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Stabilization of Boolean control networks with state-triggered impulses

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Abstract Previously, impulses were used to model abrupt changes in dynamic biological systems. This paper introduces a hybrid-index model that can characterize instantaneity of the impulsive behavior more effectively, compared with the existing impulsive Boolean network models. Using the hybrid-index model, we investigate the set stabilization of Boolean control networks with state-triggered impulses in the hybrid-domain and the time-domain. We establish necessary and sufficient conditions for set stabilizability in the hybrid and time domains, using the methods of k-domain and quotient mapping, respectively. Further, we obtain algorithms for constructing all hybrid-optimal and time-optimal set stabilizers by partitioning the state space into layers. The relationships between different set stabilizabilities are summarized. In addition, we have shared two examples to demonstrate the main results.

Keywords Boolean control networks, set stabilization, state-triggered impulses, hybrid-index model, semi-tensor product of matrices

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1 Introduction

Many modeling approaches, such as differential equations, Bayesian networks, Petri nets, and Boolean networks [1,2] are available for gene regulatory networks. Among these, Boolean networks proposed in [3] have attracted attention because of their simplicity and low computational costs. A Boolean network is composed of a group of nodes that take only binary values (1 or 0). Each node state is regulated by a logical function associated with the past state of its in-neighborhoods. Besides the applications in biological systems, Boolean networks have also been applied to physics, ecosystems, and manufacturing [4–6]. Moreover, many essential problems related to fixed points, cycles, basins of attractors, and transient periods have been investigated in [7–11].

In some practical biological systems, such as cancer treatment systems, external inputs should be applied to keep these systems stable or normal [12]. From this theoretical viewpoint, Akutsu et al. [13] developed a method to find control strategies for Boolean networks. However, the research on Boolean networks with control inputs has not made great progress until the emergence of the semi-tensor product (STP), which is a new product of matrices. By using the STP, a Boolean network can be equivalently converted into an algebraic form [14]. This approach has helped solve many classical control problems for Boolean control networks, such as controllability and observability [15–17], stabilization [18–21], synchronization [22–24], system decomposition and decoupling [25–29], and optimal control [30–32]. Besides the application in Boolean networks, the STP tool has also achieved success in other finite-value systems, such as finite potential games [33], discrete event dynamic systems [34], and nonlinear feedback shift registers [35, 36].

Owing to changes of interconnections between subsystems and sudden environment changes in gene regulatory networks, the abrupt changes of states may occur, triggered by specified states and/or events

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and at certain time instants. In the cell division of gene regulatory networks, the deoxyribonucleic acid (DNA) number is constant during cell growth, except the S phase in which the DNA number is doubled but immediately halved after division of each daughter cell [37]. As described in [37], this abrupt change in the DNA number can be modeled as an impulse. To mathematically characterize the abrupt changes of gene regulatory networks, the model of Boolean control networks with impulsive perturbation was introduced in [38,39]. This model is of the form:

$$\begin{cases} X(t+1) = f(X(t), U(t)), \; t \neq \tau_k, \\ X(t+1) = g(X(t)), \qquad\quad t = \tau_k, \end{cases} \tag{1}$$

where $X(t) \in \mathscr{D}^n$, $U(t) \in \mathscr{D}^m$ are the state and input variables, respectively, and $\{\tau_k\}_{k\in\mathbb{N}}$ is the set of impulsive time instants, which is called the time-triggered impulsive model. In model (1), the system dynamics experiences change at prescribed time instants $\{\tau_k\}_{k\in\mathbb{N}}$, modeled as impulses. Under the time-triggered impulsive model framework, some basic control problems have been investigated, such as controllability [39], stabilization [38,40], optimal control [41], and event-triggered control [42–44].

The time scale of an impulsive process is identical with that of the nonimpulsive process. Therefore, model (1) can be regarded as a special kind of switched systems. In other words, it does not characterize the instantaneity of these sudden and sharp changes (i.e., impulsive behaviors) well. To deal with this issue, we will employ a hybrid-index model, which was proposed in [45]. In this model, the states are defined by $X(t, j)$, where the indices t and j represent the time step and number of impulses, respectively. The evolution of the impulsive Boolean control network (IBCN) can be divided into two parts—the jump process and the step process. If the impulsive condition is triggered, the system experiences the impulse in the form of jumping, where time step t remains unchanged and the number of impulses j turns into $j + 1$. In addition, the system steps forward if the impulse does not occur. According to the properties, the hybrid-index model can not only be clearly distinguished from the time-triggered model, but it can also intuitively characterize the instantaneity of the impulsive behavior. In the control theory study, stability and stabilization are two unavoidable problems for a new model. Thus, this paper investigates set stabilization of IBCNs based on the hybrid-index model, which is a natural generalization of single-node stabilization. Many classical problems can be classified as special cases of set stabilization (for example, synchronization and output regulation). Therefore, the research on set stabilization of general Boolean networks (including Boolean control networks, switched Boolean control networks, and probabilistic Boolean control networks) has become increasingly popular [21,46–48]. From this point of view, the research on set stabilization of IBCNs based on the hybrid-index model has great theoretical significance and potential applications.

To avoid Zeno behaviors, we introduce the concept of forward completeness for the IBCN, which indicates that any solution can step forward to infinity along the time axis instead of jumping at some time instant all the time. According to the characteristics of the hybrid-index model, the solution to the IBCN is a spatial trajectory related to the hybrid-index pair (t, j), and this solution is called the hybrid-domain solution. The instantaneity of the impulsive jumping process makes it difficult to accurately measure complete information of the jumping process. Moreover, from a macro-timescale view of the system, the details of instantaneous jumping can be ignored. Accordingly, the time-domain solution is naturally introduced, which is the sequence of states after all jumps at each time instant. This solution is also a subsequence of the hybrid-domain solution.

Corresponding to these two types of solutions, we have defined two kinds of stabilizabilities: hybrid-domain stabilizability and time-domain stabilizability. Considering the hybrid-domain stabilizability, a k-domain logical control system (LCS) is constructed according to the original IBCN, which is essentially a special switched LCS without focusing on the action (stepping or jumping) at each time instant. Based on this construction, an IBCN, where the impulses are triggered by some special states (i.e., state-triggered impulses), is hybrid-domain stabilizable with respect to a set of target states if and only if (iff) it is forward complete and the k-domain LCS is stabilizable with respect to the same set. Accordingly, a necessary and sufficient condition for set stabilizability in the hybrid-domain follows from this method. Subsequently, we propose an algorithm for constructing controllers to ensure that any state can converge to the target set within its shortest transient period. The controllers are called the hybrid-optimal set stabilizers. Note that all the hybrid-optimal set stabilizers can be found by this algorithm.

To investigate time-domain stabilizability, a quotient LCS is obtained by constructing a quotient mapping that projects an IBCN onto the quotient space. This quotient space can be treated as a set of

Table 1 Notations

Notation	Definition	Notation	Definition
$\mathbb{N}$	Set of natural numbers	$\mathscr{D}$	Logic domain $\{0,1\}$
$[n:m]$	Set of integers k satisfying $n \leqslant k \leqslant m$	I_n	$n\times n$ identity matrix
δ_n^i	The ith column of I_n	Δ_n	Set of columns of I_n
$\mathscr{L}_{n\times m}$	Set of $n\times m$ logical matrices	$\mathrm{Col}_i(A)$	The ith column of matrix A
$\mathscr{B}_{n\times m}$	Set of $n\times m$ Boolean matrices	$\mathrm{Col}(A)$	Set of columns of matrix A
$[A]_{i,j}$	(i,j)-element of matrix A	A^{T}	Transpose of matrix A
$W_{[n,m]}$	Swap matrix with indices n and m	$\lvert\mathcal{M}\rvert$	Cardinal number of set $\mathcal{M}$
$\Lambda_{\mathcal{M}}$	Index set of state set $\mathcal{M}$	$\mathcal{M}^c$	Complement of set $\mathcal{M}$
M	Indicator matrix of set $\mathcal{M}$	$\mathcal{M}\setminus\mathcal{N}$	Set $\{x\in\mathcal{M}\lvert x\notin\mathcal{N}\}$
$\mathbf{1}_{m\times n}(\mathbf{0}_{m\times n})$	A matrix with all elements being 1 (0)	$\otimes$	Kronecker product
$\times_{\mathscr{B}}$	Boolean product	$\ltimes$	STP
$A^{(k)}$	The kth Boolean power of matrix A	$\ltimes_{\mathscr{B}}$	Boolean STP

untriggerable states. The resulting LCS retains the dynamics of the original IBCN in the time-domain, but the details of jumping are ignored. Therefore, an IBCN is time-domain stabilizable with respect to a target subset iff it is forward complete, and its quotient LCS is stabilizable with respect to the identical subset. Next, a necessary and sufficient condition of stabilizability in the time domain is proposed based on this idea. Next, an algorithm for constructing the time-optimal set stabilizers is proposed, which guarantees that each state converges to the target subset within its shortest transient period. Note that we can determine all the time-optimal set stabilizers according to this algorithm.

The rest of this paper is organized as follows. Section 2 presents some necessary preliminaries, the hybrid-index model, forward completeness, and definitions of set stabilizability and feedback set stabilizability in both hybrid and time domains. Sections 3 and 4 investigate the hybrid-domain and time-domain stabilizability, respectively, and propose algorithms for constructing all the hybrid-optimal and time-optimal stabilizers. The relationships between different stabilizabilities are discussed in Section 5. Section 6 provides two numerical examples to illustrate the effectiveness of the results presented in this paper. Section 7 provides a brief conclusion. The notations used in this paper are listed in Table 1.

2 Preliminaries and problem formulation

2.1 Preliminaries

Definition 1 ([14])**.** The STP of any two matrices A and B can be defined as $A\ltimes B := (A\otimes I_{l/n})(B\otimes I_{l/p})$, where A and B are $m\times n$ and $p\times q$ matrices, respectively; l is the least common multiple of n and p, and $\otimes$ represents the Kronecker product of matrices.

Note that the conventional matrix product AB is the special case with $n = p$ of the STP $A \ltimes B$. Hereinafter, we omit symbol '$\ltimes$' without raising any confusion.

Definition 2 ([14])**.** The Boolean product of any two matrices $A \in \mathscr{B}_{n\times m}$ and $A \in \mathscr{B}_{m\times p}$ is defined as $A \times_{\mathscr{B}} B := C \in \mathscr{B}_{n\times p}$, where $[C]_{i,j} = [A]_{i,1}[B]_{1,j} \vee [A]_{i,2}[B]_{2,j} \vee\cdots\vee [A]_{i,m}[B]_{m,j},\ i\in[1:n], j\in[1:p]$. Then, we define the Boolean STP of any two matrices $P\in\mathscr{B}_{m\times n}$ and $Q\in\mathscr{B}_{p\times q}$ as $P\ltimes_{\mathscr{B}} Q := (P\otimes I_{l/n})\times_{\mathscr{B}}(Q\otimes I_{l/p})$, where l represents the least common multiple of n and p. Specially, if $P\in\mathscr{B}_{m\times m}$, we define the Boolean power of P as

$$P^{(k)} := \underbrace{P\ltimes_{\mathscr{B}} P\ltimes_{\mathscr{B}}\cdots\ltimes_{\mathscr{B}} P}_{k}.$$

We set an equivalence relation between $\mathscr{D}$ and Δ_2 as follows: $X\sim x := \delta_2^{2-X}\in\Delta_2, \forall X\in\mathscr{D}$, where x is called the vector form of X. Then, the following claim holds by some basic properties of the STP.

Lemma 1 ([14])**.** Any logical function $l(X_1, X_2, \ldots, X_k)$ can be expressed in a multi-linear form $l(x_1, x_2, \ldots, x_k) = L\ltimes x_1\ltimes x_2\ltimes\cdots\ltimes x_k$, where $x_i\in\Delta$ is the vector form of the logical variable $X_i\in\mathscr{D}, i\in[1:k]$, and $L\in\mathscr{L}_{2\times 2^k}$ is the structure matrix of l, which is also uniquely determined by l.

2.2 Hybrid-index model

A Boolean control network with state-triggered impulses is described by

$$\begin{cases} X(t+1,j)=f(X(t,j),U(t)), & X(t,j)\in\mathbb{S},\\ X(t,j+1)=g(X(t,j)), & X(t,j)\in\mathbb{J}, \end{cases} \tag{2}$$

where $X(t,j)=[X_1(t,j)\ X_2(t,j)\ \cdots\ X_n(t,j)]^{\mathrm{T}}\in\mathscr{D}^n$ and $U(t)=[U_1(t)\ U_2(t)\ \cdots\ U_m(t)]^{\mathrm{T}}\in\mathscr{D}^m$ with $X_i(t,j)\in\mathscr{D}$, $U_k(t)\in\mathscr{D}, i\in[1:n], k\in[1:m]$, and quantities n and m represent the number of state and input nodes, respectively; $f:\mathscr{D}^{m+n}\to\mathscr{D}^n$ and $g:\mathscr{D}^n\to\mathscr{D}^n$ are logical functions; $\mathbb{J}\subset\mathscr{D}^n$ represents the set of states triggering impulses, and $\mathbb{S}:=\mathscr{D}^n\setminus\mathbb{J}$; $X(t,j)\in\mathbb{J}$ represents the triggering condition of the impulse. State $X(t,j)$ is with two-dimensional index, which is called the hybrid index in [45], where indices t and j represent the time instant and the number of impulses, respectively. For the convenience of description, network (2) is collectively referred to as the IBCN in this paper. The dynamics of IBCN (2) consists of two parts: the stepping process and the jumping process. When the triggering condition $X(t,j)\in\mathbb{J}$ is fulfilled, the system jumps with the regulation of the second equation of (2); otherwise, it steps forward to the next time instant $t+1$ according to the first equation of (2).

Let $x(t,j)\in\Delta_{2^n}$ and $u(t)\in\Delta_{2^m}$ be the vector forms of $X(t,j)$ and $U(t)$, respectively. Then, by Lemma 1, IBCN (2) can be rewritten in the algebraic form:

$$\begin{cases} x(t+1,j)=Fu(t)x(t,j), & x(t,j)\in\mathcal{S},\\ x(t,j+1)=Gx(t,j), & x(t,j)\in\mathcal{J}, \end{cases} \tag{3}$$

where $F\in\mathscr{L}_{2^n\times 2^{n+m}}$ and $G\in\mathscr{L}_{2^n\times 2^n}$ are the transition matrices, $\mathcal{J}$ represents the vector form of $\mathbb{J}$, and $\mathcal{S}:=\Delta_{2^n}\setminus\mathcal{J}$ is the vector form of $\mathbb{S}$. Hereinafter, our methods and results are presented directly in this algebraic form.

The hybrid-domain solution to IBCN (3) with an initial state x_0 and an input sequence $\boldsymbol{u}:=\{u(t)\}_{t\in\mathbb{N}}$ is defined by $x(t_k,j_k;\boldsymbol{u},x_0), k\in\mathbb{N}$, and the following conditions are fulfilled:

(a) $x(t_{k+1},j_{k+1})=Fu(t_k)x(t_k,j_k)$, $t_{k+1}=t_k+1, j_{k+1}=j_k$, if $x(t_k,j_k)\in\mathcal{S}$;

(b) $x(t_{k+1},j_{k+1})=Gx(t_k,j_k)$, $t_{k+1}=t_k, j_{k+1}=j_k+1$, if $x(t_k,j_k)\in\mathcal{J}$.

Let $t_0=j_0=0$, and then $t_k+j_k=k,\forall k\in\mathbb{N}$. We define the set of hybrid index pairs $(t_k,j_k), k\in\mathbb{N}$ of the hybrid-domain solution $x(t_k,j_k;\boldsymbol{u},x_0), k\in\mathbb{N}$, as $\mathcal{H}(\boldsymbol{u},x_0)\subset\mathbb{N}\times\mathbb{N}$, which is termed as the hybrid-domain of $x(t_k,j_k;\boldsymbol{u},x_0), k\in\mathbb{N}$. Hence, $x(t_k,j_k;\boldsymbol{u},x_0)$ can be rewritten briefly as $x(t,j;\boldsymbol{u},x_0), (t,j)\in\mathcal{H}(\boldsymbol{u},x_0)$. Next, we define $\tau(\boldsymbol{u},x_0):=\sup\{t|\exists j \text{ s.t. } (t,j)\in\mathcal{H}(\boldsymbol{u},x_0)\}$ and set $\mathcal{T}(\boldsymbol{u},x_0):=\{0,1,2,\ldots,\tau(\boldsymbol{u},x_0)-1\}$, which is called the time-domain of the hybrid-domain solution $x(t,j;\boldsymbol{u},x_0)$. Obviously, $\mathcal{T}(\boldsymbol{u},x_0)=\mathbb{N}$ if $\tau(\boldsymbol{u},x_0)=+\infty$. For any time instant $t\in\mathcal{T}(\boldsymbol{u},x_0)$, we define

$$\boldsymbol{j}_t^+(\boldsymbol{u},x_0):=\max\{j|(t,j)\in\mathcal{H}(\boldsymbol{u},x_0)\},\quad \boldsymbol{j}_t^-(\boldsymbol{u},x_0):=\min\{j|(t,j)\in\mathcal{H}(\boldsymbol{u},x_0)\}.$$

Then, the number of successive impulses at time instant t is $\boldsymbol{j}_t^+(\boldsymbol{u},x_0)-\boldsymbol{j}_t^-(\boldsymbol{u},x_0)$. The pre-jump and after-jump states at time instant t are $x(t,\boldsymbol{j}_t^-;\boldsymbol{u},x_0)$ and $x(t,\boldsymbol{j}_t^+;\boldsymbol{u},x_0)$, respectively. We define $x(t;\boldsymbol{u},x_0):=x(t,\boldsymbol{j}_t^+;\boldsymbol{u},x_0)$ and term it the time-domain solution to IBCN (3) with an input sequence $\boldsymbol{u}$ and an initial state x_0.

2.3 Forward completeness

Definition 3. A hybrid-domain solution $x(t,j;\boldsymbol{u},x_0)$ is said to be forward complete if $\tau(\boldsymbol{u},x_0)=+\infty$, that is, $\mathcal{T}(\boldsymbol{u},x_0)=\mathbb{N}$. Under this condition, the corresponding time-domain solution $x(t;\boldsymbol{u},x_0)$ is also considered to be forward complete. Subsequently, IBCN (3) is said to be forward complete if $\mathcal{T}(\boldsymbol{u},x_0)=\mathbb{N},\forall x_0\in\Delta_{2^n}$.

Definition 4. A subset $\mathcal{C}\subseteq\Delta_{2^n}$ is said to be an invariant subset of a logical matrix $L\in\mathscr{L}_{2^n\times 2^n}$ if $Lx_\alpha\in\mathcal{C},\forall x_\alpha\in\mathcal{C}$. A subset $\hat{\mathcal{C}}\subseteq\Delta_{2^n}$ is said to be a control-invariant subset of a logical matrix $\hat{L}\in\mathscr{L}_{2^n\times 2^{n+m}}$ if for any state $x_\beta\in\hat{\mathcal{C}}$, there exists a control $u_\beta\in\Delta_{2^m}$ such that $\hat{L}u_\beta x_\beta\in\hat{\mathcal{C}}$. According to the definitions, the union of any two invariant (or control-invariant) subsets of L (or $\hat{L}$) is still invariant (or control-invariant). Moreover, the union of all invariant (or control-invariant) subsets contained in a given set $\mathcal{M}\subseteq\Delta_{2^n}$ is called the largest invariant (or control-invariant) subset of L (or $\hat{L}$) contained in $\mathcal{M}$, which is denoted by $I_L(\mathcal{M})$ (or $I_{\hat{L},C}(\mathcal{M})$).

Definition 5. For a given subset $\mathcal{M} \subseteq \Delta_{2^n}$ with the index set $\Lambda_{\mathcal{M}} := \{i | \delta_{2^n}^i \in \mathcal{M}\}$, the indicator matrix of $\mathcal{M}$ is a Boolean matrix $M \in \mathscr{B}_{2^n \times 2^n}$ such that $\mathrm{Col}_s(M) = \delta_{2^n}^s$ if $\delta_{2^n}^s \in \mathcal{M}$ and $\mathrm{Col}_s(M) = \mathbf{0}_{2^n}$ if $\delta_{2^n}^s \notin \mathcal{M}$.

Lemma 2 ([21])**.** Consider a logical matrix $L \in \mathscr{L}_{2^n \times 2^n}$ and a subset $\mathcal{M} \subseteq \Delta_{2^n}$. Let $M \in \mathscr{B}_{2^n \times 2^n}$ denote the indicator matrix of $\mathcal{M}$. Then, it holds that $I_L(\mathcal{M}) = \{\delta_{2^n}^s | \mathrm{Col}_s[(ML)^{|\mathcal{M}|} M] \neq \mathbf{0}_{2^n}\}$.

Lemma 3 ([21])**.** Given a logical matrix $\hat{L} \in \mathscr{L}_{2^n \times 2^{n+m}}$ and a subset $\mathcal{M} \subseteq \Delta_{2^n}$, we have $I_{\hat{L},C}(\mathcal{M}) = \{\delta_{2^n}^s | \mathrm{Col}_s[(M \ltimes_{\mathscr{B}} C_1)^{(|\mathcal{M}|)} \ltimes_{\mathscr{B}} M] \neq \mathbf{0}_{2^n}\}$, where M is the indicator matrix of $\mathcal{M}$, and $C_1 := \hat{L} \ltimes_{\mathscr{B}} \mathbf{1}_{2^m}$.

Proposition 1 ([45])**.** IBCN (3) is forward complete iff $I_G(\mathcal{J}) = \emptyset$, equivalently, $(JG)^{|\mathcal{J}|} J = \mathbf{0}_{2^n \times 2^n}$, where J is the indicator matrix of $\mathcal{J}$.

2.4 Definitions of stabilizability

Definition 6. Consider a target subset $\mathcal{M} \subset \Delta_{2^n}$. IBCN (3) is said to be hybrid-domain $\mathcal{M}$-stabilizable if it is forward complete and if for any initial state $x_0 \in \Delta_{2^n}$, there exists a control sequence $\boldsymbol{u}$ and an integer $\eta(\boldsymbol{u}, x_0) > 0$ such that

$$x(t, j; \boldsymbol{u}, x_0) \in \mathcal{M}, \quad \forall t + j \geqslant \eta(\boldsymbol{u}, x_0). \tag{4}$$

Particularly, IBCN (3) is said to be hybrid-domain $\mathcal{M}$-stabilizable by feedback if there exists a feedback controller

$$u(t) = Kx(t) = Kx(t, \boldsymbol{j}_t^+), \quad \forall x(t, \boldsymbol{j}_t^+) \in \mathcal{S}, \tag{5}$$

with $K \in \mathscr{L}_{2^m \times 2^n}$ such that for any initial state $x_0 \in \Delta_{2^n}$, $x(t, j; \boldsymbol{u}_K, x_0) \in \mathcal{M}, \forall t + j \geqslant \eta(\boldsymbol{u}_K, x_0)$, where the control sequence $\boldsymbol{u}_K$ is generated by (5), and $\eta(\boldsymbol{u}_K, x_0)$ is a nonnegative integer. Let $\eta_{\mathcal{M}}(\boldsymbol{u}, x_0)$ denote the shortest time such that Eq. (4) holds, which is named the shortest hybrid-domain transient period of x_0 under the control sequence $\boldsymbol{u}$. Moreover, $\mathcal{U}(x_0)$ is the set of stabilizing control sequences for state x_0 in the hybrid-domain. Next, we use $\eta_{\mathcal{M}}(x_0) := \min_{\boldsymbol{u} \in \mathcal{U}(x_0)} \eta_{\mathcal{M}}(\boldsymbol{u}, x_0)$ and $\eta_{\mathcal{M}} := \max_{x_0 \in \Delta_{2^n}} \eta_{\mathcal{M}}(x_0)$ to denote the shortest hybrid-domain transient period of x_0 and the shortest hybrid-domain transient period for IBCN (3), respectively.

Definition 7. Consider a target subset $\mathcal{M} \subset \Delta_{2^n}$. IBCN (3) is said to be time-domain $\mathcal{M}$-stabilizable if it is forward complete and if for any initial state $x_0 \in \Delta_{2^n}$, there is an input sequence $\boldsymbol{u}$ and an integer $\lambda(\boldsymbol{u}, x_0) > 0$ such that

$$x(t; \boldsymbol{u}, x_0) \in \mathcal{M}, \quad \forall t \geqslant \lambda(\boldsymbol{u}, x_0). \tag{6}$$

In particular, IBCN (3) is said to be time-domain $\mathcal{M}$-stabilizable by feedback if there is a state feedback controller

$$u(t) = \hat{K}x(t), \quad \forall x(t) \in \mathcal{S}, \tag{7}$$

with $\hat{K} \in \mathscr{L}_{2^m \times 2^n}$ such that for any initial state $x_0 \in \Delta_{2^n}$, $x(t; \boldsymbol{u}_{\hat{K}}, x_0) \in \mathcal{M}, \forall t \geqslant \lambda(\boldsymbol{u}_{\hat{K}}, x_0)$, where the control sequence $\boldsymbol{u}_{\hat{K}}$ is generated by (7) and $\lambda(\boldsymbol{u}_{\hat{K}}, x_0)$ is a nonnegative integer. Let $\lambda_{\mathcal{M}}(\boldsymbol{u}, x_0)$ denote the shortest time such that Eq. (6) holds, which is called the time-domain transient period of x_0 under the input sequence $\boldsymbol{u}$. Besides, $\hat{\mathcal{U}}(x_0)$ is the set of stabilizing control sequences for state x_0. Subsequently, we use $\lambda_{\mathcal{M}}(x_0) := \min_{\boldsymbol{u} \in \hat{\mathcal{U}}(x_0)} \lambda_{\mathcal{M}}(\boldsymbol{u}, x_0)$ and $\lambda_{\mathcal{M}} := \max_{x_0 \in \Delta_{2^n}} \lambda_{\mathcal{M}}(x_0)$ to denote the shortest time-domain transient period of x_0 and the shortest time-domain transient period for stabilizing IBCN (3) from all states, respectively.

Remark 1. When the target subset $\mathcal{M}$ is a singleton, Definitions 6 and 7 degenerate into the definitions of single-state stabilizability in the hybrid-domain and the time-domain, respectively. Thus, for brevity, we only focus on set stabilizability in this paper.

3 Stabilization in the hybrid domain

3.1 Stabilizability in the hybrid domain

If we do not distinguish the stepping and jumping processes, we can represent the hybrid-domain solution to IBCN (3) as a sequence of k with $k = t_k + j_k \in \mathbb{N}$, that is, $z(k; \boldsymbol{u}, x_0) := x(t_k, j_k; \boldsymbol{u}, x_0)$. Thus, by using (3) of IBCN, we can obtain

$$z(k+1)=\begin{cases}Fu_H(k)z(k), & z(k)\in\mathcal{S},\\ Gz(k), & z(k)\in\mathcal{J},\end{cases} \tag{8}$$

where $z(0):=x_0$ and $u_H(k):=u(t)$ when $k=t+\boldsymbol{j}_t^+$. System (8), called the k-domain LCS, can be regarded as a switched LCS obtained by not distinguishing different actions (stepping or jumping).

Notice that the jumping process is not affected by the control inputs, and $u_H(k)$ in the k-domain LCS (8) is not defined for $k\neq t+\boldsymbol{j}_t^+$. For convenience, we extend $u_H(k)$ to the input sequences defined on $\mathbb{N}$. Then, $u(t)$ is a subsequence of $u_H(k)$ with

$$u(t)=u_H(t+\boldsymbol{j}_t^+), \tag{9}$$

and the k-domain LCS can be converted into

$$z(t+1)=\begin{cases}Fu_H(k)z(k), & z(k)\in\mathcal{S},\\ \bar{G}u_H(k)z(k), & z(k)\in\mathcal{J},\end{cases}$$

where $\bar{G}=\mathbf{1}_{2^m}^{\mathrm{T}}\otimes G$. Let $F=[F_1\ F_2\ \cdots\ F_{2^m}]$, where $F_i\in\mathscr{L}_{2^n\times 2^n}, i\in[1:2^m]$, is the i-th block of F. Let $F':=[F_1\ltimes S\ F_2\ltimes S\ \cdots\ F_{2^m}\ltimes S]\in\mathscr{B}_{2^n\times 2^{n+m}}$ and

$$G':=[\underbrace{G\ltimes J\ G\ltimes J\ \cdots\ G\ltimes J}_{2^m}]\in\mathscr{B}_{2^n\times 2^{n+m}},$$

where S and J are the indicator matrices of $\mathcal{S}$ and $\mathcal{J}$, respectively. Then, the k-domain LCS (8) can be equivalently expressed as

$$z(k+1)=Eu_H(k)z(k), \tag{10}$$

where $E=F'+G'\in\mathscr{L}_{2^n\times 2^{n+m}}$.

Based on the construction of the k-domain LCS, we can easily obtain the following lemma.

Lemma 4. Given a subset $\mathcal{M}\subseteq\Delta_{2^n}$, IBCN (3) is $\mathcal{M}$-stabilizable in the hybrid domain iff it is forward complete and its k-domain LCS is $\mathcal{M}$-stabilizable.

Remark 2. Based on the definition of the indicator matrices, the condition $\mathcal{S}=\mathcal{J}^c$ and the construction of matrix E, we have the following results from (10):

- If $z(t)\in\mathcal{S}$, it holds that $G'u_H(k)z(k)=\mathbf{0}_{2^n}$, $\forall u_H(k)\in\Delta_{2^m}$ and $z(k+1)=F'u_H(k)z(k)=Fu(t)z(k)$;
- If $z(t)\in\mathcal{J}$, it holds that $F'u_H(k)z(k)=\mathbf{0}_{2^n}$ and $z(k+1)=G'u_H(k)z(k)=Gz(t),\forall u_H(k)\in\Delta_{2^m}$.

Thus, system (10) is equivalent to the k-domain LCS (8). We assume that $z(k;\boldsymbol{u}_H,x_0)$ is the solution to the k-domain LCS (10) with the initial state x_0 and control sequence $\boldsymbol{u}_H$. Then, the hybrid-domain solution to IBCN (3) under the controller $u(t)=u_H(t+\boldsymbol{j}_t^+)$ is given by $x(t,j;\boldsymbol{u},x_0)=z(t+j;\boldsymbol{u}_H,x_0)$.

Lemma 5. Consider a target subset $\mathcal{M}\subseteq\Delta_{2^n}$. The k-domain LCS (10) is $\mathcal{M}$-stabilizable iff it is $I_{E,C}(\mathcal{M})$-stabilizable, equivalently,

$$\sum_{i\in\Lambda_\star}[(E\ltimes_{\mathscr{B}}\mathbf{1}_{2^m})^{(2^n)}]_{i,j}>0,\quad\forall j\in[1:2^n], \tag{11}$$

where $\Lambda_\star$ is the index set of $I_{E,C}(\mathcal{M})$.

Proof. The first part of this claim follows Lemma 4 in [21]. Next, we only need to prove the second part of the claim, that is, the k-domain LCS (10) is $I_{E,C}(\mathcal{M})$-stabilizable iff Eq. (11) holds. Eq. (11) states that for any state $x_0=\delta_{2^n}^j\in\Delta_{2^n}$, the k-domain LCS (10) has a trajectory of length 2^n from x_0 to $I_{E,C}(\mathcal{M})$ under certain control sequence. Thus, if the k-domain LCS (10) is $I_{E,C}(\mathcal{M})$-stabilizable, then Eq. (11) holds. On the contrary, if Eq. (11) holds, then, by the property of control-invariant subsets, we have $\sum_{i\in\Lambda_\star}[(E\ltimes_{\mathscr{B}}\mathbf{1}_{2^m})^{(\beta)}]_{i,j}>0,\forall j\in[1:2^n],\forall\beta\geqslant 2^n+1$, which means that for any state $x_0=\delta_{2^n}^j$, there is an input sequence $\boldsymbol{u}_H$ such that $z(t;\boldsymbol{u}_H,x_0)\in I_{E,C}(\mathcal{M}),\forall t\geqslant 2^n$. That is, system (10) is $I_{E,C}(\mathcal{M})$-stabilizable.

By Lemmas 4 and 5 and Proposition 1, the following theorem holds directly.

Theorem 1. Given a target subset $\mathcal{M} \subseteq \Delta_{2^n}$, IBCN (3) is hybrid-domain $\mathcal{M}$-stabilizable iff
(a) $(JG)^{|\mathcal{J}|}J = \mathbf{0}_{2^n \times 2^n}$, and
(b) $\sum_{i\in\Lambda_\star}[(E \ltimes_{\mathscr{B}} \mathbf{1}_{2^m})^{(2^n)}]_{i,j} > 0, \forall j \in [1:2^n]$, where $\Lambda_\star$ is the index set of $I_{E,C}(\mathcal{M})$.

Proposition 2. Suppose that IBCN (3) is $\mathcal{M}$-stabilizable in the hybrid domain. Then, it holds that for any $s \in [1:2^n]$,

$$\eta_{\mathcal{M}}(\delta_{2^n}^s) = \eta_{I_{E,C}(\mathcal{M})}(\delta_{2^n}^s) = \min_{k\in\mathbb{N}}\{k|\operatorname{Col}_s(M_\star \boldsymbol{C}_k) \neq \mathbf{0}_{2^n}\}, \tag{12}$$

where $M_\star$ is the indicator matrix of $I_{E,C}(\mathcal{M})$ and $\boldsymbol{C}_k := (E \ltimes_{\mathscr{B}} \mathbf{1}_{2^m})^{(k)}$ is the k-step controllability matrix of the k-domain LCS (10) with $\boldsymbol{C}_0 := I_{2^n}$.

Proof. The claim $\eta_{\mathcal{M}}(\delta_{2^n}^s) = \eta_{I_{E,C}(\mathcal{M})}(\delta_{2^n}^s)$ follows Lemma 4 in [21]. We only need to prove

$$\eta_{I_{E,C}(\mathcal{M})}(\delta_{2^n}^s) = \min_{k\in\mathbb{N}}\{k|\operatorname{Col}_s(M_\star \boldsymbol{C}_k) \neq \mathbf{0}_{2^n}\}.$$

First, for any $\delta_{2^n}^s \in I_{E,C}(\mathcal{M})$, it obviously holds that $\eta_{I_{E,C}(\mathcal{M})}(\delta_{2^n}^s) = 0$. On the other hand, by the fact that $M_\star$ is the indicator matrix of $I_{E,C}(\mathcal{M})$, for any $\delta_{2^n}^s \in I_{E,C}(\mathcal{M})$, $\operatorname{Col}_s(M_\star \boldsymbol{C}_0) = \operatorname{Col}_s(M_\star I_{2^n}) = \operatorname{Col}_s(M_\star) \neq \mathbf{0}_{2^n}$. Thus, the claim holds for all $\delta_{2^n}^s \in I_{E,C}(\mathcal{M})$. In the following, we assume that $\delta_{2^n}^s \notin I_{E,C}(\mathcal{M})$. Notice that $[\boldsymbol{C}_k]_{j,s} \neq 0$ iff $\delta_{2^n}^j$ is k-step reachable from $\delta_{2^n}^s$. Thus, $\operatorname{Col}_s(M_\star \boldsymbol{C}_k) \neq \mathbf{0}_{2^n}$ iff $\delta_{2^n}^s$ can reach $I_{E,C}(\mathcal{M})$ in k steps. This proves $\eta_{I_{E,C}(\mathcal{M})}(\delta_{2^n}^s) = \min_{k\in\mathbb{N}}\{k|\operatorname{Col}_s(M_\star \boldsymbol{C}_k) \neq \mathbf{0}_{2^n}\}$.

3.2 Design of state-feedback stabilizers in the hybrid domain

In this subsection, we describe an algorithm to devise hybrid-domain $\mathcal{M}$-stabilizing feedback controllers of IBCN (3).

Proposition 3. IBCN (3) is hybrid-domain $\mathcal{M}$-stabilizable iff it is hybrid-domain $\mathcal{M}$-stabilizable by feedback.

Proof. By Lemma 4, we only need to prove that the k-domain LCS (10) is $\mathcal{M}$-stabilizable iff it is $\mathcal{M}$-stabilizable by feedback, which directly follows Remark 2 in [21].

A k-domain-optimal $\mathcal{M}$-stabilizing feedback

$$u_H(k) = \tilde{K}z(k), \quad \tilde{K} \in \mathscr{L}_{2^m \times 2^n}, \tag{13}$$

for the k-domain system can be obtained by Algorithm 1, which is based on the method of state space partition proposed in [21]. Then, a hybrid-optimal $\mathcal{M}$-stabilizing feedback $u(t) = Kx(t)$ for IBCN (3), which guarantees that any closed-loop hybrid-domain solution converges to $\mathcal{M}$ in the least steps, can be obtained by the following proposition.

Algorithm 1 An algorithm that determines the $\mathcal{M}$-stabilizability of k-domain LCS (10) and the feedback gain matrix $\tilde{K}$

Input: The k-domain LCS (10) with $E = [E_1\ E_2\ \cdots\ E_{2^m}]$ and $E_k \in \mathscr{L}_{2^n \times 2^n}$, $k \in [1:2^m]$, the one-step controllability matrix $\boldsymbol{C}_1 = E \ltimes_{\mathscr{B}} \mathbf{1}_{2^m}$, a subset $\mathcal{M}$ with $I_{E,C}(\mathcal{M}) \neq \emptyset$;
Output: "Yes", if the k-domain LCS (10) is $\mathcal{M}$-stabilizable, and "No", otherwise; $\tilde{K}$;
1: $\mathcal{M}_0 = I_{E,C}(\mathcal{M})$, $\mathcal{M}_0' = \Delta_{2^n} \setminus \mathcal{M}_0$;
2: **for** $\eta = 0 : (2^n - |\mathcal{M}_0|)$ **do**
3: **for** all $i \in \Lambda_{\mathcal{M}_\eta}$ **do**
4: $\kappa(\delta_{2^n}^i) = \{\delta_{2^m}^k | M_\eta E_k \delta_{2^n}^i \neq \mathbf{0}_{2^n}\}$ where M_η is the indicator matrix of $\mathcal{M}_\eta$;
5: **end for**
6:

$$\mathcal{M}_{\eta+1} = \left\{\delta_{2^n}^i \in \mathcal{M}_\eta' \middle| \sum_{j\in\Lambda_{\mathcal{M}_\eta}} [\boldsymbol{C}_1]_{j,i} > 0\right\}, \quad \mathcal{M}_{\eta+1}' = \mathcal{M}_\eta' \setminus \mathcal{M}_{\eta+1};$$

7: **end for**
8: **if** $\mathcal{M}'_{2^n-|\mathcal{M}_0|+1} = \emptyset$ **then**
9: **return** "Yes", and $\tilde{K}$ satisfying $\operatorname{Col}_i(\tilde{K}) \in \kappa(\delta_{2^n}^i), \forall i \in [1:2^n]$;
10: **else**
10: **return** "No";
12: **end if**

Proposition 4. Let IBCN (3) be forward complete. The state feedback controller (5) is a hybrid-optimal $\mathcal{M}$-stabilizer of IBCN (3) iff the controller (5) satisfies

$$\operatorname{Col}_s(K) = \operatorname{Col}_s(\tilde{K}), \quad \forall s \in \Lambda_{\mathcal{S}}, \tag{14}$$

where $u_H(k) = \tilde{K}z(k)$ is a k-domain-optimal $\mathcal{M}$-stabilizer of LCS (10).

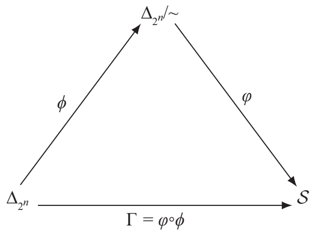

Figure 1 Relationships between different mappings.

Proof. By the connection between IBCN (3) and the k-domain LCS (10), the sufficiency holds directly. We prove the necessity as follows. Suppose that Eq. (5) is a hybrid-optimal $\mathcal{M}$-stabilizer of IBCN (3). Then, we can construct a controller $u_H(k) = \tilde{K}z(k)$ for the k-domain LCS (10) as

$$\mathrm{Col}_s(\tilde{K}) = \begin{cases} \mathrm{Col}_s(K), & s \in \Lambda_{\mathcal{S}}, \\ \delta_{2^m}^1, & s \in \Lambda_{\mathcal{J}}. \end{cases} \tag{15}$$

Thus, under this feedback, it holds that

- For any state $z_0 \in \mathcal{S}$, z_0 can converge to $\mathcal{M}$ in $\eta_{\mathcal{M}}(z_0)$ steps; and
- For any state $z_0 \in \mathcal{J}$, z_0 converges to $\mathcal{M}$ in $\eta_{\mathcal{M}}(z_0)$ steps by resorting to the connection between E and G.

Thus, the feedback satisfying (15) is a k-domain-optimal $\mathcal{M}$-stabilizer of LCS (10), and $\mathrm{Col}_s(K) = \mathrm{Col}_s(\tilde{K})\ \forall s \in \mathcal{S}$ holds obviously.

Remark 3. From Proposition 4, all the hybrid-optimal $\mathcal{M}$-stabilizers of IBCN (3) can be obtained by using Algorithm 1.

4 Stabilization in the time domain

4.1 Stabilizability in the time domain

In this subsection, to effectively analyze the time-domain stabilizability, we construct a quotient mapping from Δ_{2^n} to $\mathcal{S}$ and then directly get the time-domain solution to IBCN (3). First, we define a logical matrix

$$L_Q = \left[\mathrm{Col}_1(G^{\psi(\delta_{2^n}^1)})\ \mathrm{Col}_2(G^{\psi(\delta_{2^n}^2)})\ \cdots\ \mathrm{Col}_{2^n}(G^{\psi(\delta_{2^n}^{2^n})})\right] \in \mathscr{L}_{2^n \times 2^n},$$

where $G \in \mathscr{L}_{2^n \times 2^n}$ is the state transition matrix of the jumping process, and $\psi(x) := \min\{k \in \mathbb{N} | G^k x \in \mathcal{S}\}$ denotes the least number of jumps needed to escape from the impulse-triggering set $\mathcal{J}$. Then, we can define a mapping $\Gamma : \Delta_{2^n} \to \mathcal{S}$ as $\Gamma(x) = L_Q x$ to fulfill our requirement.

Remark 4. According to the above construction, the following should be noted.

- The mapping ψ is well defined on condition that IBCN (3) is forward complete.
- Essentially, Γ is a quotient mapping. Set an equivalence relation in Δ_{2^n} as $x \sim y$ if $\Gamma(x) = \Gamma(y)$. Then, let $[x] := \{y | x \sim y\}$ and $\Delta_{2^n}/\sim := \{[x] | x \in \Delta_{2^n}\}$ denote the equivalence class and quotient space, respectively. Hence, a quotient mapping $\phi : \Delta_{2^n} \to \Delta_{2^n}/\sim$ can be defined as $\phi(x) = [x]$. Furthermore, on the basis of $|\Delta_{2^n}/\sim| = |\mathcal{S}|$, a bijective is constructed naturally by $\varphi([x]) := \Gamma(x)$. Therefore, we obtain $\Gamma = \varphi \circ \phi$, which is termed the quotient mapping, and $\mathcal{S}$ can be also regarded as the quotient space by the effect of the bijective. The relationships between different mappings are displayed in Figure 1.

According to the definition of the quotient mapping Γ, for any hybrid-domain solution with an initial state x_0 and an input sequence $\boldsymbol{u}$, we can obtain

$$\begin{aligned} \Gamma(x(t, j; \boldsymbol{u}, x_0)) &= L_Q x(t, j; \boldsymbol{u}, x_0) = L_Q x(t, \boldsymbol{j}_t^+; \boldsymbol{u}, x_0) \\ &= x(t, \boldsymbol{j}_t^+; \boldsymbol{u}, x_0) \in \mathcal{S}, \quad \forall (t, j) \in \mathcal{H}_{(\boldsymbol{u}, x_0)}. \end{aligned} \tag{16}$$

The quotient mapping Γ maps IBCN (3) to an LCS defined on the quotient space, termed as the quotient LCS, which is derived as follows. We define

$$\chi(t) := \Gamma(x(t, j)) = L_Q x(t, j), \quad \forall (t, j) \in \mathcal{H}_{(\boldsymbol{u}, x_0)}. \tag{17}$$

By using (16), $\chi(t)$ is well-defined for all time instant t. Then, we can obtain that

$$\begin{aligned}\chi(t+1) &= L_Q x(t+1,j) = L_Q x(t+1,\boldsymbol{j}_t^+) = L_Q F u(t) x(t,\boldsymbol{j}_t^+)\\ &= L_Q F u(t) L_Q x(t,\boldsymbol{j}_t^+) = L_Q F u(t)\chi(t).\end{aligned}$$

Thus, the quotient LCS can be expressed directly as

$$\chi(t+1) = L_Q F u(t)\chi(t), \quad \chi(0)\in\mathcal{S}, \tag{18}$$

where $L_Q F \in \mathscr{L}_{2^n\times 2^{n+m}}$. Let $\chi(t;\boldsymbol{u},\chi_0)$ denote the solution to quotient LCS (18) with an initial condition $\chi(0)=\chi_0$ and an input sequence $\boldsymbol{u}$.

Lemma 6. The time-domain solutions to IBCN (3) are connected with those to quotient LCS (18) as follows:

$$\chi(t;\boldsymbol{u},\chi_0) = x(t;\boldsymbol{u},x_0), \quad \forall \chi_0 = L_Q x_0. \tag{19}$$

Proof. Based on the definitions of $\psi(x)$ and $\boldsymbol{j}_t^+(\boldsymbol{u},x)$, it holds that $\psi(x)=\boldsymbol{j}_0^+(\boldsymbol{u},x)$. Therefore, we can obtain

$$\begin{aligned}x(0;\boldsymbol{u},x_0) &= x(0,\boldsymbol{j}_0^+(\boldsymbol{u},x_0);\boldsymbol{u},x_0) = x(0,\psi(x_0);\boldsymbol{u},x_0)\\ &= L_Q x_0 = \chi_0 = \chi(0;\boldsymbol{u},\chi_0),\end{aligned}$$

when $t=0$. If Eq. (19) holds at t, then at $t+1$ we have

$$\begin{aligned}x(t+1;\boldsymbol{u},x_0) &= x(t+1,\boldsymbol{j}_{t+1}^+(\boldsymbol{u},x_0);\boldsymbol{u},x_0)\\ &= \Gamma(x(t+1,\boldsymbol{j}_{t+1}^+(\boldsymbol{u},x_0);\boldsymbol{u},x_0))\\ &= L_Q F u(t) x(t,\boldsymbol{j}_t^+(\boldsymbol{u}_{[0:t-1]},x_0);\boldsymbol{u},x_0)\\ &= L_Q F u(t) x(t;\boldsymbol{u}_{[0:t-1]},x_0)\\ &= L_Q F u(t)\chi(t;\boldsymbol{u}_{[0:t-1]},\chi_0)\\ &= \chi(t+1;\boldsymbol{u},x_0),\end{aligned}$$

where $\boldsymbol{u}_{[0:t-1]}$ represents the input subsequence $\{u_i\}_{i\in[0:t-1]}$.

Combining Definition 7 and Lemma 6, we easily obtain the following results.

Lemma 7. IBCN (3) is $\mathcal{M}$-stabilizable in the time domain iff it is forward complete, and the quotient LCS (18) is $\mathcal{M}$-stabilizable.

Lemma 8. Suppose that $\mathcal{M}$ is a target subset of Δ_{2^n}. The quotient LCS (18) is $\mathcal{M}$-stabilizable iff it is $I_{L_QF,C}(\mathcal{M})$-stabilizable, equivalently,

$$\sum_{i\in\Lambda_*}[(L_Q F \ltimes_{\mathscr{B}} \mathbf{1}_{2^m})^{(|\mathcal{S}|)}]_{i,j} > 0, \quad \forall j\in\Lambda_{\mathcal{S}}, \tag{20}$$

where Λ_* and $\Lambda_{\mathcal{S}}$ are the index sets of $I_{L_QF,C}(\mathcal{M})$ and $\mathcal{S}$, respectively.

Proof. The first part of this claim follows Lemma 4 in [21]. Next, we only need to prove the second part of the claim, that is, the quotient LCS (18) is $I_{L_QF,C}(\mathcal{M})$-stabilizable iff Eq. (20) holds. Eq. (20) indicates that for any state $x_0=\delta_{2^n}^j\in\Delta_{2^n}$, the quotient LCS (18) has a trajectory of length 2^n from x_0 to $I_{L_QF,C}(\mathcal{M})$ under certain control sequence. Therefore, if the quotient LCS (18) is $I_{L_QF,C}(\mathcal{M})$-stabilizable, then Eq. (20) holds. On the contrary, if Eq. (20) holds, then, by the property of control-invariant subsets, we have $\sum_{i\in\Lambda_*}[(L_QF\ltimes_{\mathscr{B}}\mathbf{1}_{2^m})^{(\alpha)}]_{i,j}>0, \forall j\in[1:2^n], \forall\alpha\geqslant|\mathcal{S}|+1$, which states that for any state $x_0=\delta_{2^n}^j$, there is an input sequence $\boldsymbol{u}$ such that $x(t;\boldsymbol{u},x_0)\in I_{L_QF,C}(\mathcal{M}), \forall t\geqslant|\mathcal{S}|$. That is, the quotient LCS (18) is $I_{L_QF,C}(\mathcal{M})$-stabilizable.

Theorem 2. Given $\mathcal{M}\subseteq\Delta_{2^n}$, IBCN (3) is $\mathcal{M}$-stabilizable in the time domain iff

(a) $(JG)^{|\mathcal{J}|}J=\mathbf{0}_{2^n\times 2^n}$, and

(b) $\sum_{i\in\Lambda_*}[(L_QF\ltimes_{\mathscr{B}}\mathbf{1}_{2^m})^{(|\mathcal{S}|)}]_{i,j}>0, \forall j\in\Lambda_{\mathcal{S}}$, where Λ_* and $\Lambda_{\mathcal{S}}$ are the index sets of $I_{L_QF,C}(\mathcal{M})$ and $\mathcal{S}$, respectively.

Proof. From Lemmas 7 and 8 and Proposition 1, this theorem holds directly.

Proposition 5. If IBCN (3) is $\mathcal{M}$-stabilizable in the time domain, then for any $i \in \Lambda_{\mathcal{S}}$,

$$\lambda_{\mathcal{M}}(\delta_{2^n}^i) = \lambda_{I_{L_QF,C}(\mathcal{M})}(\delta_{2^n}^i) = \min_{t\in\mathbb{N}}\{t|\,\mathrm{Col}_i(\hat{M}_*\hat{\boldsymbol{C}}_t) \neq \mathbf{0}_{2^n}\}, \tag{21}$$

where $\hat{M}_*$ is the indicator matrix of $I_{L_QF,C}(\mathcal{M})$ and $\hat{\boldsymbol{C}}_t := (L_QF \ltimes_{\mathscr{B}} \mathbf{1}_{2^m})^{(t)}$ is the t-step controllability matrix of the quotient LCS (18) with $\hat{\boldsymbol{C}}_0 := I_{2^n}$.

Proof. The claim $\lambda_{\mathcal{M}}(\delta_{2^n}^i) = \lambda_{I_{L_QF,C}(\mathcal{M})}(\delta_{2^n}^i)$ follows Lemma 4 in [21]. Next, we only need to prove $\lambda_{I_{L_QF,C}(\mathcal{M})}(\delta_{2^n}^i) = \min_{t\in\mathbb{N}}\{t|\,\mathrm{Col}_i(\hat{M}_*\hat{\boldsymbol{C}}_t) \neq \mathbf{0}_{2^n}\}$. First, for any $\delta_{2^n}^i \in I_{L_QF,C}(\mathcal{M})$, it obviously holds that $\lambda_{I_{L_QF,C}(\mathcal{M})}(\delta_{2^n}^i) = 0$. On the other hand, by the fact that $\hat{M}_*$ is the indicator matrix of $I_{L_QF,C}(\mathcal{M})$, for any $\delta_{2^n}^i \in I_{L_QF,C}(\mathcal{M})$, $\mathrm{Col}_i(\hat{M}_*\hat{\boldsymbol{C}}_0) = \mathrm{Col}_i(\hat{M}_*I_{2^n}) = \mathrm{Col}_i(\hat{M}_*) \neq \mathbf{0}_{2^n}$. Thus, the claim holds for all $\delta_{2^n}^i \in I_{L_QF,C}(\mathcal{M})$. In the following, we consider the case $\delta_{2^n}^i \notin I_{L_QF,C}(\mathcal{M})$. Notice that $[\hat{\boldsymbol{C}}_t]_{j,i} \neq 0$ iff $\delta_{2^n}^j$ is t-step reachable from $\delta_{2^n}^i$. Thus, $\mathrm{Col}_i(\hat{M}_*\hat{\boldsymbol{C}}_t) \neq \mathbf{0}_{2^n}$ iff $\delta_{2^n}^i$ can reach $I_{L_QF,C}(\mathcal{M})$ in t steps. This proves $\lambda_{I_{L_QF,C}(\mathcal{M})}(\delta_{2^n}^i) = \min_{t\in\mathbb{N}}\{t|\,\mathrm{Col}_i(\hat{M}_*\hat{\boldsymbol{C}}_t) \neq \mathbf{0}_{2^n}\}$.

4.2 Design of state-feedback stabilizers in the time domain

In this subsection, we describe an algorithm to design time-domain $\mathcal{M}$-stabilizing feedback controllers for IBCN (3).

Proposition 6. IBCN (3) is time-domain $\mathcal{M}$-stabilizable iff it is time-domain $\mathcal{M}$-stabilizable by feedback.

Proof. By Lemma 7, we only need to prove that the quotient LCS (18) is $\mathcal{M}$-stabilizable iff it is $\mathcal{M}$-stabilizable by feedback, which directly follows Remark 2 in [21].

By employing the method of state space partition proposed in [21], a time-optimal $\mathcal{M}$-stabilizing feedback

$$u(t) = \bar{K}\chi(t), \quad \bar{K} \in \mathscr{L}_{2^m\times 2^n}, \tag{22}$$

for the quotient LCS (18) can be obtained by using Algorithm 2.

Algorithm 2 An algorithm that determines the $\mathcal{M}$-stabilizability of quotient LCS (18) and the feedback gain matrix $\bar{K}$

Input: The quotient LCS (18) with $L_QF = [(L_QF)_1\ (L_QF)_2\ \cdots\ (L_QF)_{2^m}]$ and $(L_QF)_k \in \mathscr{L}_{2^n\times 2^n}$, $k \in [1:2^m]$, the one-step controllability matrix $\hat{\boldsymbol{C}}_1 = L_QF \ltimes_{\mathscr{B}} \mathbf{1}_{2^m}$, a subset $\mathcal{M}$ with $I_{L_QF,C}(\mathcal{M}) \neq \emptyset$;

Output: "Yes", if the quotient LCS (18) is $\mathcal{M}$-stabilizable, and "No", otherwise; $\bar{K}$;

$\hat{\mathcal{M}}_0 = I_{L_QF,C}(\mathcal{M})$, $\hat{\mathcal{M}}_0' = \Delta_{2^n} \setminus \hat{\mathcal{M}}_0$;

for $\lambda = 0 : (2^n - |\hat{\mathcal{M}}_0|)$ **do**

  for all $i \in \Lambda_{\hat{\mathcal{M}}_\lambda}$ **do**

    $\hat{\kappa}(\delta_{2^n}^i) = \{\delta_{2^m}^k | \hat{M}_\lambda (L_QF)_k \delta_{2^n}^i \neq \mathbf{0}_{2^n}\}$ where $\hat{M}_\lambda$ is the indicator matrix of $\hat{\mathcal{M}}_\lambda$;

  end for

$$\hat{\mathcal{M}}_{\lambda+1} = \left\{\delta_{2^n}^i \in \hat{\mathcal{M}}_\lambda' \Big| \sum_{j\in\Lambda_{\hat{\mathcal{M}}_\lambda}} [\hat{\boldsymbol{C}}_1]_{j,i} > 0\right\}, \quad \hat{\mathcal{M}}_{\lambda+1}' = \hat{\mathcal{M}}_\lambda' \setminus \hat{\mathcal{M}}_{\lambda+1};$$

end for

if $\hat{\mathcal{M}}'_{2^n-|\hat{\mathcal{M}}_0|+1} = \emptyset$ **then**

  return "Yes", and $\bar{K}$ satisfying $\mathrm{Col}_i(\bar{K}) \in \hat{\kappa}(\delta_{2^n}^i), \forall i \in [1:2^n]$;

else

  return "No";

end if

Proposition 7. Assume that IBCN (3) is forward complete. The state feedback controller (7) is a time-optimal stabilizer of IBCN (3) iff the controller (7) satisfies

$$\mathrm{Col}_s(\hat{K}) = \mathrm{Col}_s(\bar{K}), \quad \forall s \in \mathcal{S}, \tag{23}$$

where $u(t) = \bar{K}\chi(t)$ is a time-optimal $\mathcal{M}$-stabilizer of LCS (18).

Proof. Based on the connection between IBCN (3) and the quotient LCS (18), the sufficiency holds directly. Next, we will verify the necessity. Note that the state feedback (7) is a time-optimal stabilizer of IBCN (3). Then, we construct a controller $u(t) = \bar{K}\chi(t)$ for the quotient LCS (18) satisfying $\mathrm{Col}_s(\bar{K}) = \mathrm{Col}_s(\hat{K}), \forall s \in \mathcal{S}$, where $\hat{K}$ is the feedback gain matrix of (7). Thus, under this feedback, we have

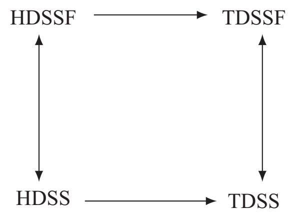

Figure 2 Relationships between different stabilizabilities. HDSS, HDSSF, TDSS, and TDSSF represent hybrid-domain set stabilizability, hybrid-domain set stabilizability by feedback, time-domain set stabilizability, and time-domain set stabilizability by feedback, respectively. Two-way arrows and one-way arrows represent equivalence and implication, respectively.

that χ_0 can converge to $\mathcal{M}$ in $\lambda_{\mathcal{M}}(\chi_0)$ steps for any state $\chi_0 \in \mathcal{S}$. Hence, the feedback satisfying $\mathrm{Col}_s(\bar{K}) = \mathrm{Col}_s(\hat{K}), \forall s \in \mathcal{S}$ is a time-optimal $\mathcal{M}$-stabilizer of LCS (18), and Eq. (23) holds obviously.

Remark 5. According to Proposition 7, all the time-optimal $\mathcal{M}$-stabilizers of IBCN (3) can be determined by Algorithm 2.

Remark 6. In Algorithms 1 and 2, it holds that $0 \leqslant \eta \leqslant 2^n - |\mathcal{M}_0| \leqslant 2^n$, $0 \leqslant i \leqslant |\mathcal{M}_\eta| \leqslant 2^n$, and $0 \leqslant k \leqslant 2^m$. Thus, a rough estimation of the computational complexity of Algorithms 1 and 2 is 2^{2n+m}. How to reduce the computational complexity is a challenging problem for Boolean control networks because the number of states increases exponentially with the increase of the number of nodes. Further research is needed to reduce the computational complexity.

5 Relationships between different stabilizabilities

Proposition 8. If IBCN (3) is hybrid-domain $\mathcal{M}$-stabilizable, then it is also time-domain $\mathcal{M}$-stabilizable.

Proof. This claim directly follows the fact that, for any initial state x_0 and any input sequence $\boldsymbol{u}$, the time-domain solution $x(t; \boldsymbol{u}, x_0) = x(t, \boldsymbol{j}_t^+; \boldsymbol{u}, x_0)$ is a subsequence of the hybrid-domain solution $x(t_k, j_k; \boldsymbol{u}, x_0)$.

As illustrated in Figure 2, we can summarize the relationships between different stabilizabilities as follows:

(1) The hybrid-domain set stabilizability is equivalent to the hybrid-domain set stabilizability by feedback (Proposition 3).

(2) The time-domain set stabilizability and the time-domain set stabilizability by feedback are equivalent (Proposition 6).

(3) The hybrid-domain set stabilizability implies the time-domain set stabilizability (Proposition 8), but the reverse does not hold, as illustrated in Example 2. However, a hybrid-optimal feedback stabilizer is not necessarily time-optimal, as illustrated in Example 1.

(4) Based on Propositions 3, 6, and 8, the hybrid-domain set stabilizability by feedback implies the time-domain set stabilizability by feedback. However, the reverse is not true in general.

6 Examples

Example 1. Consider IBCN (3) with four state nodes ($n = 4$) and two input nodes ($m = 2$), and $\mathcal{J} = \{\delta_{16}^3, \delta_{16}^4, \delta_{16}^6, \delta_{16}^9, \delta_{16}^{10}, \delta_{16}^{13}, \delta_{16}^{14}\}$, $\mathcal{S} = \Delta_{16} \setminus \mathcal{J}$,

$$\begin{aligned} F = \delta_{16}[&2\ \ 3\ \ \diamond\ \ \diamond\ \ 3\ \ \diamond\ \ 6\ \ 10\ \ \diamond\ \ \diamond\ \ 11\ \ 13\ \ \diamond\ \ \diamond\ \ 13\ \ 3 \\ &5\ \ 12\ \ \diamond\ \ \diamond\ \ 1\ \ \diamond\ \ 5\ \ 8\ \ \diamond\ \ \diamond\ \ 5\ \ 16\ \ \diamond\ \ \diamond\ \ 11\ \ 15 \\ &6\ \ 16\ \ \diamond\ \ \diamond\ \ 3\ \ \diamond\ \ 8\ \ 8\ \ \diamond\ \ \diamond\ \ 7\ \ 14\ \ \diamond\ \ \diamond\ \ 15\ \ 16 \\ &13\ \ 7\ \ \diamond\ \ \diamond\ \ 12\ \ \diamond\ \ 13\ \ 15\ \ \diamond\ \ \diamond\ \ 2\ \ 3\ \ \diamond\ \ \diamond\ \ 6\ \ 13], \\ G = \delta_{16}[&\diamond\ \ \diamond\ \ 12\ \ 3\ \ \diamond\ \ 9\ \ \diamond\ \ \diamond\ \ 14\ \ 6\ \ \diamond\ \ \diamond\ \ 5\ \ 3\ \ \diamond\ \ \diamond]. \end{aligned}$$

The diamond symbols represent the irrelevant elements that can be arbitrarily chosen. Assume that the target set $\mathcal{M} = \{\delta_{16}^1, \delta_{16}^3, \delta_{16}^9, \delta_{16}^{12}, \delta_{16}^{16}\}$. Because $(JG)^6 J = \mathbf{0}_{16\times 16}$, this IBCN is forward complete. In the following, the hybrid-domain stabilizability will be checked first. According to the construction of the

Table 2 Admissible-control sets $\kappa(x)$, where $x \in \mathcal{S}$ in Example 1

State x	$\kappa(x)$	State x	$\kappa(x)$	State x	$\kappa(x)$
δ_{16}^1	$\{\delta_4^1, \delta_4^2\}$	δ_{16}^7	$\{\delta_4^2\}$	δ_{16}^{12}	$\{\delta_4^2, \delta_4^4\}$
δ_{16}^2	$\{\delta_4^1, \delta_4^2, \delta_4^3\}$	δ_{16}^8	$\{\delta_4^4\}$	δ_{16}^{15}	$\{\delta_4^1, \delta_4^2\}$
δ_{16}^5	$\{\delta_4^1, \delta_4^3, \delta_4^4\}$	δ_{16}^{11}	$\{\delta_4^2, \delta_4^4\}$	δ_{16}^{16}	$\{\delta_4^1, \delta_4^3\}$

k-domain LCS (10), its transition matrix E can be given as

$$\begin{aligned} E &= [F_1 \ltimes S\ F_2 \ltimes S\ F_3 \ltimes S\ F_4 \ltimes S] + [G \ltimes J\ G \ltimes J\ G \ltimes J\ G \ltimes J] \\ &= \delta_{16}[2\ \ 3\ \ 12\ 3\ \ 3\ 9\ \ 6\ \ 10\ 14\ 6\ \ 11\ 13\ 5\ 3\ 13\ \ 3 \\ &\quad 5\ \ 12\ 12\ 3\ \ 1\ 9\ \ 5\ \ \ 8\ \ 14\ 6\ \ 5\ \ 16\ 5\ 3\ 11\ 15 \\ &\quad 6\ \ 16\ 12\ 3\ \ 3\ 9\ \ 8\ \ \ 8\ \ 14\ 6\ \ 7\ \ 14\ 5\ 3\ 15\ 16 \\ &\quad 13\ \ 7\ 12\ 3\ 12\ 9\ 13\ 15\ \ 14\ 6\ \ 2\ \ 3\ \ 5\ 3\ \ 6\ 13], \end{aligned}$$

where $F_i \in \mathscr{L}_{16\times 16}$, $i \in [1:4]$, is the i-th block of F, and S and J are the indicator matrices of $\mathcal{S}$ and $\mathcal{J}$, respectively. Based on Lemma 3, we have $I_{E,C}(\mathcal{M}) = \{\delta_{16}^3, \delta_{16}^{12}, \delta_{16}^{16}\}$. In addition, $\sum_{i\in\Lambda_\star}[(E \ltimes_{\mathscr{B}} \mathbf{1}_4)^{(16)}]_{i,j} > 0, \forall j \in [1:16]$, where $\Lambda_\star = \{3, 12, 16\}$ is the index set of $I_{E,C}(\mathcal{M})$. Therefore, by using Theorem 1, the IBCN is hybrid-domain $\mathcal{M}$-stabilizable. By Proposition 3, it is also hybrid-domain $\mathcal{M}$-stabilizable by feedback. Hence, we can design a hybrid-optimal $\mathcal{M}$-stabilizer for the IBCN as follows. Based on Proposition 2, we can calculate $\eta_{\mathcal{M}} = \max\{\eta_{\mathcal{M}}(x) | x \in \Delta_{16}\} = 4$. Thus, the state space Δ_{16} can be divided into 5 disjoint nonempty layers as

$$\begin{aligned} &\mathcal{M}_0 = \{\delta_{16}^3, \delta_{16}^{12}, \delta_{16}^{16}\},\ \mathcal{M}_1 = \{\delta_{16}^2, \delta_{16}^4, \delta_{16}^5, \delta_{16}^{14}\},\ \mathcal{M}_2 = \{\delta_{16}^1, \delta_{16}^7, \delta_{16}^9, \delta_{16}^{11}, \delta_{16}^{13}\}, \\ &\mathcal{M}_3 = \{\delta_{16}^6, \delta_{16}^{15}\}, \qquad \mathcal{M}_4 = \{\delta_{16}^8, \delta_{16}^{10}\}. \end{aligned}$$

Then, the hybrid-optimal feedback gain matrix K can be calculated by Algorithm 1, wherein $\mathrm{Col}_s(K) \in \kappa(\delta_{16}^s)$, $s \in \mathcal{S}$, and the admissible-control sets $\kappa(x), x \in \mathcal{S}$, are shown in Table 2. Hence, a hybrid-optimal state feedback $\mathcal{M}$-stabilizer can be given as $u(t) = Kx(t, \boldsymbol{j}_t^+)$, where

$$K = \delta_4[2\ 1\ \diamond\ \diamond\ 3\ \diamond\ 2\ 4\ \diamond\ \diamond\ 4\ 2\ \diamond\ \diamond\ 2\ 1].$$

Under the above feedback, the closed-loop system becomes

$$\begin{cases} x(t+1, j) = F_K x(t, j),\ x(t, j) \in \mathcal{S}, \\ x(t, j+1) = Gx(t, j), \quad x(t, j) \in \mathcal{J}, \end{cases} \tag{24}$$

where $F_K = FKM_{r,16} = \delta_{16}[5\ 3\ \diamond\ \diamond\ 3\ \diamond\ 5\ 15\ \diamond\ \diamond\ 2\ 16\ \diamond\ \diamond\ 11\ 3]$ and $M_{r,16} \in \mathscr{L}_{256\times 16}$ is the power-reducing matrix satisfying $\mathrm{Col}_i(M_{r,16}) = \delta_{16}^i \ltimes \delta_{16}^i$, $i \in [1:16]$. The state transfer graph of the closed-loop system is depicted in Figure 3. According to this figure, we find that all states can converge to $I_{E,C}(\mathcal{M})$ and δ_{16}^8 transfers to $I_{E,C}(\mathcal{M})$ within 4 time steps. On the other hand, based on the construction of L_Q, we have $L_Q = \delta_{16}[1\ 2\ 12\ 12\ 5\ 12\ 7\ 8\ 12\ 12\ 11\ 12\ 12\ 12\ 15\ 16]$ and

$$\begin{aligned} L_Q F = \delta_{16}[&2\ \ 12\ \ \diamond\ \diamond\ \ 12\ \diamond\ 12\ \ 12\ \diamond\ \diamond\ 11\ \ 12\ \ \diamond\ \diamond\ \ 12\ 12 \\ &5\ \ 12\ \diamond\ \diamond\ \ 1\ \ \diamond\ \ 5\ \ \ 8\ \ \diamond\ \diamond\ \ 5\ \ 16\ \ \diamond\ \diamond\ \ 11\ 15 \\ &12\ 16\ \diamond\ \diamond\ \ 12\ \diamond\ \ 8\ \ \ 8\ \ \diamond\ \diamond\ \ 7\ \ 12\ \ \diamond\ \diamond\ 15\ 16 \\ &12\ \ 7\ \diamond\ \diamond\ \ 12\ \diamond\ \ 12\ 15\ \ \diamond\ \diamond\ \ 2\ \ 12\ \ \diamond\ \diamond\ 12\ 12]. \end{aligned}$$

Based on Lemma 3, we can obtain $I_{L_QF,C}(\mathcal{M}) = \{\delta_{16}^{12}, \delta_{16}^{16}\}$. By Proposition 5, we can calculate $\lambda_{\mathcal{M}}(\delta_{16}^8) = 1 < 4$. Thus, the feedback control is not a time-optimal $\mathcal{M}$-stabilizer for this IBCN.

Example 2. We consider IBCN (3) with four state nodes and two input nodes, and $\mathcal{J} = \{\delta_{16}^1, \delta_{16}^4, \delta_{16}^7, \delta_{16}^{11}, \delta_{16}^{15}\}$, $\mathcal{S} = \Delta_{16} \setminus \mathcal{J}$,

$$\begin{aligned} F = \delta_{16}[&\diamond\ 5\ \ 3\ \diamond\ \ 7\ \ 10\ \diamond\ 6\ \ 3\ 14\ \diamond\ 10\ 13\ 15\ \diamond\ \ 1 \\ &\diamond\ 7\ \ 9\ \diamond\ 11\ \ \ 7\ \diamond\ 2\ \ 2\ \ 3\ \ \diamond\ 12\ 12\ \ 1\ \ \diamond\ \ 7 \\ &\diamond\ 9\ 12\ \diamond\ \ 2\ \ \ 1\ \diamond\ 7\ \ 6\ 13\ \diamond\ \ 9\ \ 3\ \ 15\ \diamond\ 11 \\ &\diamond\ 3\ 10\ \diamond\ 14\ 15\ \diamond\ 1\ 10\ \ 6\ \diamond\ 13\ 13\ 12\ \diamond\ 14], \end{aligned}$$

$$G = \delta_{16}[5\ \diamond\ \diamond\ 7\ \ \diamond\ \diamond\ 8\ \diamond\ \ \diamond\ \ \diamond\ 15\ \diamond\ \ \diamond\ \ \diamond\ 16\ \ \diamond].$$

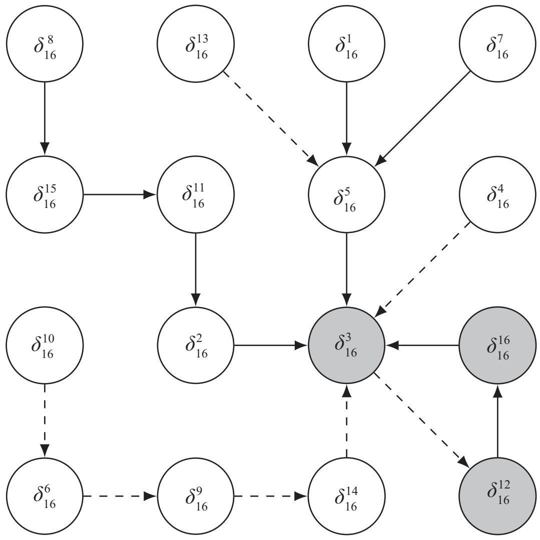

Figure 3 The state transfer graph of the closed-loop IBCN (24), where the solid arrows and the dashed arrows represent the behaviors of stepping and jumping, respectively.

Suppose that the target set $\mathcal{M} = \{\delta_{16}^1, \delta_{16}^5, \delta_{16}^8, \delta_{16}^{10}, \delta_{16}^{15}, \delta_{16}^{16}\}$. This IBCN is forward complete because $(JG)^5J = \mathbf{0}_{16\times 16}$. In the following, we first check the hybrid-domain stabilizability. Based on the construction of the transition matrix E for the k-domain LCS (10), we have

$$\begin{aligned} E &= [F_1 \ltimes S\ F_2 \ltimes S\ F_3 \ltimes S\ F_4 \ltimes S] + [G \ltimes J\ G \ltimes J\ G \ltimes J\ G \ltimes J] \\ &= \delta_{16}[5\ 5\ \ 3\ \ 7\ \ 7\ 10\ 8\ 6\ \ 3\ \ 14\ 15\ 10\ 13\ 15\ \ 16\ \ 1 \\ &\qquad 5\ 7\ \ 9\ 7\ \ 11\ 7\ \ 8\ 2\ \ 2\ \ \ 3\ 15\ 12\ 12\ \ 1\ \ 16\ \ 7 \\ &\qquad 5\ 9\ 12\ 7\ \ 2\ \ 1\ \ 8\ 7\ \ 6\ \ 13\ 15\ \ 9\ \ 3\ \ 15\ 16\ 11 \\ &\qquad 5\ 3\ 10\ 7\ 14\ 15\ 8\ 1\ \ 10\ \ 6\ 15\ 13\ 13\ 12\ 16\ 14], \end{aligned}$$

where $F_i \in \mathscr{L}_{16\times 16}$ is the i-th block of F with $i \in [1:4]$, and S and J are the indicator matrices of $\mathcal{S}$ and $\mathcal{J}$, respectively. By Lemma 3, it is easily calculated that $I_{E,C}(\mathcal{M}) = \emptyset$. Thus, by Theorem 1, this IBCN is not $\mathcal{M}$-stabilizable in the hybrid domain.

Next, we check the time-domain stabilizability. Based on the construction of L_Q, we can calculate $L_Q = \delta_{16}[2\ 2\ 3\ 8\ 5\ 6\ 8\ 8\ 9\ 10\ 16\ 12\ 13\ 14\ 16\ 16]$. Then we have

$$\begin{aligned} L_QF = \delta_{16}[&\diamond\ 5\ \ 3\ \diamond\ \ 8\ \ 10\ \diamond\ 6\ \ 3\ 14\ \diamond\ 10\ 13\ 16\ \diamond\ \ 5 \\ &\diamond\ 8\ \ 9\ \diamond\ 16\ \ \ 8\ \diamond\ 2\ \ 2\ \ 3\ \ \diamond\ 12\ 12\ \ 5\ \diamond\ \ 8 \\ &\diamond\ 9\ 12\ \diamond\ \ 2\ \ \ 5\ \diamond\ 8\ \ 6\ 13\ \diamond\ \ 9\ \ 3\ \ 16\ \diamond\ 16 \\ &\diamond\ 3\ 10\ \diamond\ 14\ 16\ \diamond\ 5\ 10\ \ 6\ \diamond\ 13\ 13\ 12\ \diamond\ 14]. \end{aligned}$$

By Lemma 3, we can obtain $I_{L_QF,C}(\mathcal{M}) = \{\delta_{16}^5, \delta_{16}^8, \delta_{16}^{16}\}$. Moreover, it is easily checked that $\sum_{i\in\Lambda_*}[(L_QF \ltimes_{\mathscr{B}} \mathbf{1}_4)^{(11)}]_{i,j} > 0, \forall j \in \Lambda_{\mathcal{S}}$, where Λ_* and $\Lambda_{\mathcal{S}}$ are the index sets of $I_{L_QF,C}(\mathcal{M})$ and $\mathcal{S}$, respectively. Thus, this IBCN is time-domain $\mathcal{M}$-stabilizable by Theorem 2. Then, based on Proposition 6, we can construct a time-optimal controller to stabilize the IBCN with respect to $\mathcal{M}$. By Proposition 5, it holds that $\lambda_{\mathcal{M}} = \max\{\lambda_{\mathcal{M}}(x) | x \in \mathcal{S}\} = 4$. Subsequently, we can partition $\mathcal{S}$ into 5 layers, i.e.,

$$\hat{\mathcal{M}}_0 = \{\delta_{16}^5, \delta_{16}^8, \delta_{16}^{16}\},\ \hat{\mathcal{M}}_1 = \{\delta_{16}^2, \delta_{16}^6, \delta_{16}^{14}\},\ \hat{\mathcal{M}}_2 = \{\delta_{16}^9, \delta_{16}^{10}\},\ \hat{\mathcal{M}}_3 = \{\delta_{16}^3, \delta_{16}^{12}\},\ \hat{\mathcal{M}}_4 = \{\delta_{16}^{13}\}.$$

Further, by Algorithm 2, the time-optimal feedback gain matrix K is calculated wherein $\mathrm{Col}_s(\hat{K}) \in \hat{\kappa}(\delta_{16}^s),\ s \in \Lambda_{\mathcal{S}}$, and each admissible-control set $\hat{\kappa}(x)$ is illustrated in Table 3. Therefore, a feedback that stabilizes the IBCN with respect to $\mathcal{M}$ in shortest transient period corresponds to $u(t) = \hat{K}x(t, \boldsymbol{j}_t^+)$ with

$$\hat{K} = \delta_4[\diamond\ 2\ 4\ \diamond\ 1\ 3\ \diamond\ 4\ 2\ 1\ \diamond\ 3\ 3\ 1\ \diamond\ 2].$$

Table 3 Admissible-control sets $\hat{\kappa}(x)$, where $x \in \mathcal{S}$ in Example 2

State x	$\hat{\kappa}(x)$	State x	$\hat{\kappa}(x)$	State x	$\hat{\kappa}(x)$	State x	$\hat{\kappa}(x)$
δ_{16}^{2}	$\{\delta_4^1, \delta_4^2\}$	δ_{16}^{6}	$\{\delta_4^2, \delta_4^3\}$	δ_{16}^{10}	$\{\delta_4^1, \delta_4^4\}$	δ_{16}^{14}	$\{\delta_4^1, \delta_4^2, \delta_4^3\}$
δ_{16}^{3}	$\{\delta_4^2, \delta_4^4\}$	δ_{16}^{8}	$\{\delta_4^3, \delta_4^4\}$	δ_{16}^{12}	$\{\delta_4^1, \delta_4^3\}$	δ_{16}^{16}	$\{\delta_4^1, \delta_4^2, \delta_4^3\}$
δ_{16}^{5}	$\{\delta_4^1, \delta_4^2\}$	δ_{16}^{9}	$\{\delta_4^2, \delta_4^3\}$	δ_{16}^{13}	$\{\delta_4^2, \delta_4^3\}$		

7 Conclusion

In this paper, we investigated set stabilization of Boolean control networks with state-triggered impulses in the hybrid domain and the time domain under the framework of a hybrid-index model. Compared with the existing impulse triggering model based on time or events, the hybrid-index model focuses more on describing the instantaneity of the impulse process. We used the k-domain method to show that the hybrid-domain set stabilizability of an IBCN is equivalent to the set stabilizability of the corresponding k-domain LCS on the condition that the IBCN is forward complete. Moreover, we demonstrated that the time-domain set stabilizability of the IBCN is equivalent to set stabilizability of the resulting quotient LCS constructed by a quotient mapping if the forward completeness of an IBCN is fulfilled. According to these ideas, we obtained the necessary and sufficient conditions for the hybrid-domain and time-domain set stabilizability. Furthermore, based on the idea of partitioning the state space into layers that are disjoint and interrelated, we have proposed algorithms for determining all the hybrid-optimal and time-optimal feedback stabilizers. We have also summarized relationships between different set stabilizabilities.

Our future work will focus on the impulsive probabilistic Boolean control networks based on the proposed model.

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