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Sufficient conditions and limitations of equivalent partition in multiagent controllability

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Abstract The emergence of the graphical characterization of multiagent controllability has raised several issues concerning how to directly establish controllability from topology structures. Arguably, one of the most serious challenges to this research field is the means through which equivalent partition, which plays an important role in graphical characterization, obtains sufficient controllability conditions; hence, how equivalent partition influences controllability has garnered considerable attention. This article specifically focuses on the sufficient conditions and limitations of equivalent partition in multiagent controllability. We provide two sufficient conditions: (i) the absence of the system matrix's eigenvectors that make the equation formed by the eigenvalues and eigenvectors hold and (ii) the addition of leaders by reducing the same number of followers. The first condition particularly exhibits a relation between two apparently unrelated parts: Tao's equation and controllability. We further propose a necessary and sufficient condition for controllability under n-node graphs ($n \leqslant 5$) by taking advantage of iso-neighbor nodes, and analyze the resulting difficulties when n is greater than 5. Immediate corollaries of our results are obtained. Finally, we reveal the limitation of equivalent partition in controllability analysis. Several constructive examples demonstrate our results.

Keywords multiagent system, controllability, equivalent partition, Tao's equation, graph theory

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1 Introduction

As studies on multiagent systems (e.g., [1–16]) continue to increase, so does the focus on network controllability (e.g., [17–36]). Notably, Tanner [8] studied a so-called leader-follower control dynamic framework in which a subset of nodes were selected as leaders, aiming to change the natural dynamics of the network to solve specific control problems (e.g., [11–27]). The remaining agents, called followers, were indirectly controlled by the leaders through network connectivity. This led to many achievements under this structure by [28,37]. Tanner's framework is similar to fixed-time group tracking control with unknown inherent nonlinear dynamics [36], with the important similarity being that a subset of nodes were selected as leaders. For high-order dynamic agents, Wang et al. [9] provided in-depth discussions. Recently, to better understand how topological structure shapes the controllability of the leader-follower system based on the Laplacian matrix, concepts such as the controllability graph, the completely uncontrollable graph, and the conditionally controllable graph were introduced [38]. Controllability was described in [39], and a leader selection method for symbolic networks was developed. Serving an important role in graphical characterization, controllability is typically investigated via equivalent partition and relaxed equivalent partition. Cardoso et al. [40] proposed a necessary and sufficient condition for relaxed equivalent partition and clarified the relation between the Laplacian matrix L and the generalized Laplacian matrix L_π. Meanwhile, Lou et al. [41] suggested a graph theory method to determine the controllability of multiagent systems with one nontrivial cell taking the leader's role. The controllability of an undirected graph with four nodes was analyzed in detail. In addition, the limitation of equivalent partition in controllability

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analysis was identified for the first time. Qu et al. [42] studied the controllability of multiagent systems under equivalent partition and further argued that selecting $|C_i|-1$ leaders from each nontrivial cell C_i is a necessary condition for system controllability. In addition, Ji et al. [43] presented a systematic process of designing and identifying complete graphical characterization by taking advantage of controllability destructive nodes.

The above work guides a further study of controllability. Here, we concentrate on two problems pertaining to a network system with a fixed undirected-graph topology: one refers to obtaining sufficient controllability conditions, and the other involves solving the limitations of equivalent partition. After devoting much thought to the first problem, we see that the sufficient conditions for controllability are less available, especially those based on graph theory. Nonetheless, finding one or more classes of destructive topologies is relatively easy, which only leads to the necessary conditions of controllability. This is because to obtain sufficient conditions, we need to find all destructive topologies, especially for the selection of all leaders, which is an extremely difficult problem. For instance, we will consider $21\sum_{i=1}^{4}C_5^i+6\sum_{i=1}^{3}C_4^i+2\sum_{i=1}^{2}C_3^i=726$ kinds of graphs (there are 21, 6, 2 different topologies in the five, four, three-node graphs, respectively) when we study the n-node ($n\leqslant 5$) graphs. If n is greater than 5, we encounter thousands of graphs. Therefore, directly constructing controllable topologies is more difficult than finding several destructive topologies. Meanwhile, limitations to equivalent partition refer to topological structures that equivalent partition methods cannot determine for controllability.

We specifically focus on two ways to assess controllability from the perspective of equivalent partition under a fixed topology: one is choosing the leader first and then applying equivalent partition to evaluate the system's controllability; the other is creating an equivalent partition first and then selecting the corresponding leaders according to the equivalent partition results. In this paper, the number of leaders selected is taken from 1 to $n-1$, and all leader positions are considered. Therefore, these two ideas are almost the same. But if we use only one of them, assessing a system's controllability would be different. Such differences are not insignificant, however: for the first idea, it is no longer the original topology that we need to partition after leader selection, but it remains the original topology under the second idea. We derive results based on the latter.

Next, we discuss this paper's contents more concretely:

(1) To judge the controllability of any graph with any selection of leaders directly based on the Laplacian matrix, we first analyze the topology with fewer than six nodes from an equivalent partition perspective. We study the n-node graph ($n\leqslant 5$) and obtain a sufficient and necessary condition for the system to be controllable. The important value of this conclusion is that through the concepts of automorphism and iso-neighbor nodes, the graph theory characteristics of the controllability of all n-node graphs ($n\leqslant 5$) are completely described. We use the term "iso-neighbor" to denote the nodes with equal neighbor sets. Compared with the algebraic method, this is another complete description of the theoretical characteristics of multiagent systems. More specifically, among the five-node graph's 21 different topologies, if different positions and numbers of leaders are selected, 630 different topologies will be generated, all of which can be judged according to the sufficient and necessary condition obtained in this paper. Following this conclusion, we observe other results: First, we obtain a leader selection method to maintain the system's controllability. Second, under a fixed topology, system controllability remains invariant when leaders and followers are exchanged. Finally, integrating the proposition in [39], we provide a theoretical basis for a further discussion of the topological structure of more nodes. After all, many multiple-node graphs can be formed by some n-node graphs ($n\leqslant 5$). Beyond that, we identify the reason for using this method to resolve problems when n is greater than 5.

(2) We reveal the limitation of the equivalent partition of n-node graphs ($n\leqslant 5$) originally proposed in [41]. While we only solve the case where n less than 6, this is not an easy task, because it requires us to calculate, analyze, and generalize 756 topologies. Besides, no better methods have been determined so far. Even when n is equal to 6, we need to compute thousands of topologies. Thus, this is a major challenge we have faced in this study. Through a comprehensive analysis of five-node graphs, we identify and recognize the limitation of equivalent partition when solving controllability problems. Then, the method to overcome this limitation is provided, and the short board of the equivalent partition to solve the controllability problem is completed, which is also a critical aspect in bridging the gap between equivalent partition and system controllability.

(3) Despite the breadth of controllability findings, none of them can combine controllability with Tao's equation — $|v_{i,j}|^2\prod_{k=1;k\neq i}^{n}(\lambda_i(A)-\lambda_k(A))=\prod_{k=1}^{n-1}(\lambda_i(A)-\lambda_k(M_j))$, which was discovered by three physicists in the study of neutrino oscillations [44]. Tao's equation is essentially a method to calculate

eigenvectors using given eigenvalues. Furthermore, it plays an important role in Theorem 2; that is, the Laplacian matrix L is proven to possess the same eigenvalues as the matrix L_f composed of followers. Through this result, we find that this conclusion contains the equation of the product of eigenvectors and L_{fl}, where L_{fl} is exactly the controllable matrix of the system. This is consistent with $\alpha^{\mathrm{T}}B$ in the well-known Popov-Belevitch-Hautus (PBH) test. More importantly, we apply Tao's equation to system controllability for the first time, which will generate more ideas for system controllability research because of the equation's strong physical background.

The rest of this article is structured as follows: In Section 2, we establish some symbols and provide the necessary definitions from a graph theory perspective. In Section 3, we discuss the motivation for this article and present main results. In Section 4, we summarize and discuss ideas regarding future work.

2 Problem statement

In this paper, agents and communication relationships between pairs of agents are regarded as nodes and edges between pairs of nodes, respectively. An undirected graph G consists of node set $V(G)$ and edge set $E(G) \subset V(G) \times V(G)$. If graph G is a finite graph with n nodes, $V(G)$ and $E(G)$ can be represented as $V(G) = \{1, 2, \ldots, n\}$, $E(G) = \{(i, j)|i, j \in V(G)\}$, respectively. The neighbor set of i is defined as $N(i) = \{j \in V(G)|(i, j) \in E(G)\}$.

A path of length $n-1$ in G is a concatenation of distinct edges $\{(1, 2), (2, 3), \ldots, (n-1, n)\}$. A circle is a path with identical starting and ending node, i.e., $1 = n$. A graph G is connected if there is a path between any two different nodes i and j $(i \neq j)$. Graphs without circle and multiple edges (two or more edges that are incident to the same two nodes) are called simple graphs. The adjacency matrix of graph G is defined as $A(G) = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1, & (i, j) \in E(G), \\ 0, & (i, j) \notin E(G). \end{cases}$$

The Laplacian matrix of G is defined as $L(G) = D(G) - A(G)$, where $D(G) = \mathrm{diag}([d_i]_{i=1}^n)$, d_i is the cardinality of the neighbor set of i. We only study connected undirected simple graphs in this paper.

Let $x(t) = [x_1(t), \ldots, x_n(t)]^{\mathrm{T}} \in \mathbb{R}^n$ denotes the stacked system states, where each entry $x_i(t) \in \mathbb{R}$ represents the state of agent i. We assume that the system states evolve according to the following Laplacian dynamics:

$$\dot{x}(t) = -L(G)x(t).$$

Suppose the node set V is divided into leader set V_l and follower set V_f with $V = V_l \bigcup V_f$. Without loss of generality, we assume that the follower set consists of the former m agents and the leader set consists of the remaining $n-m$ agents. We can rewrite $x(t)$ as $[x_f^{\mathrm{T}}(t), x_l^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^n$, where $x_f(t) \in \mathbb{R}^m$ and $x_l(t) \in \mathbb{R}^{n-m}$ represent the states of followers and leaders, respectively. Similar to [39], the Laplacian matrix of a graph can be partitioned as

$$L(G) = \begin{pmatrix} L_f & L_{fl} \\ L_{lf} & L_l \end{pmatrix}.$$

It follows that the dynamic equation of the followers is

$$\dot{x}_f = -L_f x_f - L_{fl} x_l. \tag{1}$$

We use A, B, x, and u to represent $-L_f$, $-L_{fl}$, x_f, x_l, respectively. Then Eq. (1) can be expressed as

$$\dot{x} = Ax + Bu. \tag{2}$$

Similar to [39], we introduce the following formula:

$$\dot{x} = -Lx + Bu, \tag{3}$$

where B is an $n \times (n-m)$ matrix satisfying $B = [e_{m+1}, \ldots, e_n]$. We let $e_i, i = 1, \ldots, n$ be the i-th element of the canonical basis, e.g., $e_1 = [1\ 0\ \cdots\ 0]^{\mathrm{T}}$. It is normally appropriate to treat these two systems as if they were equivalent under the different control inputs.

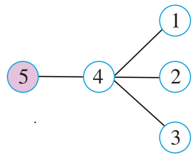

Figure 1 (Color online) A five-node graph with the minimum number of edges when $a_{14} = 1$, $a_{24} = 1$, $a_{34} = 1$.

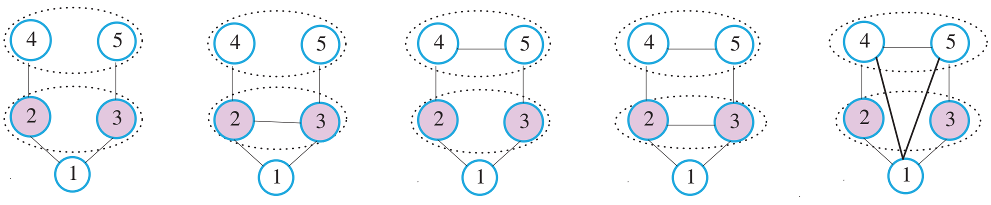

Figure 2 (Color online) A topology without iso-neighbor nodes.

3 Controllability analysis of the system

In this part, in order to judge the controllability of any graph with any selection of leaders directly based on Laplacian matrix, we first analyze the topology with fewer than six nodes from the perspective of equivalent partition and then get some criterion about controllability. Before stating main results, we undertake the necessary task of introducing some useful definitions and lemmas.

Definition 1 ([37])**.** If there is a nonidentity permutation J such that $JA = AJ$, where $A = L_f$, then system (1) is symmetric for leader i. If a system does not allow any leader to have this permutation, we call it an asymmetric system.

Proposition 1 ([37])**.** If system (1) is leader symmetric, then it is uncontrollable.

Through the above description of leader symmetric, we can illustrate its meaning with concrete examples. In Figure 1, if we select only node 4 as the leader, Figure 1 is leader symmetric with respect to a single leader 4. The reader is invited to verify that there exists a nonidentity permutation J such that $JA = AJ$. At this time, system (1) cannot be controlled, because nodes 1, 2, 3, and 5 have equal status, which makes the leader unable to control them to different target positions.

Definition 2 (Automorphism mapping [45])**.** Let $\sigma(i) = j$ mean that permutation σ maps node i to node j. σ is an automorphism mapping of graph G if $(i, j) \in E \Leftrightarrow (\sigma(i), \sigma(j)) \in E$.

According to the definition of automorphism mapping, we can get the method to judge the automorphism node. As shown in Figure 2(d), nodes 2 and 3, 4 and 5 all constitute a group of automorphism nodes. The reason is as follows: when we interchange the labels of these two sets of nodes, we do the following permutation: $\sigma(1) = 1, \sigma(2) = 3, \sigma(3) = 2, \sigma(4) = 5, \sigma(5) = 4$. In this case, $(i, j) \in E \Leftrightarrow (\sigma(i), \sigma(j)) \in E$, $i, j = 1, 2, \ldots, 5$.

Proposition 2 ([37])**.** System (1) is leader symmetric if and only if the follower subset G_f has a nonidentity automorphism, so that the indication function remains unchanged under its action.

The concepts of leader symmetry and automorphism are introduced through the above definitions and propositions. In case of single leader, these two concepts are equivalent.

Definition 3 (Equivalent partition [41])**.** A k-partition of vertex set V means that all vertices in V are divided into k non-empty cells $C_1, \ldots, C_k$, which satisfy $C_i \cap C_j = \emptyset$, $i, j = 1, 2, \ldots, k$. If for every pair of nodes s, t in C_i, $d_s(C_j) = d_t(C_j)$, $j = 1, 2, \ldots, k$, where $d_s(C_j)$ represents $|N_s \cap C_j|$, and $|N_s \cap C_j|$ is the number of elements in the set $N_s \cap C_j$, the partition is said to be a relaxed equivalent partition. If for every pair of nodes s, t in C_i, $d_s(C_j) = d_t(C_j)$, $j = 1, 2, \ldots, i-1, i+1, \ldots, k$, the partition is said to be an equivalent partition.

Consider the graph G shown in Figure 2(a) with 5 nodes and the node partition π where $C_1 = \{1\}$, $C_2 = \{2, 3\}$, $C_3 = \{4, 5\}$. Let (i, C_j) represent the number of connecting edges between node i and all points in cell C_j. It can be obtained by verification that $(2, C_j) = (3, C_j)$, $(4, C_j) = (5, C_j)$, $j = 1, 2, 3$. Therefore, partition π constitutes an equivalent partition.

Definition 4 (Nontrivial maximal relaxed equivalent partition [41])**.** Let π be a relaxed equivalent partition of graph G. If the number of cells in π is the smallest, then π is called the maximum relaxed equivalent partition under the fixed leader.

We analyze the system controllability of n-node graphs in detail by the concepts of equivalent partition and relaxed equivalent partition.

Definition 5 (Iso-neighbor nodes). Nodes i and j $(i \neq j)$ in the graph G are called iso-neighbor nodes if their neighbor sets are all equal. Specially, iso-neighbor nodes include the case that nodes i and j are neighbors.

Iso-neighbor nodes must be automorphism nodes, not vice versa. There are automorphism nodes in Figure 2, but no iso-neighbor nodes. In Figure 1, nodes 1, 2, 3 are iso-neighbor nodes according to the definition.

Lemma 1 (Rank criterion [46]). For a linear time-invariant continuous system (2), the system is controllable if and only if

$$\operatorname{rank}(Q) = \operatorname{rank}[B \ AB \ A^2B \ \cdots \ A^{n-1}B] = n.$$

Lemma 2 (Controllability and binary complements [38]). Consider the following system

$$\dot{x} = -Lx + bu, \tag{4}$$

where $b = [b_1, \ldots, b_n]^{\mathrm{T}} \in \{0, 1\}$, $n \geqslant 2$. Let $\bar{b} = 1_n - b$, where 1_n is an n-dimensional column vector of 1. Then system (4) is controllable if and only if the pair $(L, \bar{b})$ is controllable.

Lemma 3 ([41]). If there is a relaxed equivalent partition under the graph G of system (2), any nontrivial cell C_i is chosen as the leader set. Then system (2) is uncontrollable if G_f contains at least two automorphism nodes.

Lemma 4 (PBH criterion [46]). For a linear time-invariant continuous system, system (2) is controllable if and only if there is no non-zero vector α such that the following equations are satisfied:

$$\alpha^{\mathrm{T}} A = \lambda_i \alpha^{\mathrm{T}}, \quad \alpha^{\mathrm{T}} B = 0,$$

where λ_i is an eigenvalue of A.

Lemma 5 ([10]). The elements of the eigenvectors of $L(G)$ correspond to zero with the position of the leader if and only if system (3) is uncontrollable.

We usually use the above lemmas to judge the controllability of a system. Next, we will introduce Tao's equation (5), which plays an important role in this research.

Lemma 6 ([44]). The eigenvalues $\lambda_i(A)$ of the Hermitian matrix A have the following relationship to the elements of eigenvectors $v_{i,j}$:

$$|v_{i,j}|^2 \prod_{k=1;k\neq i}^{n} (\lambda_i(A) - \lambda_k(A)) = \prod_{k=1}^{n-1} (\lambda_i(A) - \lambda_k(M_j)), \tag{5}$$

where M_j is the $(n-1) \times (n-1)$ submatrix of A that results from deleting the j-th column and the j-th row, with eigenvalues $\lambda_k(M_j)$.

Lemma 7 ([41]). There is a nonidentity permutation J such that $J^{\mathrm{T}} L_f J = L_f$ if and only if there are automorphism nodes in topology.

By means of the above lemmas and concepts, especially the concepts of automorphism and iso-neighbor nodes, we make a comprehensive analysis of n-node graphs $(n \leqslant 5)$. The following theorem reveals the graph theory characteristics of controllability of n-node graphs $(n \leqslant 5)$. This is a complete description from an aspect of the graph theory characteristics of multiagent system controllability compared to what we had before.

Proposition 3 ([41]). The method of equivalent partition has no effect on judging the controllability of the following two topologies:

(1) The follower subgraph G_f has only trivial equivalent partitions if the leader is selected from the trivial cells under the maximal relaxed equivalent partition;

(2) G_f has no automorphism nodes if the leader is selected from the nontrivial cells under the maximal relaxed equivalent partition.

Theorem 1. Let G be a graph with n nodes ($n \leqslant 5$). Below are two necessary and sufficient conditions for system (2) to be controllable.

- If there is only one leader, the follower subgraph G_f does not contain automorphism nodes;
- If there are n ($2 \leqslant n < 5$) leaders, the follower subgraph G_f does not contain iso-neighbor nodes.

Proof. **(1)** If there is only one leader in graph G, Proposition 1 shows that the necessity of controllability is true. Next, we prove the sufficient condition. Let us prove its converse-negative proposition: If system (2) is uncontrollable, there are automorphism nodes in the follower subgraph G_f.

We prove the case of five-node graphs first, and without loss of generality assume that the leader is node 5.

Case 1. If node 5 has only one neighbor, and without loss of generality, we assume that the neighbor is node 4, then A, B of system (2) are taken as

$$A = \begin{pmatrix} -d_1 & a_{12} & a_{13} & a_{14} \\ a_{12} & -d_2 & a_{23} & a_{24} \\ a_{13} & a_{23} & -d_3 & a_{34} \\ a_{14} & a_{24} & a_{34} & -d_4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

where $a_{ij} = 1$ if i is connected to j and $a_{ij} = 0$ otherwise. By calculation,

$$AB = \begin{pmatrix} a_{14} \\ a_{24} \\ a_{34} \\ -d_4 \end{pmatrix}.$$

Suppose that system (2) is uncontrollable. The rank of the controllability matrix Q is less than four. Since AB is unknown, we consider the values of a_{14}, a_{24}, a_{34} first.

(i) If $a_{14} = 0$, $a_{24} = 0$, $a_{34} = 1$, we calculate the values of a_{ij}: The first case is $a_{13} = 1$, $a_{23} = 1$, $a_{12} = 0$, and the second one is $a_{13} = 1$, $a_{23} = 1$, $a_{12} = 1$. At the same time, we can get L_f under these two cases, respectively:

$$L_f = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \quad \text{or} \quad L_f = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

From Lemma 7, the above two matrices satisfy the necessary and sufficient condition for automorphism nodes. Thus, we have proved Theorem 1 in these two cases.

Calculations show that if $a_{14} = 0$, $a_{24} = 1$, $a_{34} = 0$ or $a_{14} = 1$, $a_{24} = 0$, $a_{34} = 0$, we have the same results as that of $a_{14} = 0$, $a_{24} = 0$, $a_{34} = 1$. Therefore, these two cases' analysis is omitted here.

(ii) If $a_{14} = 0$, $a_{24} = 1$, $a_{34} = 1$, we analyze the values of a_{ij}: The first case is $a_{13} = 1$, $a_{23} = 0$, $a_{12} = 1$, and the second one is $a_{13} = 1$, $a_{23} = 1$, $a_{12} = 1$. At the same time, we can get L_f under these two cases, respectively:

$$L_f = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 3 \end{pmatrix} \quad \text{or} \quad L_f = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 3 \end{pmatrix}.$$

These above two matrices satisfy Lemma 7. Thus, we have proved Theorem 1 in these two cases.

By calculation, if $a_{14} = 1$, $a_{24} = 1$, $a_{34} = 0$ and $a_{14} = 1$, $a_{24} = 0$, $a_{34} = 1$, we have the same results as that of $a_{14} = 0$, $a_{24} = 1$, $a_{34} = 1$. Therefore, these two cases' analysis is omitted here.

(iii) If $a_{14} = 1$, $a_{24} = 1$, $a_{34} = 1$, the structure of the five-node graph with the minimum number of edges is shown in Figure 1. It can be seen from Figure 1 that no matter how a_{12}, a_{13}, a_{23} are valued, system (2) is uncontrollable according to Lemma 1, because there are automorphism nodes in all cases. Therefore, we have proved the theorem in this case.

Case 2. If node 5 has two neighbors, the control vector B of system (2) can be set as

$$B = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

By calculation, we have the following product of matrix A and control vector B:

$$AB = \begin{pmatrix} a_{13} + a_{14} \\ a_{23} + a_{24} \\ -a_{13} - a_{23} \\ -a_{14} - a_{24} \end{pmatrix}.$$

We assume that system (2) is uncontrollable. Then the rank of the controllability matrix Q is less than four. Since AB is unknown, we consider the values of a_{13}, a_{14}, a_{23}, a_{24} first.

(i) If $a_{13} = 1$, $a_{14} = 1$, $a_{23} = 0$, $a_{24} = 0$, the values of a_{34} are computed as follows: The first case is $a_{34} = 1$, and the second one is $a_{34} = 0$. At the same time, we can get L_f under the two cases, respectively:

$$L_f = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 3 & -1 \\ -1 & 0 & -1 & 3 \end{pmatrix} \quad \text{or} \quad L_f = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix}.$$

According to Lemma 7, we know that there are automorphism nodes in these two cases. Thus, we have proved Theorem 1 in these two cases. By calculation, we find that the next five cases, in which a_{13}, a_{14}, a_{23}, a_{24} are taken as 1,0,1,0; 1,0,0,1; 0,1,1,0; 0,1,0,1; 0,0,1,1, respectively, have the same results as this case. Therefore, these five cases' analysis will be omitted here.

(ii) In the same way, we find that if $a_{13} = 1$, $a_{14} = 1$, $a_{23} = 1$, $a_{24} = 0$ or $a_{13} = 1$, $a_{14} = 1$, $a_{23} = 1$, $a_{24} = 1$, we have the same results as the case of $a_{13} = 1$, $a_{14} = 1$, $a_{23} = 0$, $a_{24} = 0$. Thus, the two cases are omitted here.

Case 3. If the leader has three neighbors, and without loss of generality, we assume that these three neighbors are nodes 2–4, then we can get the control vector $B = [0\ 1\ 1\ 1]^{\mathrm{T}}$. Next, the analysis method is the same as case 1, because the control vector $[0\ 1\ 1\ 1]^{\mathrm{T}}$ is the complement of $[1\ 0\ 0\ 0]^{\mathrm{T}}$, which is the control vector in case 1. Since we have discussed all cases of follower subgraphs in a comprehensive way in case 1, the theorem also holds in this case according to Lemma 2.

Case 4. If the leader (node 5) has four neighbors, no matter how the followers are connected, the system is uncontrollable, because there are automorphism nodes. Thus, we have proved Theorem 1 in this case.

In conclusion, under the assumption of Theorem 1, all the topological structures of the five-node graph can be verified. Three-node graphs and four-node graphs have similar proofs as five-node graphs, which are omitted here. Therefore, sufficiency has been proved.

(2) We are going to prove converse-negative proposition of Theorem 1: If more than one leaders are selected, the system is uncontrollable if and only if there are iso-neighbor nodes in the follower subgraph.

The sufficiency of the converse-negative proposition can be obtained from Lemma 3. Next, we prove the necessity of the converse-negative proposition. We assume that the follower subgraph does not contain iso-neighbor nodes.

We suppose that nodes 4 and 5 are leaders and the remaining nodes are followers. Then we can get

$$-L_f = \begin{pmatrix} -d_1 & a_{12} & a_{13} \\ a_{12} & -d_2 & a_{23} \\ a_{13} & a_{23} & -d_3 \end{pmatrix}, \quad -L_{fl} = \begin{pmatrix} a_{14} & a_{15} \\ a_{24} & a_{25} \\ a_{34} & a_{35} \end{pmatrix}.$$

If the follower subgraph does not contain iso-neighbor nodes, nodes 1 and 2 are not iso-neighbor nodes. In other words, nodes 1 and 2 do not have the same neighbors, i.e., $a_{13} \neq a_{23}$, $a_{14} \neq a_{24}$, $a_{15} \neq a_{25}$.

(i) If $a_{13} = 1$, $a_{14} = 1$, $a_{15} = 1$, we obtain $a_{23} = 0$, $a_{24} = 0$, $a_{25} = 0$, $a_{12} = 1$, $d_1 = 1$, $d_2 = 4$. According to Lemma 4, if the system is uncontrollable, there must exist a non-zero eigenvector $v_i = [v_{i1}\ v_{i2}\ v_{i3}]^{\mathrm{T}}$ of A such that $v_i^{\mathrm{T}} B = 0$, and $(A - \lambda_i I)v_i = 0$, where I is the three-dimensional identity matrix. We can get the following equations:

$$\begin{cases} v_{i1}a_{14} + v_{i2}a_{24} + v_{i3}a_{34} = 0, \\ v_{i1}a_{15} + v_{i2}a_{25} + v_{i3}a_{35} = 0, \\ v_{i1}(-d_1 - \lambda_i) + v_{i2}a_{12} + v_{i3}a_{13} = 0, \\ v_{i1}a_{12} + v_{i2}(-d_2 - \lambda_i) + v_{i3}a_{23} = 0, \\ v_{i1}a_{13} + v_{i2}a_{23} + v_{i3}(-d_3 - \lambda_i) = 0. \end{cases} \tag{6}$$

We substitute the coefficients into (6), and get that the values of a_{34} and a_{35} are both zero. In other words, nodes 2 and 3 are iso-neighbor nodes, which is in contradiction with the assumption. Therefore, we have proved Theorem 1 in this case.

(ii) If $a_{13} = 1$, $a_{14} = 1$, $a_{15} = 0$, we get that the values of v_{i1}, v_{i2} and v_{i3} are all zero according to (6), which is inconsistent with nonzero v_i; thus, the assumption is not valid.

Analogously, similar analysis can also be done in other cases, omitted here. We can prove three-node graphs and four-node graphs in the same way. Therefore, we have proved Theorem 1.

Example 1. Among all of the topologies of a five-node graph, on the one hand, only five topologies do not have iso-neighbor nodes, as shown in Figure 2. Take Figure 2(a) as an example. If a nontrivial cell $C_1 = \{2, 3\}$ is chosen as the leader set, system (2) is controllable from Theorem 1, because there are no iso-neighbor nodes in the follower subgraph. The same result is obtained if the cell is composed of nodes 4 and 5. On the other hand, the automorphism nodes will exist only if node 1 is chosen as the leader. Therefore, according to Theorem 1, it is uncontrollable at this time. However, if one of the remaining four nodes is chosen as the leader, the system is controllable. Similarly, Figures 2(b)–(e) have the same property.

Remark 1. There are two aspects in the study of the multiagent controllability: one is to get the controllability of any graph after selecting one or several leaders; the other is to get the controllability after selecting any leader in the graph of fixed number of nodes. The second case is studied in this paper. In this case, even an n-node graph ($n \leqslant 5$) needs to analyze 726 cases before the final conclusion can be made. This also reflects the special nature of the graph when the number of nodes n is less than or equal to 5. In particular, we are looking at the result of a matrix with a weight of 1. If the weight is not 1, then the definitions of automorphism node, iso-neighbor node, equivalent partition, relaxed equivalent partition will all change, and then the Theorem 1 obtained based on these definitions will also change. It is worth noting that, for n-node graphs, although a very practical and simple method to judge the controllability of system (1) is obtained when $n \leqslant 5$, Theorem 1 is not necessarily true when $n > 5$, as shown in the six-node and seven-node graphs in Figure 3.

• In the case of single leader, automorphism nodes are not included in the follower subgraph G_f of Figure 3, but system (1) is uncontrollable. Specially, the Laplacian matrix L contains an eigenvector $[1\ 1\ 1\ 1\ 1\ 1\ 1]^{\mathrm{T}}$, and the leader in Figure 3(c) is connected to all followers; thus, system (1) is uncontrollable according to PBH criterion.

• In the case of multiple leaders, the follower subgraph G_f of Figure 3 does not contain iso-neighbor nodes, but system (1) is uncontrollable. The main reason lies in the existence of a cell composed of automorphism nodes.

No matter in which case of Theorem 1, we note that there exist six-node graphs and seven-node graphs that do not satisfy the condition of Theorem 1. Thus, five is the limitation of the number of graph nodes of Theorem 1.

Corollary 1. Let π be a maximal relaxed equivalent partition, $\pi = C_1 \cup C_2 \cup \cdots \cup C_s \cup C_{s+1} \cup \cdots \cup C_m$, $1 \leqslant s \leqslant m \leqslant 5$. If there are s cells consisting only of iso-neighbor nodes, and without loss of generality, they are C_j, $j = 1, 2, \ldots, s$, then system (1) is controllable if only 0 or 1 follower is selected in each C_j.

Proof. This corollary destroys relaxed equivalent partition by selecting followers, which is a direct consequence of Theorem 1.

Example 2. Figure 4 shows all sixteen topologies with iso-neighbor nodes in five-node graphs. The red dotted-line circle represents the cell composed of all iso-neighbor nodes. For example, in Figure 4(a)

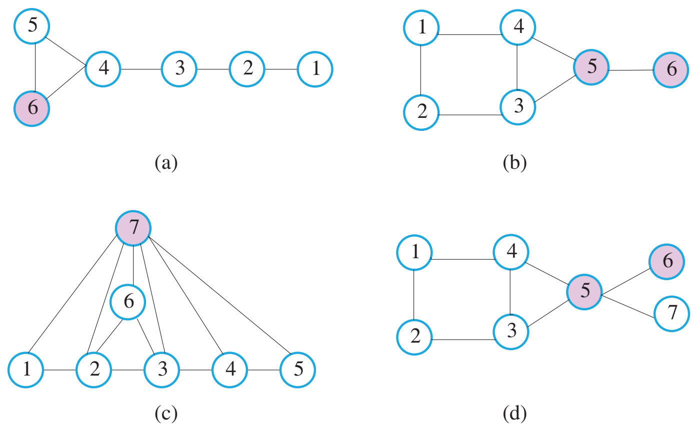

Figure 3 (Color online) Four graphs that do not satisfy the conclusion of Theorem 1.

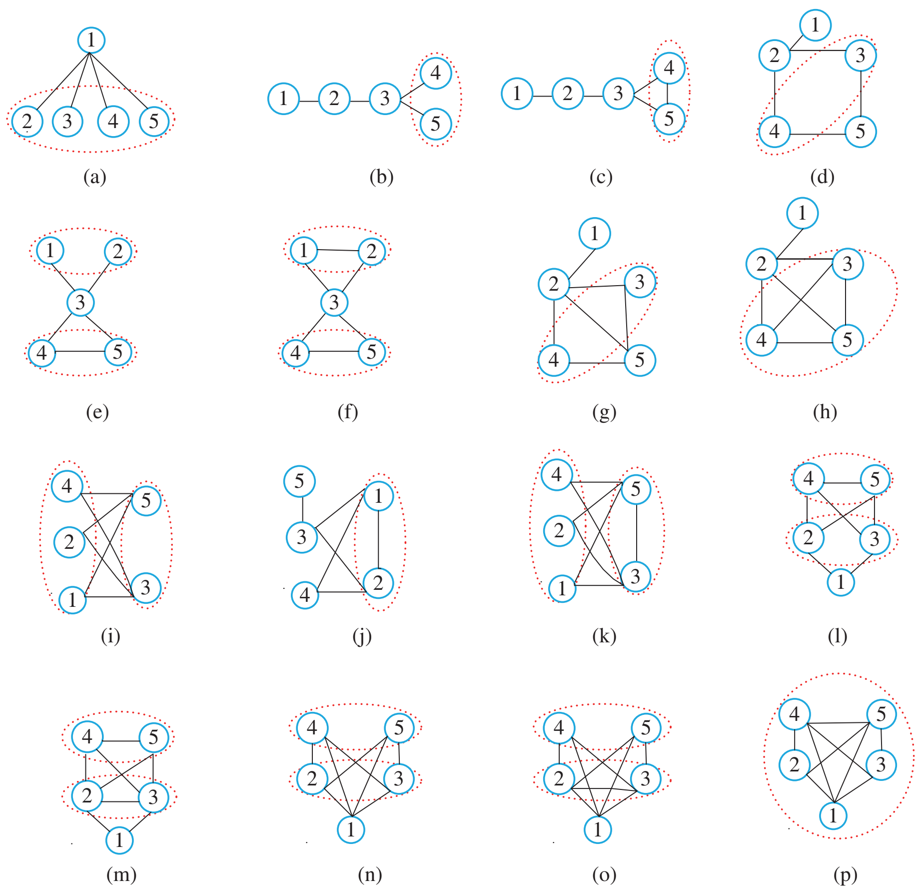

Figure 4 (Color online) All five-node graphs with iso-neighbor nodes.

where $C_1 = \{1\}$ and $C_2 = \{2, 3, 4, 5\}$, only if we choose three or four leaders from C_2 we can guarantee that there are only zero or one follower in C_2. According to Corollary 1, system (1) is controllable. However, if there are more than one follower in C_2, system (1) is uncontrollable.

Remark 2. Figures 2 and 4 constitute all topologies of five-node graphs. In these twenty-one different topologies, 630 different cases will be generated if different positions and numbers of leaders (less than four) are selected. In these sixteen topologies of Figure 4, 480 different cases can be obtained, all of which can be judged by Corollary 1 without any calculation.

Corollary 2. If $n \leqslant 5$, we assume that system (2) is controllable under a fixed topology, where the number of followers and leaders is p and q, respectively, $p < q$. If the role of followers and leaders is switched, and there is no nontrivial equivalent partition, then system (2) is controllable.

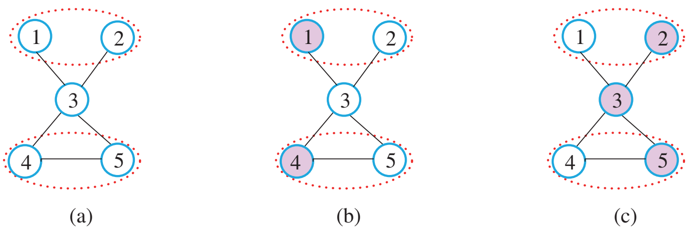

Figure 5 (Color online) The leaders and followers switch roles in a five-node graph.

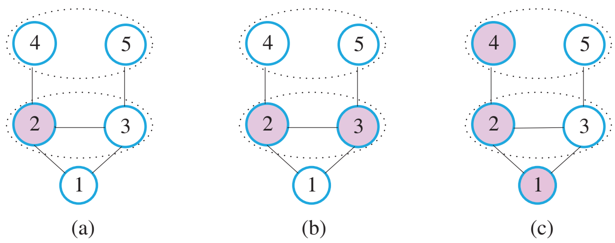

Figure 6 (Color online) The addition of leaders.

Proof. Since there is no nontrivial equivalent partition after interchanging, system (2) is controllable according to Theorem 1.

Example 3. Under a fixed topology of Figure 5(a), if there are two leaders and three followers, as shown in Figure 5(b), system (2) is controllable. Since $2 < 3$, after all leaders and followers are switched, as shown in Figure 5(c), it can be known from Corollary 2 that system (2) is still controllable. Similarly, if there are one leader and four followers, it will have the same property.

If $n > 5$, Corollary 2 is not necessarily true, which indicates that under a fixed topology, the interchange of roles of leaders and followers does not produce such a property as Lemma 2.

Proposition 4. Suppose system (1) is controllable under a fixed topology. If the number of leaders is added by reducing the same number of followers, then system (1) is still controllable.

Proof. We assume that each external control input is different; then systems (1) and (3) can be equivalently reformulated into each other, which yields the same controllability. Thus, we replace system (1) with system (3). At this point, the agents containing control inputs are regarded as the leaders. Suppose we add m leaders. Since the topology is fixed, when adding the number of leaders by reducing the same number of followers, the Laplacian matrix L does not change. However, the input matrix B increases m unit column vectors. Since the original system is controllable, the controllability matrix Q is full row rank. When B increases m unit column vectors, the rank of Q does not decrease. Therefore, the system is still controllable.

Example 4. As shown in Figure 6(a), when node 2 is selected as the leader, system (1) is controllable; if nodes 2 and 3 are both leaders (see Figure 6(b)), system (1) is also controllable from Proposition 4. Similarly, if nodes 1, 2, and 4 are leaders (as shown in Figure 6(c)), system (1) is controllable according to Proposition 4 as well.

Lemma 8 ([39])**.** In a leader-follower network G, if the connections of follower-to-follower and leader-to-follower are unaltered, then the controllability of system (1) remains unchanged no matter how we change the connections of leader-to-leader.

Remark 3. In a topological structure with a large number of nodes, the controllability of system (1) can be judged more conveniently by reducing the number of leaders. For example, in Figure 6(c), we can remove the connections of the node 2 to node 4. We know that the controllability is invariant according to Lemma 8. Thus, the system controllability is up to the corresponding four-node graph. Similarly, in a six-node graph with two or three leaders, it is possible to make judgments based on all the structures of the corresponding four-node graph and five-node graph. More multi-node graphs can also be followed in turn.

Corollary 2 and Proposition 4 both pay attention to the relationship between systems (1) and (3). Specially, we applied the relationship between these two systems in the proof of Proposition 4. Next, the system matrices L and L_f of these two systems are considered respectively, which can lay a foundation for analyzing the relationship between these two systems.

Theorem 2. If the leader is single, a sufficient condition for system (1) to be controllable is that there does not exist α_i such that

$$\alpha_i^{\mathrm{T}} L_{fl} = -\lambda_i \sum_{j=1}^{j=n-1} \alpha_{ij},$$

where $\alpha_i, i = 1, 2, \ldots, n-1$ is an eigenvector corresponding to the i-th eigenvalue λ_i of L_f, and α_i is made up of $\alpha_{ij}, j = 1, 2, \ldots, n-1$.

Proof. For the convenience of description, we prove its converse-negative proposition: If there is only one leader, a necessary condition for system (1) to be uncontrollable is that there exists α_i such that $\alpha_i^{\mathrm{T}} L_{fl} = -\lambda_i \sum_{j=1}^{j=n-1} \alpha_{ij}$.

Without loss of generality, we assume the n-th agent is the leader, and β_k, α_i are eigenvectors corresponding to the k-th eigenvalue of L and the i-th eigenvalue of L_f, respectively.

$$\beta_k = \begin{pmatrix} \beta_{k1} \\ \vdots \\ \beta_{kn} \end{pmatrix}, \quad \alpha_i = \begin{pmatrix} \alpha_{i1} \\ \vdots \\ \alpha_{i,n-1} \end{pmatrix}.$$

Since system (1) is uncontrollable, we get $\beta_{kn} = 0$ according to Lemma 5. Further, there must exist $\lambda_i(L_f)$ such that $\lambda_i = \lambda_k$, according to Lemma 6. Let us define a new matrix L'_f and a new vector α'_i as shown below:

$$L'_f = \begin{pmatrix} a_{11} & \ldots & a_{1,n-1} & 0 \\ \vdots & & \vdots & \vdots \\ a_{n-1,1} & \ldots & a_{n-1,n-1} & 0 \\ 0 & \ldots & 0 & 0 \end{pmatrix}, \quad \alpha'_i = \begin{pmatrix} \alpha_{i1} \\ \vdots \\ \alpha_{i,n-1} \\ 0 \end{pmatrix}.$$

According to $(L - \lambda_k I)\beta_k = 0$, we have

$$(a_{mm} - \lambda_k)\beta_{km} + \sum_{j=1, j\neq m}^{n} a_{mj}\beta_{kj} = 0, \ m = 1, 2, \ldots, n. \tag{7}$$

Based on $(L'_f - \lambda_i I)\alpha'_i = 0$, we obtain an equation as

$$(a_{mm} - \lambda_i)\alpha_{im} + \sum_{j=1, j\neq m}^{n} a_{mj}\alpha_{ij} = 0, \ m = 1, 2, \ldots, n. \tag{8}$$

Combining (7) and (8), the following equalities are obtained:

$$(a_{mm} - \lambda_i)(\beta_{km} - \alpha_{im}) + \sum_{j=1, j\neq m}^{n-1} a_{mj}(\beta_{kj} - \alpha_{ij}) = 0, \ m = 1, 2, \ldots, n-1. \tag{9}$$

$$(a_{nn} - \lambda_k)\beta_{kn} + \sum_{j=1}^{n-1} a_{nj}\beta_{kj} = 0. \tag{10}$$

According to (10), we have

$$(a_{nn} - \lambda_k)(\beta_{kn} - \alpha_{in}) + \sum_{j=1}^{n-1} a_{nj}(\beta_{kj} - \alpha_{ij}) = -\sum_{j=1}^{n-1} a_{nj}\alpha_{ij} - (a_{nn} - \lambda_k)\alpha_{in}. \tag{11}$$

From (9) and (11), we obtain an equation as follows:

$$\sum_{m=1}^{n} \left[(a_{mm} - \lambda_i)(\beta_{km} - \alpha_{im}) + \sum_{j=1, j\neq m}^{n-1} a_{mj}(\beta_{kj} - \alpha_{ij}) \right] = -\sum_{j=1}^{n-1} a_{nj}\alpha_{ij} - (a_{nn} - \lambda_k)\alpha_{in}, \tag{12}$$

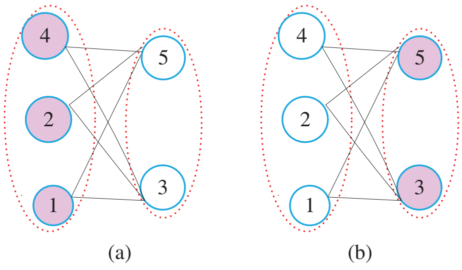

Figure 7 (Color online) Multi-leader symmetry.

where $\alpha_{in} = 0$, $-\sum_{j=1}^{n-1} a_{nj}\alpha_{ij} = -\alpha_i^{\mathrm{T}} L_{fl}$. Thus, the right side of (12) is $-\alpha_i^{\mathrm{T}} L_{fl}$, and the left side of (12) is $-\lambda_i \sum_{m=1}^{n}(\beta_{km} - \alpha_{im}) = \sum_{m=1}^{n} \lambda_i \alpha_{im} = \sum_{m=1}^{n-1} \lambda_i \alpha_{im}$.

Therefore, we have proved Theorem 2 according to (12).

It is worth noting that we get a sufficient condition for controllability by Theorem 2. Sufficient conditions for controllability are very rare, because from a topological point of view it is much easier to get necessary conditions for controllability. Moreover, Theorem 2 contains an equation of the product of eigenvectors and L_{fl}, where L_{fl} is exactly the controllable matrix in system (1). This is consistent with $\alpha^{\mathrm{T}} B$ in the PBH test, which is another important value of Theorem 2. The equation in Theorem 2 shows that we can judge the controllability of the system from the point of view of the eigenvalues of the system matrix and the specific elements of the eigenvectors.

Through the analysis of the five-node graph, we also find that when a nontrivial cell is selected as the leader set, it will produce similar properties with the leader symmetry, as described in Proposition 6 below.

Definition 6 (Multi-leader symmetry)**.** Under a maximum relaxed equivalent partition, if there is a nontrivial cell with all leaders, and the follower subgraph has automorphism nodes, then the system is multi-leader symmetric.

Remark 4. The definition of leader symmetry emphasizes the case that there is only one leader. However, multi-leader symmetry is the application of maximum relaxed equivalent partition to extend single leader to multiple leaders. Thus, it can be seen that if system (1) is multi-leader symmetric, the system is uncontrollable.

Example 5. As shown in Figure 7, $C_1 = \{1, 2, 4\}$, $C_2 = \{3, 5\}$. If C_1 is selected as the leader set, since nodes 3 and 5 are automorphism nodes, nodes 3 and 5 are multi-leader symmetric according to Definition 7 (see Figure 7(a)). Similarly, if C_2 is chosen as the leader set, nodes 1, 2, and 4 are multi-leader symmetric. According to Remark 4, system (1) is uncontrollable.

Below, we state a feature of five-node graphs. In a topology which contains leaders and followers, for a controllable system, if we add the number of edges between any two nodes, and no automorphism nodes are generated, then the new structured system is still controllable.

Example 6. As shown in Figures 8(a)–(c), there are five-node graphs with four, five, and six edges, respectively. If the leader is single in Figure 8(a), system (1) is always controllable when we chose the nodes 2–5 as the leader, respectively. Moreover, when more than two nodes are chosen as leaders, system (1) can always be controlled. Figure 8(b) is formed by adding one edge between nodes 2 and 3 of Figure 8(a). Similarly, Figure 8(c) is formed by adding one edge between nodes 4 and 5 of Figure 8(b). Figure 8(c) has exactly the same controllability as Figure 8(b). For example, if only node 4 is selected as the leader in Figures 8(b) and (c), system (1) can be controlled. The same goes for choosing nodes 2 and 3 as leaders.

So far, for general graphs, we have obtained two sufficient conditions for controllability. As far as the current research is concerned, there are few conclusions in this aspect. In order to solve this kind of problem, we have a new way to start from the case with fewer nodes in the proof process. What is more difficult is the induction and analysis of 726 kinds of graphs. At the same time, we thoroughly revealed the limitation of equivalent partition under the n-node graphs ($2 \leqslant n \leqslant 5$).

The limitation of equivalent partition lies in the existence of topological structures that cannot be judged by equivalent partition. An immediate consequence is that most of conditions obtained by using equivalent partition are necessary rather than sufficient. However, we get a necessary and sufficient condition when n is less than 6 by finding all topological structures that cannot be judged controllability

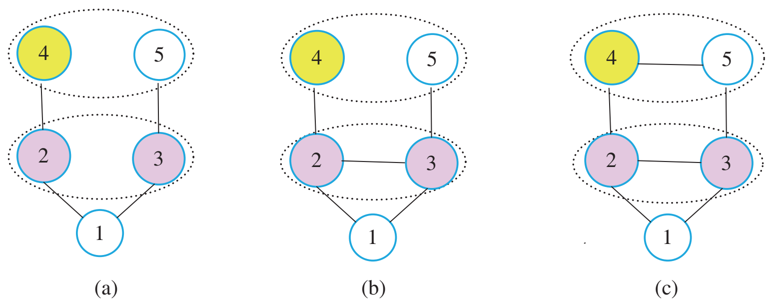

Figure 8 (Color online) Five-node graphs with different number of edges.

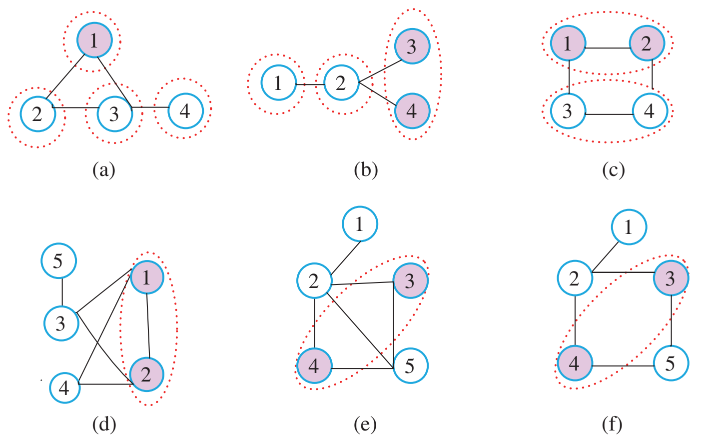

Figure 9 (Color online) Six graphs whose controllability cannot be determined by equivalent partition.

according to equivalent partition. More specifically, we analyzed and summarized 726 topological structures generated by n-node graphs (n less than 6), and then subdivided the cell produced by equivalent partition into a special kind of cell, that is, the cell with iso-neighbor nodes, so as to solve the limitation of the equivalent partition of n-node graphs (n less than 6). Nevertheless, for the case of n greater than or equal to 6, we need to analyze thousands of additional topologies. Thus, so far we can only reach the conclusion when n is less than 6: The n-node graphs ($n < 6$) do not have the limitation of equivalent partition. In other words, when n is less than 6, if some nontrivial cells are selected to play a leader role, and there are no automorphism nodes in the follower subgraph under the maximal relaxed equivalent partition, we only need to see if each cell contains iso-neighbor nodes. If there exist iso-neighbor nodes, then the system is uncontrollable; conversely, it is controllable. Theorem 1 verifies this conclusion.

Example 7. In order to further illustrate the limitation of equivalent partition, we consider six graphs in Figure 9. All of them satisfy the condition of Proposition 3, so we cannot use equivalent partition to judge controllability. In this case, one may judge controllability by calculation. But according to Theorem 1, we can judge controllability without any calculation. For example, Figure 9(a) contains only one leader, and the follower subgraph does not contain automorphism nodes. According to Theorem 1, system (1) is controllable. Similarly, the controllability of Figures 9(b)–(f) can be determined according to Theorem 1.

When we use equivalent partition to judge the controllability of a system, there are some limitations as described in Proposition 3. Fortunately, Theorem 1 solves the limitation of equivalent partition under the n-node graph ($2 \leqslant n \leqslant 5$): On the one hand, Theorem 1 points out and recognizes the limitations of equivalent partition for solving controllability problems; on the other hand, the method to overcome this limitation is given, and the short board of equivalent partition to solve the controllability problem is completed. Crossing this gap is the greatest value of Theorem 1.

4 Conclusion and future work

When evaluating the controllability of system (1) under the n-node graph ($n \leqslant 5$), we can classify all topologies into two: those with iso-neighbor nodes and those without iso-neighbor nodes. Thus, we establish a necessary and sufficient condition for system controllability. Next, we solve the limitation of equivalent partition in the n-node graph ($n \leqslant 5$). In addition, we propose a solution to the limitation of equivalent partition in the n-node graph. In other words, controllability can be examined by analyzing the corresponding path graph. Moreover, we obtain a leader selection method from an equivalent partition perspective to ensure system controllability. Next, we analyze the change in system controllability after all leaders and followers switch roles in a fixed topology. We also adopt the method of analyzing the controllability of graphs with fewer nodes to simplify the controllability analysis of graphs with more nodes. More importantly, we obtain a sufficient condition of multiagent controllability according to Tao's equation. Finally, we extend leader symmetry to multi-leader symmetry.

It is also important to describe what we are not doing. First, for the equivalent partition limitation, only n-node graphs ($n \leqslant 5$) have been solved for now. In the case of $n \geqslant 6$, we will continue in-depth research. We know that the smallest asymmetric graph is a six-node graph; thus, we can establish a relation with asymmetric graphs. Furthermore, to deal with this problem, we can consider pinning some nodes similar to what has been done in [47]. In addition, in a fixed topology with leaders and followers, we can further study the correlation of controllability between the original and the new structures by adding the number of edges between any two nodes as shown in Example 6. What deserves our attention is the discovery of Tao's equation by three physicists working on neutrino oscillations. Therefore, this equation has a strong physical background. We will further study its influence when it is integrated into system controllability.

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