *pth* moment stable in the intersection of the disk centered at $(q, 0)$ with radius r $(|q| + |r| < 1)$ and the vertical strip (a, b), i.e., all eigenvalues of $\hat{A}_{n \cdot p}$ lie in the intersection of the disk and the vertical strip, if and only if the linear matrix inequalities

$$\begin{bmatrix} -r\hat{P} & q\hat{P} + \hat{A}_{n\cdot p}\hat{P} \\ q\hat{P} + \hat{P}\hat{A}_{n\cdot p}^T & -r\hat{P} \end{bmatrix} < 0, \tag{B7}$$

$$\hat{A}_{n\cdot p}\hat{P}+\hat{P}\hat{A}_{n\cdot p}^T-2a\hat{P}>0 \tag{B8}$$

and

$$\hat{A}_{n\cdot p}\hat{P}+\hat{P}\hat{A}_{n\cdot p}^T-2b\hat{P}<0 \tag{B9}$$

admit a solution $\hat{P}>0$.

Another frequently-used intersection of regions for control purpose is the set $R((q_1,0),r_1,(q_2,0),r_2)$. Confining the closed-loop poles to this region ensures that the damping ratio lies in $[cos\alpha\ 1]$, the decay rate lies in $[|q_2-r_2|\ f(q_1,r_1,q_2,r_2]$, where

$$cos\alpha=\frac{r_2^2-r_1^2-q_2^2+q_1^2}{2(2(q_1-q_2)(q_1r_2^2-q_1q_2^2-q_2r_1^2+q_2q_1^2))^{1/2}},$$
$$f(q_1,r_1,q_2,r_2)=(\frac{2(q_1r_2^2-q_1q_2^2-q_2r_1^2+q_2q_1^2)}{q_1-q_2})^{1/2}.$$

The region $R((q_1,0),r_1,(q_2,0)$ can be characterized as the following corollary.

Corollary 5. The discrete-time linear stochastic system (2) is pth moment stable in the intersection of the disk centered at $(q,0)$ with radius r $(|q|+|r|<1)$ and the vertical strip (a,b), i.e., all eigenvalues of $\hat{A}_{n\cdot p}$ lie in the intersection of the disk and the vertical strip, if and only if the linear matrix inequalities

$$\begin{bmatrix} -r\hat{P} & q\hat{P}+\hat{A}_{n\cdot p}\hat{P} \\ q\hat{P}+\hat{P}\hat{A}_{n\cdot p}^T & -r\hat{P} \end{bmatrix}<0, \tag{B10}$$

$$\hat{A}_{n\cdot p}\hat{P}+\hat{P}\hat{A}_{n\cdot p}^T-2a\hat{P}>0 \tag{B11}$$

and

$$\hat{A}_{n\cdot p}\hat{P}+\hat{P}\hat{A}_{n\cdot p}^T-2b\hat{P}<0 \tag{B12}$$

admit a solution $\hat{P}>0$.

The following corollary presents the problem of a region $R(a,r,\theta)$ intersecting of a disk with radius $r(r<1)$, a sector with inner angle 2θ, and a half plane less than a.

Corollary 6. The linear discrete stochastic system (2) is pth moment stable in $R(a,r,\theta)$ i.e., all eigenvalues of $\hat{A}_{n\cdot p}$ lie in $R(a,r,\theta)$, if and only if there exists a symmetric matrix $\hat{P}>0$, such that the linear matrix inequalities

$$\hat{P}\hat{A}_{n\cdot p}+\hat{A}_{n\cdot p}^T\hat{P}+2a\hat{P}<0, \tag{B13}$$

$$\begin{bmatrix} -r\hat{P} & \hat{A}_{n\cdot p}\hat{P} \\ \hat{P}\hat{A}_{n\cdot p}^T & -r\hat{P} \end{bmatrix}<0 \tag{B14}$$

and

$$\begin{bmatrix} sin\theta(\hat{A}_{n\cdot p}\hat{P}+\hat{P}\hat{A}_{n\cdot p}^T) & cos\theta(\hat{A}_{n\cdot p}\hat{P}-\hat{P}\hat{A}_{n\cdot p}^T) \\ cos\theta(\hat{P}\hat{A}_{n\cdot p}^T-\hat{A}_{n\cdot p}\hat{P}) & sin\theta(\hat{A}_{n\cdot p}\hat{P}+\hat{P}\hat{A}_{n\cdot p}^T) \end{bmatrix}<0. \tag{B15}$$

Appendix C Numerical Example

Example 1. Consider the second-order discrete-time stochastic system (2) with matrices

$$A=\begin{bmatrix} 1 & -4 \\ -3 & -2 \end{bmatrix}, B=\begin{bmatrix} 0.9 & 0.1 \\ -1.5 & 0.6 \end{bmatrix}.$$

According to Theorem 1, we can get

$$\hat{A}_{2\cdot 2}=\begin{bmatrix} 1.8100 & -7.8200 & 16.0100 \\ -4.3500 & 10.3900 & 8.0600 \\ 11.2500 & 10.2000 & 4.3600 \end{bmatrix},$$

and the eigenvalue of $\hat{A}_{2\cdot 2}$

$$\sigma(\hat{A}_{2\cdot 2})=\{-14.7534, 12.9187, 18.3947\}.$$

Obviously, the discrete-time linear stochastic system with system matrices A and B is not stable in the mean square without a controller.

In what follows, we will investigate the mean square stabilization of the system (2) without any regional constrain. Associated with $C=\begin{bmatrix} -2 & -1 \\ -0.3 & 3 \end{bmatrix}$ and $u(t)=Kx(t)$, where

$$K_1=\begin{bmatrix} 0.0952 & -2.1111 \\ 1.0095 & 0.2222 \end{bmatrix}.$$

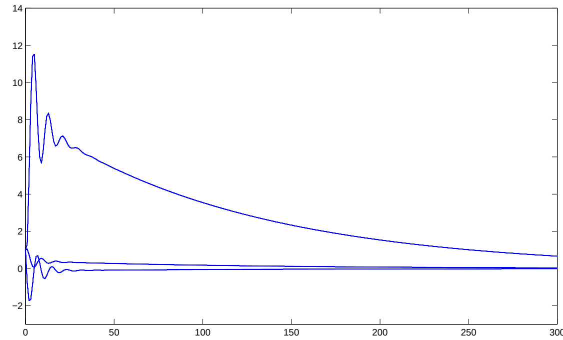

Figure C1 State response under the stabilization controller

After the addition of controller K_1, the system matrix becomes

$$\hat{A}_{K_1} = \begin{bmatrix} 0.8500 & 0.1800 & 0.0100 \\ -1.3500 & 0.5300 & 0.0600 \\ 2.2500 & -1.8000 & 0.8500 \end{bmatrix},$$

and its eigenvalue is

$$\sigma(\hat{A}_{K_1}) = \{0.6192 + 0.5836i, 0.6192 - 0.5836i, 0.9917\},$$

which is asymptotically stable in the mean square sense. However, the system converges too slowly as shown in figure C1.

In the following, we give the region $R((q,0),r,a,b) = R((0.5,0),0.3,0.4,0.7)$, in order to get better system properties. The controller

$$K_2 = \begin{bmatrix} 0 & -2.2222 \\ 1 & 0.4444 \end{bmatrix},$$

is obtained in light of Theorem 3. After the addition the controller K_2, the system matrix becomes

$$\hat{A}_{K2} = \begin{bmatrix} 0.8100 & 0.1800 & 0.0100 \\ -1.3500 & 0.3900 & 0.0600 \\ 2.2500 & -1.8000 & 0.3600 \end{bmatrix},$$

and its eigenvalue is

$$\sigma(\hat{A}_{K_2}) = \{0.4350 + 0.5356i, 0.4350 - 0.5356i, 0.6900\},$$

which is asymptotically stable in the mean square sense. However, the system converges faster than above, which is shown in figure C2.

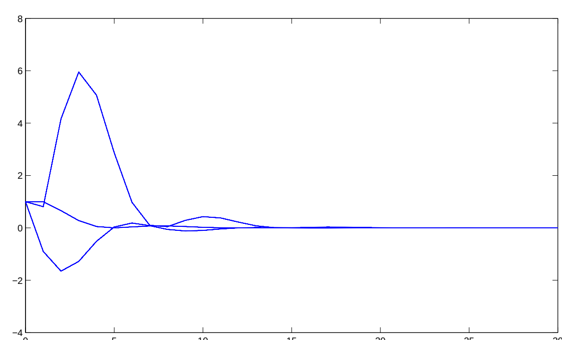

Figure C2 State response under the stabilization controller

Remark 3. In the usual stabilization control, we just require the discrete-time system to be stable, that is, the system poles are just in unit circle after adding the controller. However, those are not good enough in many practical applications. The discrete-time system pole is too close to the edge of the unit circle, the convergence speed is very slow(see figure C1). Meanwhile, the anti-interference of the discrete-time system will be very poor if the system poles are too close to original point. The probability that the poles fall into the right region with no constraints is extremely small.

Example 2. Consider the discrete-time stochastic system (2) with matrices

$$A = \begin{bmatrix} -1 & 0.8 \\ 1.2 & 2.2 \end{bmatrix}, B = \begin{bmatrix} 0.5 & 0.2 \\ 0.5 & 0.3 \end{bmatrix}.$$

According to Theorem 1, we can get

$$\hat{A}_{2\cdot 2} = \begin{bmatrix} 1.2500 & -1.4000 & 0.6800 \\ -0.9500 & -0.9900 & 1.8200 \\ 1.6900 & 5.5800 & 4.9300 \end{bmatrix},$$

and the eigenvalue of $\hat{A}_{2\cdot 2}$

$$\sigma(\hat{A}_{2\cdot 2}) = \{-2.9140,\ 1.7295,\ 6.3745\}.$$

Obviously, the discrete-time linear stochastic system with system matrices A and B is not stable in the mean square without a controller, which is shown in figure C3.

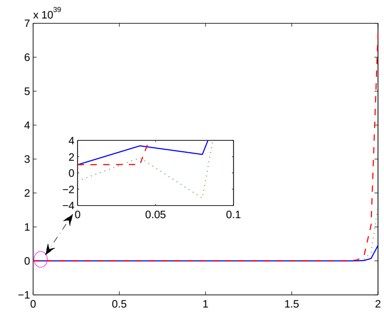

Figure C3 State response without the stabilization controller

In the following, we give the region $R((q,0),r) = R((0.4,0),0.2)$, in order to get better system properties. The controller

$$K = \begin{bmatrix} 1.1020 & -1.0903 \\ -0.3724 & -3.1214 \end{bmatrix},$$

is obtained in light of Theorem 3. After the addition the controller K, the system matrix becomes

$$\hat{A}_K = \begin{bmatrix} 0.2604 & 0.1408 & 0.1243 \\ 0.3344 & -0.0842 & 0.3275 \\ 0.9349 & -1.2251 & 0.9390 \end{bmatrix},$$

and its eigenvalue is

$$\sigma(\hat{A}_K) = \{0.5845,\ 0.2653 + 0.0292i,\ 0.2653 - 0.0292i\}.$$

The system is asymptotically stable in the mean square sense, which is shown in following figure C4.

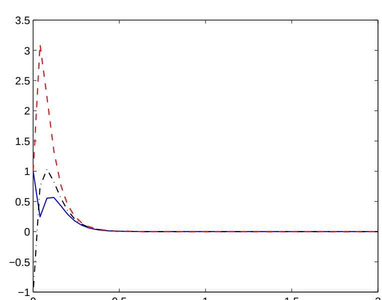

Figure C4 State response under the stabilization controller