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# Distributed convex optimization for nonlinear multi-agent systems disturbed by a second-order stationary process over a digraph

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**Abstract** In this paper, we investigate the distributed convex optimization problem for a class of nonlinear multi-agent systems disturbed by random noise over a directed graph. The target problem involves designing a continuous-time algorithm to minimize the sum of all local cost functions associated with each agent. The target noise is considered as a second-order stationary process under mild assumptions. The noise-to-state exponential stability for the multi-agent system based on random differential equations is analyzed using a random field method. Sufficient conditions corresponding to the second moment relative to the optimal solution in the form of matrix inequalities are established. Then, the grid search method is employed to determine the best system parameters such that the second moment of the estimation error has the minimum value. In addition, the obtained results are applied to solve the average consensus problem in the presence of a stationary process. Finally, a numerical example is presented to verify the effectiveness of the proposed algorithm.

**Keywords** distributed optimization, random differential equations, stationary process, noise-to-state exponential stability

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## 1 Introduction

Over the past two decades, cooperative control of multi-agent systems has received increasing attention owing to its extensive applications such as formation control, target location, resource allocation, containment control, and tracking control [1–5]. Distributed optimization is an important problem for multi-agent systems. Here, the goal is to minimize the sum of all agents' local cost functions in a distributed manner:

$$\min_x \sum_{i=1}^{n} f_i(x), \quad i = 1, 2, \dots, n, \tag{1}$$

where $x \in \mathbb{R}$ is a global variable, and $f_i(x) : \mathbb{R} \to \mathbb{R}$ denotes a local cost function for agent $i$. Many discrete-time algorithms for the distributed optimization problem (DOP) have been proposed previously over an undirected graph or a directed graph (digraph) [6–12]. Owing to the effective development of continuous-time control techniques, several continuous-time algorithms have also been proposed to search the optimal solution of the DOP [13–17]. In engineering and communication fields, noisy disturbances always exist [18–23], and such disturbances are considered as an obstacle to obtain an accurate solution to the target problem. In addition, most existing methods can only make sure that the system states converge

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to a small neighborhood of the optimal solution of the problem in the presence of such disturbances, and the estimation error between the current and optimal solutions is unknown, which means that developing the effective evaluation criteria for the solution to the optimization problem with noise has become an important issue. Some results have been reported for these two problems [18–23]. Note that the random noise in [19,22] is a white noise. White noise is considered as the formal derivative of a Wiener process [24]. Because the mean power of the white noise process is unbounded, dynamical models based on stochastic differential equations driven by a Wiener process are inappropriate for many practical problems. For systems disturbed by a stationary process whose mean power is bounded, random differential equations (RDEs) are employed to describe the dynamical models [25]. For example, it is more reasonable to employ stationary processes than white noises to model thermal noises generated by agents performing computations on the digital computers owing to the heating of electronic devices in circuit systems [24]. Accordingly, the dynamics of systems disturbed by stationary processes in this paper are modeled using RDEs. Most existing results for the DOP [19, 21] consider linear systems rather than more practical networks and more complex node dynamics models, e.g., a coupled FitzHugh-Nagumo type system [26]. In addition, nonlinearity and uncertainty are ubiquitous in real applications [27]; however, they are not considered in [19, 21]. To the best of our knowledge, the existing results regarding the DOP based on random nonlinear multi-agent systems disturbed by random noises are very rare. Addressing this challenging limitation is the motivation of this study.

In this paper, a continuous-time algorithm is proposed to solve the DOP for multi-agent systems with stationary processes, which are used to model the external disturbances present in the agents' complex computing surroundings and uncertainties induced by the local information exchange among agents over the communication network. Then, a noise-to-state (NOS) exponential stability criterion is developed to evaluate the quality of DOP solutions in the presence of stationary processes. The primary contributions of this paper are summarized as follows. First, in the DOP, the dynamics of agents are extended to nonlinear systems disturbed by random noise that can be equivalently considered as a second-order stationary process under some mild conditions, and these are different from linear systems in continuous-time algorithmic design [13–15, 21]. In addition, it is more appropriate to employ stationary processes to model the effect of a vibration environment in practical applications than white noises [28, 29]. Second, the NOS exponential stability of the target multi-agent system is analyzed using the random field method, and sufficient conditions corresponding to the second moment relative to the optimal solution in the form of matrix inequalities are obtained. These matrices contain some positive and free constants; thus, conservatism of the established criterion is alleviated. Third, the grid search method is applied to find optimal system parameters using the developed criterion such that the upper bound of the second moment of the estimation error of the solution has the minimum value. This helps us obtain more accurate solutions for multi-agent systems based on RDEs. Finally, the obtained results are applied to average consensus for nonlinear multi-agent systems with second-order stationary processes. Thus, our results provide an alternative solution for the average consensus problem subject to stationary processes.

## 2 Preliminaries and problem formulation

### 2.1 Notations and preliminaries

$C$ and $C^{\mathrm{T}}$ denote a matrix and its transpose, respectively. $C > 0$ denotes that the matrix $C$ is positive definite. $\mathbb{R}^n$ represents the $n$-dimensional real vector space. $0_n$ and $1_n$ denote the $n$-dimensional vectors with all entries 0 and 1, respectively. $I_n$ denotes the $n$-dimensional identity matrix. For a vector $\bar{c}$, $\|\bar{c}\|_2$ denotes its Euclidean norm. $\mathrm{col}\,(\hat{c}_1, \hat{c}_2, \ldots, \hat{c}_n) = [\hat{c}_1^{\mathrm{T}}, \hat{c}_2^{\mathrm{T}}, \ldots, \hat{c}_n^{\mathrm{T}}]^{\mathrm{T}}$ denotes the column vector stacked by vectors $\hat{c}_1, \hat{c}_2, \ldots, \hat{c}_n$. $\mathrm{diag}\,\{\kappa_1, \kappa_2, \ldots, \kappa_n\}$ denotes the diagonal matrix with diagonal entries $\kappa_i$. $\otimes$ denotes the Kronecker product. For two positive numbers $a$ and $c$, let $a \wedge c = \min\{a, c\}$. $*$ denotes an element of the symmetric position of the symmetric matrix. For a random variable $\varepsilon$, $\mathrm{E}\varepsilon$ denotes its expectation.

A non-empty set $K$ is said to be convex if $\bar{b}y_1 + (1-\bar{b})\bar{y} \in K$, for any $y_1,\ \bar{y} \in K$ and $\bar{b} \in [0,1]$. A differentiable function $f : K \to \mathbb{R}$ is strongly convex if there exists a positive constant $\hat{\varpi}$ such that $(y_1 - \bar{y})^{\mathrm{T}} (\nabla f(y_1) - \nabla f(\bar{y})) \geqslant \hat{\varpi} \|y_1 - \bar{y}\|_2^2,\ \forall y_1,\ \bar{y} \in K$. In addition, its gradient $\nabla f$ is Lipschitz if there exists a positive constant $\bar{l}$ such that $\|\nabla f(y_1) - \nabla f(\bar{y})\|_2 \leqslant \bar{l}\|y_1 - \bar{y}\|_2$, $\forall y_1,\ \bar{y} \in K$ in [30].

### 2.2 Graph theory

The basic concepts about graph theory can be found in [31]. The communication topology for a multi-agent system with $n$ agents is described by a weighted directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{1, 2, \ldots, n\}$ is the agent set, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ is the weighted adjacency matrix that is used to portray the interaction among agents. Particularly, $a_{ij} > 0$ if agent $i$ can receive information from agent $j$ and $a_{ij} = 0$, otherwise. In this paper, $a_{ii} = 0$. The notation $N_i = \{j| (i, j) \in \mathcal{E}\}$ denotes the set of in-neighbors of agent $i$. The graph $\mathcal{G}$ is said to be connected if there exists a path between any pair of different agents. The in-degree of agent $i$ is $d_i = \sum_{j\in N_i} a_{ij}$. Thus, the Laplacian matrix $\mathcal{L}$ of $\mathcal{G}$ is defined as $\mathcal{L} = \mathcal{D} - \mathcal{A}$ with $\mathcal{D} = \text{diag}\,\{d_1, d_2, \ldots, d_n\}$.

### 2.3 Problem formulation

Consider a multi-agent system consisting of $n$ agents operating in a vibration environment, where the dynamics of agents are always affected by some unforeseen factors, e.g., stochastic interferences and stochastic missing measurements [32]. Here, these factors are considered external disturbances present by the agents' computing surroundings, and uncertainty is induced by the local information exchange among agents over the communication network. Furthermore, nonlinearity is also an inherent feature of almost all practical systems, so the dynamics of each agent is described by the following random nonlinear dynamical model:

$$\begin{aligned}\dot{z}_i &= g_{i1}\left(z_i, x_i, w_i\right) + \varphi_{ii}^{0}\left(z_i, t\right)\sigma_{ii}^{0}\left(t\right),\\ \dot{x}_i &= g_{i2}\left(z_i, x_i, w_i\right) + \varphi_{ii}^{\bar{0}}\left(x_i, t\right)\sigma_{ii}^{\bar{0}}\left(t\right) + u_i + \bar{\Lambda}_i\left(u_i\right),\end{aligned} \tag{2}$$

where $(z_i, x_i) \in \mathbb{R}\times\mathbb{R}$ are states; $u_i$ is the control input; $w_i$ is an uncertain constant parameter in a fixed compact set $\mathbb{W} \in \mathbb{R}$; $z_i(t_0) = z_i(0)$, $x_i(t_0) = x_i(0)$; the functions $g_{i1}\left(z_i, x_i, w_i\right)$ and $g_{i2}\left(z_i, x_i, w_i\right)$ are Lipschitz in $z_i$ and $x_i$, respectively; $\sigma_{ii}^{0}\left(t\right)$ is an independent 1-dimensional random process, so is $\sigma_{ii}^{\bar{0}}\left(t\right)$; $\varphi_{ii}^{0}\left(z_i, t\right)$ is a general-force matrix, so is $\varphi_{ii}^{\bar{0}}\left(x_i, t\right)$; $\varphi_{ii}^{0}\left(z_i, t\right)\sigma_{ii}^{0}\left(t\right)$ and $\varphi_{ii}^{\bar{0}}\left(x_i, t\right)\sigma_{ii}^{\bar{0}}\left(t\right)$ are random noises, which respectively denote the impact of the external environment on $g_{i1}\left(z_i, x_i, w_i\right)$ and $g_{i2}\left(z_i, x_i, w_i\right)$; $\bar{\Lambda}_i\left(u_i\right)$ represents a random noise that depends on $u_i$, $i \in \mathcal{V}$; and the specific expression is given later. Note that the target noise considered is a second-order stationary process under some weak conditions that will be provided later.

The target problem involves designing a continuous-time algorithm $u_i$ for agent $i$ with dynamics (2) to cooperatively solve the following equivalent problem of (1):

$$\min_{\boldsymbol{x}} F(\boldsymbol{x}) = \sum\nolimits_{i=1}^{n} f_i(x_i) \ \ \text{s.t.} \ \ \lim_{t\to\infty} \mathrm{E}\left|x_i - x_j\right| = 0, \tag{3}$$

where $\boldsymbol{x} = \text{col}\left(x_1, x_2, \ldots, x_n\right) \in \mathbb{R}^n$, and $x_i$ denotes a local estimate for the global variable $x$. Suppose that the set of solutions of problem (3) is non-empty, which is marked as $X^* = \{x^*|\mathrm{E}x_i = \mathrm{E}x_j = x^*, i \neq j \in \mathcal{V}\}$.

Throughout this paper, the following assumptions are made on the communication topology and the local cost functions.

**Assumption 1.** The communication topology $\mathcal{G}$ among agents is described by a strongly-connected and weight-balanced directed graph.

**Remark 1.** This assumption is widely used [14, 21] and has the properties $1_n^{\mathrm{T}}\mathcal{L} = 0_n$ and $\mathcal{L}1_n = 0_n$. 0 is the simple eigenvalue of the Laplacian matrix $\mathcal{L}$ [31]. In addition, there is a standard non-singular matrix $Q$ such that $Q^{\mathrm{T}}\mathcal{L}Q = \text{diag}\,\{0, J\}$, where $Q = [Q_1,\ Q_2]$ with $Q_1 = \frac{1}{\sqrt{n}}1_n \in \mathbb{R}^n$ and $Q_2 \in \mathbb{R}^{n\times(n-1)}$ satisfying $Q_2^{\mathrm{T}}Q_2 = I_{n-1}$. $J$ is an upper triangular matrix, and its diagonal entries have positive real parts. As a result, the matrix $J + J^{\mathrm{T}}$ is a positive definite matrix.

**Assumption 2.** The local cost function $f_i(x_i)$ is $\varpi_i$-strongly convex and its gradient $\nabla f_i(x_i)$ is $\bar{\nu}_i$-Lipschitz, where the minimum value of the strongly convex coefficient is $\varpi = \min\{\varpi_1, \varpi_2, \ldots, \varpi_n\}$, and the maximum value of the Lipschitz constant is $\bar{\nu} = \max\{\bar{\nu}_1, \bar{\nu}_2, \ldots, \bar{\nu}_n\}$.

**Remark 2.** For Assumption 2, $f_i(x_i)$ is strongly convex, which implies that the global function $f(x)$ is strongly convex. Moreover, this assumption can guarantee that the optimal solution $\boldsymbol{x}^* = 1_n \otimes x^*$ of (3) is unique. Thus, it is derived that $X^*$ is a single point set.

For each agent $i$, $i \in \mathcal{V}$, $u_i$ is designed as

$$\begin{aligned} u_i &= \dot{\varsigma}_i + \varsigma_i, \\ \dot{q}_i &= \chi\xi \sum_{j\in N_i} a_{ij}\left(x_i - x_j\right), \\ \dot{\varsigma}_i &= -\chi\nabla f_i\left(x_i\right) - \xi \sum_{j\in N_i} a_{ij}[(x_i - x_j) + (q_i - q_j)] \end{aligned} \tag{4}$$

with $q_i(t_0) = q_i(0)$ and $\varsigma_i(t_0) = \varsigma_i(0)$, where $q_i$ and $\varsigma_i$ are the intermediate states, $\nabla f_i\left(x_i\right)$ is the gradient of $f_i(x_i)$, and $\chi$ and $\xi$ are positive constants to be determined.

The intuition behind the algorithm (4) is as follows. For the first equation of (4), the negative gradient term $-\chi\nabla f_i\left(x_i\right)$ drives agent $i$ to search the minimizer of $f_i(x_i)$, $\sum_{j=1}^{n} a_{ij}\left(q_i - q_j\right)$ is viewed as the estimation of the local intermediate states with its neighbors based on an integral feedback and the local communication, and its key role is to eliminate the deviation $\sum_{j=1}^{n} a_{ij}\left(x_i - x_j\right)$. The term $\sum_{j=1}^{n} a_{ij}\left(x_i - x_j\right)$ in the second equation is to ensure that states of agents achieve consensus. The role of the last equation is to compensate the effect of the nonlinear term $g_{i2}\left(z_i, x_i, w_i\right)$ on the first-order optimality condition $\sum_{i=1}^{n} \nabla f_i\left(x^*\right) = 0$ of problem (3) at the optimal solution $x^*$ such that $x_i$ converges to the optimal solution $x^*$ of problem (3), i.e., $g_{i2}\left(z_i^*, x^*, w_i\right) + \varsigma_i^* = 0$, where $z_i^*$ and $\varsigma_i^*$ are the steady states of $z_i$ and $\varsigma_i$, respectively.

Note that $u_i$ in (4) only uses its local information as well as the exchanged information from its in-neighbors via a directed communication topology to agent $i$. The noise we consider is additive and does not require to decay over time. This means that if there is an additive persistent random noise, it will affect not only computations of agents but also communication channels of the topology $\mathcal{G}$ [19].

(1) Computation noise. Any computation performed by each agent is subject to random noise. Particularly, when agent $i$ computes $\nabla f_i\left(x_i\right)$ at time instant $t \geqslant t_0 \geqslant 0$, it actually gets

$$\nabla f_i\left(x_i\right) + \varphi_{ii}\left(x_i, t\right)\sigma_{ii}\left(t\right), \tag{5}$$

where $\sigma_{ii}\left(t\right)$ is an independent copy of $\sigma_{ii}^{0}\left(t\right)$ in the first equation of (2), and $\varphi_{ii}\left(x_i, t\right)$ is the corresponding general-force matrix [22].

(2) Communication noise. Similarly, the communication channels between two different agents are also corrupted by random noise. Specifically, when agent $i$ receives $x_j$ from its in-degree neighbor $j$ via the communication channel $(i, j) \in \mathcal{E}$ at $t \geqslant t_0 \geqslant 0$, agent $i$ ultimately obtains

$$x_j + \varphi_{ij}\left(x_i, t\right)\sigma_{ij}\left(t\right), \quad j \in N_i, \tag{6}$$

where $\sigma_{ij}\left(t\right)$ is also an independent copy of $\sigma_{ii}^{0}\left(t\right)$, and $\varphi_{ij}\left(x_i, t\right)$ also represents the general-force matrix.

As described in [29], with the aid of dynamic-static method and relative-motion principle, the effect of random variations in surroundings to a real system can be equivalently considered as random disturbances to the control protocol. Furthermore, it can be seen that Eq. (4) consists of a three-dimensional dynamics for each agent. As mentioned earlier, the noise $\bar{\Lambda}_i\left(u_i\right)$ is induced by the first equation of (4). Similarly, the noises induced by the second and third equations of (4) are respectively recorded as $\bar{\Lambda}_i\left(\dot{q}_i\right)$ and $\bar{\Lambda}_i\left(\dot{\varsigma}_i\right)$. In what follows, we give their specific expressions and use superscript to distinguish the noise terms associated with the different terms in (4). Particularly, based on (5) and (6), in the first equation of (4), the computation noises corresponding to $\nabla f_i\left(x_i\right)$ and $\varsigma_i$ are respectively $\frac{\xi}{\chi}\varphi_{ii}^{1}\left(x_i, t\right)\sigma_{ii}^{1}\left(t\right)$ and $\varphi_{ii}^{2}\left(\varsigma_i, t\right)\sigma_{ii}^{2}\left(t\right)$ for the convenience of analysis, while the communication noises related to the consensus terms $\sum_{j\in N_i} a_{ij}\left(x_i - x_j\right)$ and $\sum_{j\in N_i} a_{ij}\left(q_i - q_j\right)$ are $\sum_{j\in N_i} a_{ij}\varphi_{ij}^{1}\left(x_i, t\right)\sigma_{ij}^{1}\left(t\right)$ and $\sum_{j\in N_i} a_{ij}\varphi_{ij}^{3}\left(q_i, t\right)\sigma_{ij}^{3}\left(t\right)$, respectively. Similarly, in the second equation of (4), $\sum_{j\in N_i} a_{ij}\left(x_i - x_j\right)$ relevant to communication noise is $\sum_{j\in N_i} a_{ij}\varphi_{ij}^{4}\left(x_i, t\right)\sigma_{ij}^{4}\left(t\right)$. Besides, in the last equation of (4), the computation noise of $\nabla f_i\left(x_i\right)$ is also $\frac{\xi}{\chi}\varphi_{ii}^{5}\left(x_i, t\right)\sigma_{ii}^{5}\left(t\right)$, the communication noises associated with $\sum_{j\in N_i} a_{ij}\left(x_i - x_j\right)$ and $\sum_{j\in N_i} a_{ij}\left(q_i - q_j\right)$ are $\sum_{j\in N_i} a_{ij}\varphi_{ij}^{5}\left(x_i, t\right)\sigma_{ij}^{5}\left(t\right)$ and $\sum_{j\in N_i} a_{ij}\varphi_{ij}^{6}\left(q_i, t\right)\sigma_{ij}^{6}\left(t\right)$, respectively. Furthermore, here the random process $\sigma_{ij}^{k}\left(t\right)$ is also an independent copy of $\sigma_{ii}^{0}\left(t\right)$, and $\varphi_{ij}^{k}\left(\cdot, t\right)$, $k = \bar{0}, 1, 2, \ldots, 6$, are also the general-force matrices associated with the relevant variables.

Based on the above analysis, additive noises $\bar{\Lambda}_i\left(u_i\right)$, $\bar{\Lambda}_i\left(\dot{q}_i\right)$ and $\bar{\Lambda}_i\left(\dot{\varsigma}_1\right)$ that depend on the control (4) are respectively

$$\begin{aligned}
&\bar{\Lambda}_i\left(u_i\right)=-\xi \sum_{j \in N_i} a_{ij}(\varphi_{ij}^1(x_i,t)\sigma_{ij}^1+\varphi_{ij}^3(q_i,t)\sigma_{ij}^3)-\xi \varphi_{ii}^1\left(x_i,t\right)\sigma_{ii}^1\left(t\right)+\xi \varphi_{ii}^2\left(\varsigma_i,t\right)\sigma_{ii}^2\left(t\right), \\
&\bar{\Lambda}_i\left(\dot{q}_i\right)=\chi \xi \sum_{j \in N_i} a_{ij}\varphi_{ij}^4\left(x_i,t\right)\sigma_{ij}^4\left(t\right), \\
&\bar{\Lambda}_i\left(\dot{\varsigma}_i\right)=-\xi \sum_{j \in N_i} a_{ij}(\varphi_{ij}^5(x_i,t)\sigma_{ij}^5+\varphi_{ij}^6(q_i,t)\sigma_{ij}^6)-\xi \varphi_{ii}^5\left(x_i,t\right)\sigma_{ii}^5\left(t\right).
\end{aligned} \tag{7}$$

The underlying complete probability space of random process $\sigma_{ij}^y(t)$ $(y=0,k)$ is taken to be the quartet $(\Omega,\mathcal{F},\mathcal{F}_t,P)$ with a filtration $\mathcal{F}_t$ satisfying the general conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $P$-null sets) [33]. Furthermore, the matrix $\varphi_{ij}^y(\cdot,t)$ satisfies the following assumption.

**Assumption 3** ([33]). $\varphi_{ij}^y(\cdot,t)$ is measurable and uniformly bounded. In addition, there exist two positive constants $L_0$ and $L_1$ such that

$$\begin{aligned}
&\left|\varphi_{ij}^y\left(0,t\right)\right|<L_0, \\
&\left|\varphi_{ij}^y\left(s_1,t\right)-\varphi_{ij}^y\left(s_2,t\right)\right| \leqslant L_1\left|s_1-s_2\right|
\end{aligned} \tag{8}$$

for $s_1,s_2 \in \mathbb{R}$.

Define the target random process as

$$\sigma^y(t)=\begin{bmatrix} \sigma_{11}^y(t) & \cdots & \sigma_{1n}^y(t) \\ \vdots & \vdots & \vdots \\ \sigma_{n1}^y(t) & \cdots & \sigma_{nn}^y(t) \end{bmatrix}$$

which is a second-order stationary process that satisfies the following two assumptions.

**Assumption 4** ([33]). Random process $\sigma^y(t)$ is $\mathcal{F}_t$-adapted and piecewise continuous, and there is a constant $S>0$ such that

$$\sup_{t \geqslant t_0} \mathrm{E}\left\|\sigma^y\left(t\right)\right\|_2^2<S, \quad \forall t_0>0. \tag{9}$$

**Assumption 5** ([33]). For random process $\sigma^y(t)$, for any $\varepsilon_1>0$ and $\delta_1>0$, there exists a $T>t_0 \geqslant 0$ such that $P\{|\frac{1}{t-t_0}\int_{t_0}^t\|\sigma^y(s)\|_2^2\mathrm{d}s-\mathrm{E}\|\sigma^y(t)\|_2^2| \geqslant \delta_1\} \leqslant \varepsilon_1$ for all $t \geqslant T$.

**Remark 3.** Random process $\sigma^y(t)$ is only restricted by Assumptions 4 and 5. Compared with the existing results [19], the weak law of large numbers we used in this study is a relaxed condition, which implies that the target random process is a stationary process, although Assumption 5 is added. It is derived from $P\{|\frac{1}{t-t_0}\int_{t_0}^t\|\sigma^y(s)\|_2^2\mathrm{d}s-\mathrm{E}\|\sigma^y(t)\|_2^2| \geqslant \delta_1\} \leqslant \varepsilon_1$ that the stationary process is a random process, where the probability distribution at a fixed time and location is the same as the probability distribution at all times and locations. Therefore, its mathematical expectations and variances do not change with time and position. In addition, the reason why the effect of vibration environment on system performance is modeled by a stationary process rather than a white noise directly in many applications, for a detailed explanation, please see [29]. The closed-loop system composed of (2), (4) and (7) is based on RDEs; thus, Itô integral and Itô formula cannot be used to analyze its dynamical characteristics [25]. As a result, we need to employ a random field method for developing a criterion to evaluate the quality of DOP solutions for the target MASs driven by a stationary process.

To proceed, the following assumptions related to the first equation of (2) are provided.

**Assumption 6.** Letting $\sigma_{ii}^0(t)=0$ for a given point $x_i=x^*$, there exists a unique point $z_i^*$ satisfying $g_{i1}\left(z_i^*,x^*,w_i\right)=0$.

**Assumption 7.** There is a continuously differentiable function $V_{\bar{z}_i}\left(\bar{z}_i\right)$ that has upper and lower bounds of $\tau_1\left|\bar{z}_i\right|_2^2$ and $\tau_2\left|\bar{z}_i\right|_2^2$ with $\tau_1 \geqslant \tau_2>0$, respectively. In addition, its time derivative along $\dot{\bar{z}}_i=\bar{g}_{i1}\left(\bar{z}_i,\bar{x}_i,w_i\right)+\bar{\varphi}_{ii}^0(\bar{z}_i,t)\sigma_{ii}^0(t)$ satisfies

$$\dot{V}_{\bar{z}_i}\left(\bar{z}_i\right) \leqslant-\hat{\theta}_i\left|\bar{z}_i\right|_2^2+\gamma_i\left|\bar{x}_i\right|_2^2+2\left|\bar{z}_i\bar{\varphi}_{ii}^0(\bar{z}_i,t)\sigma_{ii}^0(t)\right|, \tag{10}$$

where $\bar{g}_{i1}(\bar{z}_i,\bar{x}_i,w_i) = g_{i1}(z_i,x_i,w_i) - g_{i1}(z_i^*,x^*,w_i)$, $\bar{z}_i = z_i - z_i^*$, $\bar{x}_i = x_i - x^*$, $\bar{\varphi}_{ii}^0(\bar{z}_i,t) = \varphi_{ii}^0(z_i,t) - \varphi_{ii}^0(z_i^*,t)$, and $\hat{\theta}_i$ and $\gamma_i$ are positive constants.

Note that Assumption 6 is a crucial condition for obtaining the optimal solution $x^*$ of (3), that is, the state $x_i$ converges to $x^*$ if and only if $z_i$ also converges to $z_i^*$ for any $i \in \mathcal{V}$. We first need Assumption 7 to guarantee that the transformed system $\dot{\bar{z}}_i = \bar{g}_{i1}(\bar{z}_i,\bar{x}_i,w_i) + \bar{\varphi}_{ii}^0(\bar{z}_i,t)\sigma_{ii}^0(t)$ is NOS stable relative to the origin $(0,0)$, and then we only need to analyze the stability of the remainder of system (2). Moreover, Assumption 7 can degrade into a standard assumption to ensure the stability of unmodeled dynamics at the point $(z_i^*,x^*)$ and the effectiveness of feedback control when $\sigma_{ii}^0(t)=0$ [18, 34, 35].

Let $b_{ij} = a_{ij}$ if $(i,j)\in\mathcal{E}$, $b_{ij}=1$ if $i=j$, and

$$\varphi^0(z,t) = \begin{bmatrix} b_{11}\varphi_{11}^0(z_1,t) & \cdots & b_{1n}\varphi_{1n}^0(z_n,t) \\ \vdots & \vdots & \vdots \\ b_{n1}\varphi_{n1}^0(z_1,t) & \cdots & b_{nn}\varphi_{nn}^0(z_n,t) \end{bmatrix},$$

where $\varphi_{ij}^k(\cdot,t)$, $k=\bar{0},1,2,\ldots,6$, are defined in the same way, $z=\mathrm{col}(z_1,z_2,\ldots,z_n)$, $q,\boldsymbol{x},\varsigma,\nabla f(\boldsymbol{x})$, $g_1(z,\boldsymbol{x},w)$ and $g_2(z,\boldsymbol{x},w)$ are defined in the same way. Thus, the closed-loop system composed of (2), (4) and (7) can be rewritten in a compact form:

$$\begin{aligned} &\dot{z} = g_1(z,\boldsymbol{x},w) + \Phi(z,t), \\ &\dot{\boldsymbol{x}} = g_2(z,\boldsymbol{x},w) - \chi\nabla f(\boldsymbol{x}) - \xi\mathcal{L}(\boldsymbol{x}+q) + \varsigma - \xi\left(-1/\xi\Psi(\boldsymbol{x},t) + \Delta(\boldsymbol{x},t) - \Lambda(\varsigma,t) + H(q,t)\right), \\ &\dot{q} = \chi\xi\mathcal{L}\boldsymbol{x} + \chi\xi\Gamma(\boldsymbol{x},t), \\ &\dot{\varsigma} = -\chi\nabla f(\boldsymbol{x}) - \xi\mathcal{L}(\boldsymbol{x}+q) - \xi\left(\Pi(\boldsymbol{x},t) + B(q,t)\right), \end{aligned} \tag{11}$$

where $\Phi(z,t)$, $\Psi(\boldsymbol{x},t)$, $\Delta(\boldsymbol{x},t)$, $\Lambda(\varsigma,t)$, $H(q,t)$, $\Gamma(\boldsymbol{x},t)$, $\Pi(\boldsymbol{x},t)$ and $B(q,t)$ are the column vectors stacked by the main diagonal elements of matrices $\varphi^0(z,t)(\sigma^0(t))^{\mathrm{T}}$, $\varphi^{\bar{0}}(\boldsymbol{x},t)(\sigma^{\bar{0}}(t))^{\mathrm{T}}$, $\varphi^1(\boldsymbol{x},t)(\sigma^1(t))^{\mathrm{T}}$, $\varphi^2(\varsigma,t)(\sigma^2(t))^{\mathrm{T}}$, $\varphi^3(q,t)(\sigma^3(t))^{\mathrm{T}}$, $\varphi^4(\boldsymbol{x},t)(\sigma^4(t))^{\mathrm{T}}$, $\varphi^5(\boldsymbol{x},t)(\sigma^5(t))^{\mathrm{T}}$ and $\varphi^6(q,t)(\sigma^6(t))^{\mathrm{T}}$, respectively.

## 3 Noise-to-state exponential stability analysis

In this section, the NOS exponential stability of system (11) is analyzed using a random field method. Particularly, the relationship between the optimal solution of (11) without a stationary process and the NOS stable solution of system (11) in the second moment is discussed. The following definition relative to the NOS stability of random systems is provided.

**Definition 1** ([33])**.** System $\dot{x}(t) = f(x(t),t,\sigma(t)) \in \mathbb{R}^n$ is said to be NOS stable in the second moment if there exist a class-$\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and a class-$\mathcal{K}$ function $\rho(\cdot)$ such that

$$\mathrm{E}\|x(t)\|_2^2 \leqslant \beta(\|x_0\|_2, t-t_0) + \rho\left(\sup_{t_0\leqslant s\leqslant t}\mathrm{E}\|\sigma(s)\|_2^2\right) \tag{12}$$

for $x_0\in\mathbb{R}^n$ and $t\in[t_0,\infty)$, where $\sigma(t)$ is a stationary process satisfying Assumptions 4 and 5. Furthermore, system $\dot{x} = f(x(t),t,\sigma(t))$ is said to be exponentially NOS stable in the second moment if the class-$\mathcal{KL}$ function $\beta(\cdot,\cdot)$ can be represented by $h_1\|x_0\|_2 \mathrm{e}^{-h_2(t-t_0)}$ for two positive constants $h_1$ and $h_2$.

The following lemma is about the existence and uniqueness of the global solution for the closed-loop system (11).

**Lemma 1** ([33])**.** Under Assumptions 3–5, system (11) has a unique solution $\tilde{x}(t)$ on $[t_0,\infty)$.

**Remark 4.** It is obtained from Definition 1 and Lemma 1 that the solution $\tilde{x}(t)$ of problem (3) is unique and NOS stable in the second moment, i.e., $\mathrm{E}\|x(t)-\tilde{x}(t)\|_2^2 \leqslant \beta(\|x_0\|_2, t-t_0) + \rho(\sup_{t_0\leqslant s\leqslant t}\mathrm{E}\|\sigma(s)\|_2^2)$. Thus, we know that the second moment of the estimation error between the optimal solution of system (11) without random noises and the NOS stable solution of system (11) is bounded. Note that the size of the second moment of the estimation error is dependent on the initial states and the parameters of the concerned system as well as the magnitude of random noise. Nevertheless, an oversized upper bound for analyzing the stability of (11) is insignificant. Thereby, we need to employ the grid search method for finding the appropriate parameters to decrease the estimation error concerned.

The following lemmas related to the main results are provided.

**Lemma 2** ([36])**.** Suppose that $\tilde{y}(t)$ is an absolutely continuous function for $t \geqslant t_0$ and its derivative satisfies the inequality $\dot{\tilde{y}}(t) \leqslant e(t)\tilde{y}(t) + m(t)$ for almost all $t \geqslant t_0 \geqslant 0$, where $e(t)$ and $m(t)$ are almost everywhere continuous and integrable functions over every finite interval. Thus, the inequality $\tilde{y}(t) \leqslant \tilde{y}(t_0)\mathrm{e}^{\int_{t_0}^{t} e(s)\mathrm{d}s} + \int_{t_0}^{t} \mathrm{e}^{\int_{s}^{t} e(c)\mathrm{d}c} m(s)\mathrm{d}s$ holds for $t \geqslant t_0 \geqslant 0$.

**Lemma 3** ([37])**.** Given two vectors $\tilde{y}_1$ and $\tilde{y}_2$ with compatible dimensions, the inequality $\tilde{y}_1^{\mathrm{T}}\tilde{y}_2 \leqslant r\|\tilde{y}_1\|_2^2 + \|\tilde{y}_2\|_2^2/4r$ holds for any positive scalar $r$.

When a system is disturbed by a stationary process, there is no exactly optimal solution of the target problem. For this case, we need to develop a new evaluation criterion for the quality of DOP solutions (3). To this end, assume that the equilibrium point of system (11) in the absence of a stationary process is $\mathrm{col}(z^*, \boldsymbol{x}^*, q^*, \varsigma^*)$. Define new coordinate transformations as

$$\vartheta = Q^{\mathrm{T}}\bar{z}, \quad \nu = Q^{\mathrm{T}}\bar{\boldsymbol{x}}, \quad \phi = Q^{\mathrm{T}}\bar{q}, \quad \eta = Q^{\mathrm{T}}\bar{\varsigma}, \tag{13}$$

where $Q = [Q_1, \ Q_2]$ is defined in Remark 1, $\bar{z} = z - z^*$, $\bar{\boldsymbol{x}} = \boldsymbol{x} - \boldsymbol{x}^*$, $\bar{q} = q - q^*$, $\bar{\varsigma} = \varsigma - \varsigma^* - \bar{\boldsymbol{x}}$, $\vartheta = [\vartheta_1, \ \vartheta_2^{\mathrm{T}}]^{\mathrm{T}}, \nu = [\nu_1, \ \nu_2^{\mathrm{T}}]^{\mathrm{T}}, \phi = [\phi_1, \ \phi_2^{\mathrm{T}}]^{\mathrm{T}}, \eta = [\eta_1, \ \eta_2^{\mathrm{T}}]^{\mathrm{T}}$ with $\vartheta_1, \ \nu_1, \ \phi_1, \ \eta_1 \in \mathbb{R}$ and $\vartheta_2, \ \nu_2, \ \phi_2, \ \eta_2 \in \mathbb{R}^{n-1}$. Thus, system (11) can be equivalently decomposed to the following form:

$$\begin{aligned}
&\dot{\vartheta} = Q^{\mathrm{T}}\bar{g}_1 + Q^{\mathrm{T}}\bar{\Phi}(\bar{z}, t), \\
&\dot{\nu}_1 = Q_1^{\mathrm{T}}\bar{g}_2 - \chi Q_1^{\mathrm{T}} h(\bar{\boldsymbol{x}}) + \eta_1 + \nu_1 - \xi Q_1^{\mathrm{T}}\left(\bar{\Delta}(\bar{\boldsymbol{x}}, t) + \bar{H}(\bar{q}, t) - \bar{\Lambda}(\bar{\varsigma}, t)\right), \\
&\dot{\nu}_2 = Q_2^{\mathrm{T}}\bar{g}_2 - \chi Q_2^{\mathrm{T}} h(\bar{\boldsymbol{x}}) - \xi J(\phi_2 + \nu_2) + \eta_1 + \nu_1 - \xi Q_2^{\mathrm{T}}\left(-1/\xi\Psi(\bar{\boldsymbol{x}}, t) + \bar{\Delta}(\bar{\boldsymbol{x}}, t) + \bar{H}(\bar{q}, t) - \bar{\Lambda}(\bar{\varsigma}, t)\right), \\
&\dot{\phi}_1 = \chi\xi Q_1^{\mathrm{T}}\bar{\Gamma}(\bar{\boldsymbol{x}}, t), \\
&\dot{\phi}_2 = \chi\xi J\nu_2 + \chi\xi Q_2^{\mathrm{T}}\bar{\Gamma}(\bar{\boldsymbol{x}}, t), \\
&\dot{\eta}_1 = -Q_1^{\mathrm{T}}\bar{g}_2 - \eta_1 - \nu_1 - \xi Q_1^{\mathrm{T}}\hat{\Gamma}(\cdot)\sigma'(t), \\
&\dot{\eta}_2 = -Q_2^{\mathrm{T}}\bar{g}_2 - \eta_2 - \nu_2 - \xi Q_2^{\mathrm{T}}\hat{\Gamma}(\cdot)\sigma'(t),
\end{aligned} \tag{14}$$

where $\bar{g}_1 = g_1(z, \boldsymbol{x}, w) - g_1(z^*, \boldsymbol{x}^*, w)$, $\bar{g}_2 = g_2(z, \boldsymbol{x}, w) - g_2(z^*, \boldsymbol{x}^*, w)$, $h(\bar{\boldsymbol{x}}) = \nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{x}^*)$, $\bar{\Phi}(\bar{z}, t) = \Phi(z, t) - \Phi(z^*, t)$, $\bar{\Psi}(\bar{\boldsymbol{x}}, t) = \Psi(\boldsymbol{x}, t) - \Psi(\boldsymbol{x}^*, t)$, $\bar{\Delta}(\bar{\boldsymbol{x}}, t) = \Delta(\boldsymbol{x}, t) - \Delta(\boldsymbol{x}^*, t)$, $\bar{H}(\bar{q}, t) = H(q, t) - H(q^*, t)$, $\bar{\Lambda}(\bar{\varsigma}, t) = \Lambda(\varsigma, t) - \Lambda(\varsigma^* + \bar{\boldsymbol{x}}, t)$, $\bar{\Gamma}(\bar{\boldsymbol{x}}, t) = \Gamma(\boldsymbol{x}, t) - \Gamma(\boldsymbol{x}^*, t)$, $\bar{\Pi}(\bar{\boldsymbol{x}}, t) = \Pi(\boldsymbol{x}, t) - \Pi(\boldsymbol{x}^*, t)$, $\bar{B}(\bar{q}, t) = B(q, t) - B(q^*, t)$ and $\hat{\Gamma}(\cdot)\sigma'(t) = \bar{\Pi}(\bar{\boldsymbol{x}}, t) + \bar{B}(\bar{q}, t) + 1/\xi\Psi(\bar{\boldsymbol{x}}, t) - \bar{\Delta}(\bar{\boldsymbol{x}}, t) - \bar{H}(\bar{q}, t) + \bar{\Lambda}(\bar{\varsigma}, t)$. According to Assumption 1 and Remark 1, it is derived that $\phi_1 \equiv 0$ when system (11) is not affected by a random process.

**Theorem 1.** Under Assumptions 1–7, there exist positive constants $l_1, l_2, c_1, c_2, c_3, c_4, p_1, \xi, \chi$ and the diagonal positive-definite matrix $\hat{P} \in \mathbb{R}^{(n-1)\times(n-1)}$ such that the solution of system (14) is exponentially NOS stable in the second moment if the following condition holds:

$$-P_1\theta + l_1c_4I_n < 0, \quad \Omega = \mathrm{diag}\left\{-P_1\hat{\theta} + l_1c_4I_n, \ \Omega^*\right\} < 0 \tag{15}$$

with

$$\Omega^* = \begin{bmatrix} \Omega_{11} & 0 & 0 & \frac{l_1}{2} & 0 \\ * & \Omega_{22} & \Omega_{23} & 0 & \frac{l_1\chi I_{n-1}}{2} - \frac{l_2\xi J}{2} \\ * & * & \Omega_{33} & 0 & \frac{l_1 I_{n-1}}{2} - \frac{l_2\xi J}{2} \\ * & * & * & \Omega_{44} & 0 \\ * & * & * & * & \Omega_{44}I_{n-1} \end{bmatrix},$$

where $\Omega_{11} = l_1c_1 + l_1(\chi+1) + (\frac{l_1\chi}{4} + \frac{l_2\chi}{4})\bar{\nu}^2 - \chi(l_1 + l_2 + l_1\chi)\varpi + p_1\gamma_1$, $\Omega_{22} = (\Omega_{11} - p_1\gamma_1)I_{n-1} - \xi l_2 J - \chi l_1 J + \hat{P}\gamma$, $\Omega_{23} = -\chi\xi l_1 J - \xi l_2 J + l_1 I_{n-1}/2$, $\Omega_{33} = c_3 l_1 I_{n-1} + \chi l_1 I_{n-1} - l_1\xi J$, $\Omega_{44} = c_2 l_1 + \chi l_2 - l_1$, $\hat{P} = \mathrm{diag}\{p_2, \ldots, p_n\}$, $P_1 = \mathrm{diag}\{p_1, \hat{P}\}$, $\hat{\theta} = \mathrm{diag}\{\hat{\theta}_1, \ldots, \hat{\theta}_n\}$ and $\gamma = \mathrm{diag}\{\gamma_2, \ldots, \gamma_n\}$.

*Proof.* Consider the Lyapunov function candidate:

$$V(\psi) = \psi_1^{\mathrm{T}}\Theta\psi_1 + \sum_{i=1}^{n} p_i V_{\bar{z}_i}(\bar{z}_i) \tag{16}$$

with

$$\Theta=\begin{bmatrix} l+\frac{l_1\chi}{2} & 0 & 0 & \frac{1}{2}l_2 & 0\\ * & (l+\frac{l_1\chi}{2})I_{n-1} & \frac{l_1}{2}I_{n-1} & 0 & \frac{l_2}{2}I_{n-1}\\ * & * & \frac{l_1}{\chi}I_{n-1} & 0 & 0\\ * & * & * & l & 0\\ * & * & * & * & lI_{n-1}\end{bmatrix},$$

where $\psi=\mathrm{col}\,(\vartheta,\psi_1)$ with $\psi_1=\mathrm{col}(\nu_1,\nu_2,\phi_2,\eta_1,\eta_2)$, $l=\frac{l_1+l_2}{2}$, $V_{\bar{z}_i}\,(\bar{z}_i)$ is defined in Assumption 7 and $p_i$ is a positive constant.

Thus, it is obtained that

$$\hbar_1\left\|\psi\right\|_2^2\leqslant V(\psi)\leqslant\hbar_2\left\|\psi\right\|_2^2, \tag{17}$$

where

$$\hbar_1\left\|\psi\right\|_2^2=\lambda_{\min}\left(\Theta\right)\left\|\psi_1\right\|_2^2+\tau_2\left[p_i\right]_{i\in\mathcal{V}}^{\min}\left\|\vartheta\right\|_2^2,$$
$$\hbar_2\left\|\psi\right\|_2^2=\lambda_{\max}\left(\Theta\right)\left\|\psi_1\right\|_2^2+\tau_1\left[p_i\right]_{i\in\mathcal{V}}^{\max}\left\|\vartheta\right\|_2^2.$$

Taking the derivative of (16) along the solutions of (14) yields

$$\begin{aligned}\dot{V}(\psi)\leqslant{}& l_1((1+\chi)Q\nu-Q\eta+Q_2\phi_2)^{\mathrm{T}}\bar{g}_2+l_1\left(\chi+1\right)\nu^{\mathrm{T}}\nu-l_2\chi\eta_2^{\mathrm{T}}J\nu_2-\xi l_2\nu_2^{\mathrm{T}}J\left(\nu_2+\phi_2\right)\\ &-l_1\chi\nu_2^{\mathrm{T}}J\nu_2+l_1\eta_2^{\mathrm{T}}\phi_2-\chi(l_1Q\nu+l_2Q\nu+l_2Q\eta+l_1\chi Q\nu+l_2Q_2\phi_2)^{\mathrm{T}}h\left(\bar{x}\right)\\ &-\chi\xi l_1\nu_2^{\mathrm{T}}J\phi_2+p_1\gamma_1\nu_1^{\mathrm{T}}\nu_1+l_1\chi\nu^{\mathrm{T}}\eta-\xi l_2\eta_2^{\mathrm{T}}J\phi_2-l_1\xi\phi_2^{\mathrm{T}}J\phi_2+l_1\phi_2^{\mathrm{T}}\nu_2-\vartheta^{\mathrm{T}}P_1\hat{\theta}\vartheta\\ &+\gamma\nu_2^{\mathrm{T}}\hat{P}\nu_2-l_1\eta^{\mathrm{T}}\eta-2\psi_1^{\mathrm{T}}\Theta M\left(\psi_1,t\right)+\sum_{i=1}^{n}2p_i\left|\bar{z}_i\bar{\varphi}_{ii}^0(\bar{z}_i,t)\sigma_{ii}^0(t)\right|,\end{aligned} \tag{18}$$

where matrices $\hat{P}$, $P_1$, $\hat{\theta}$ and $\gamma$ are defined in (15), $M\left(\psi_1,t\right)=\mathrm{col}(Q_1^{\mathrm{T}}M_1,Q_2^{\mathrm{T}}M_1,-\chi Q_2^{\mathrm{T}}\bar{\Gamma}\left(\bar{\boldsymbol{x}},t\right),$ $Q_1^{\mathrm{T}}\hat{\Gamma}\left(\cdot\right)\sigma'\left(t\right),Q_2^{\mathrm{T}}\hat{\Gamma}\left(\cdot\right)\sigma'\left(t\right))$ with $M_1=-1/\xi\Psi(\bar{\boldsymbol{x}},t)+\bar{\Delta}\left(\bar{\boldsymbol{x}},t\right)+\bar{H}\left(\bar{q},t\right)-\bar{\Lambda}\left(\bar{\varsigma},t\right)$. Note that $Q^{\mathrm{T}}Q=I_n$ and $Q_2^{\mathrm{T}}Q_2=I_{n-1}$. It is derived from Assumption 2 and Lemma 3 that

$$\begin{aligned}&-\left(Q_2\phi_2\right)^{\mathrm{T}}h\left(\bar{\boldsymbol{x}}\right)\leqslant\phi_2^{\mathrm{T}}\phi_2+0.25\bar{\nu}^2\nu^{\mathrm{T}}\nu,\\ &-\left(Q\eta\right)^{\mathrm{T}}h\left(\bar{\boldsymbol{x}}\right)\leqslant\eta^{\mathrm{T}}\eta+0.25\bar{\nu}^2\nu^{\mathrm{T}}\nu,\\ &\left(Q\nu\right)^{\mathrm{T}}h\left(\bar{\boldsymbol{x}}\right)\geqslant\varpi\nu^{\mathrm{T}}\nu,\end{aligned} \tag{19}$$

and

$$\begin{aligned}l_1((1+\chi)Q\nu-Q\eta+Q_2\phi_2)^{\mathrm{T}}\bar{g}_2&=l_1((1+\chi)Q\nu-Q\eta+Q\phi)^{\mathrm{T}}\bar{g}_2\\ &\leqslant l_1\left(c_1\nu^{\mathrm{T}}\nu+c_2\eta^{\mathrm{T}}\eta+c_3\phi_2^{\mathrm{T}}\phi_2+c_4\vartheta^{\mathrm{T}}\vartheta\right)\end{aligned} \tag{20}$$

hold for some positive constants $c_1$, $c_2$, $c_3$ and $c_4$. The above inequalities are obtained by using Lemma 7.8 in [35] and $\phi_1\equiv0$.

According to Assumption 3 and (17), there are two positive constants $d_1$ and $d_2$ such that

$$-2\psi_1^{\mathrm{T}}\Theta M\left(\psi_1,t\right)+\sum_{i=1}^{n}2p_i\left|\bar{z}_i\bar{\varphi}_{ii}^0(\bar{z}_i,t)\sigma_{ii}^0(t)\right|\leqslant2d_1\left(\hbar_2+L_1^2\right)\left\|\psi\right\|_2^2+2d_2\|\hat{\sigma}\left(t\right)\|_2^2, \tag{21}$$

where $\hat{\sigma}\left(t\right)=\mathrm{col}(\sigma^0(t),\sigma^k(t))$, $k=\bar{0},1,2,\ldots,6$. Furthermore, it is obtained from (18), (19), (20) and (21) that

$$\dot{V}\leqslant\psi^{\mathrm{T}}\Omega\psi+2d_1\left(\hbar_2+L_1^2\right)\left\|\psi\right\|_2^2+2d_2\|\hat{\sigma}\left(t\right)\|_2^2\leqslant\left(\lambda_{\max}\left(\Omega\right)+2d_1\left(\hbar_2+L_1^2\right)\right)\left\|\psi\right\|_2^2+2d_2\|\hat{\sigma}\left(t\right)\|_2^2, \tag{22}$$

where matrix $\Omega$ defined in (15) is negative-definite, and $\lambda_{\max}(\Omega)$ denotes its maximum eigenvalue. For obtaining the condition of the NOS exponential stability of system (14), let $\lambda_{\max}(\Omega)+2d_1\left(\hbar_2+L_1^2\right)\leqslant -d_3$ hold with a positive constant $d_3$, and then, the equation $\lambda_{\max}(\Omega)+2d_1\left(\hbar_2+L_1^2\right)+d_3=-\kappa$ holds for a positive constant $\kappa$. Owing to $2d_1\left(\hbar_2+L_1^2\right)>0$ and $d_3>0$, using the mean value inequality in [38], $\sqrt{2d_1d_3\left(\hbar_2+L_1^2\right)}\leqslant\frac{1}{2}\left(2d_1\left(\hbar_2+L_1^2\right)+d_3\right)<-\frac{1}{2}\lambda_{\max}(\Omega)$ means that $d_1d_3\leqslant\frac{1}{8\left(\hbar_2+L_1^2\right)}\lambda_{\max}^2(\Omega)$, where the equality holds when $d_3=-\lambda_{\max}(\Omega)/2$ and $d_1=-\lambda_{\max}(\Omega)/4\left(\hbar_2+L_1^2\right)$. In what follows, substituting (17) into (22) leads to

$$\dot{V}(\psi(t))\leqslant-\frac{\kappa}{\hbar_2}V(\psi(t))+2d_2\left\|\hat{\sigma}(t)\right\|_2^2. \tag{23}$$

Taking integrals first in the time-interval $[t_0,t\wedge\upsilon_b)$ and then expectations on both sides of (23), we have

$$\mathrm{E}V(\psi(t\wedge\upsilon_b))-V(\psi(0))\leqslant-\frac{\kappa}{\hbar_2}\mathrm{E}\int_{t_0}^{t\wedge\upsilon_b}V(\psi(s))\,\mathrm{d}s+2d_2\mathrm{E}\int_{t_0}^{t}\left\|\hat{\sigma}(s)\right\|_2^2\mathrm{d}s, \tag{24}$$

where stopping time $\upsilon_b=\inf\{t\geqslant t_0|\,\|\psi(t)\|_2^2\geqslant b\}$ for any $b>0$, $\upsilon_\infty=\lim_{b\to\infty}\upsilon_b=\infty$ and $\inf\emptyset=\infty$ almost surely.

Combining (9) with (24) yields

$$\mathrm{E}V(\psi(t\wedge\upsilon_b))\leqslant V(\psi(0))+16d_2St\leqslant(V(\psi(0))+16d_2)\,St. \tag{25}$$

Therefore, based on Lemma 5 (note that we replace $\mathrm{e}^{ct}$ with $St$ owing to Assumption 6) in [33], the existence of solutions of system (14) on the time interval $[t_0,\infty)$ is guaranteed, that is, $\upsilon_\infty=\infty$ almost surely. It follows from $\upsilon_\infty=\infty$ almost surely and (23) that $V(\psi(t))<\infty$ and $\dot{V}(\psi(t))<\infty$. In light of Fubini's theorem [39], the equation $\mathrm{E}\frac{\mathrm{d}V(\psi)}{\mathrm{d}t}=\frac{\mathrm{dE}V(\psi)}{\mathrm{d}t}$ holds, which means the exchangeability of expectation and derivative. According to Lemma 2, it is derived from (23) that $\mathrm{E}V(\psi(t))\leqslant V(\psi(0))\mathrm{e}^{-\kappa/\hbar_2(t-t_0)}+2d_2\hbar_2/\kappa(1-\mathrm{e}^{-\kappa/\hbar_2(t-t_0)})\sup_{t_0\leqslant s\leqslant t}\mathrm{E}\left\|\hat{\sigma}(s)\right\|_2^2\leqslant V(\psi(0))\mathrm{e}^{-\kappa/\hbar_2(t-t_0)}+\mathrm{e}^{-\kappa/\hbar_2(t-t_0)}\sup_{t_0\leqslant s\leqslant t}\mathrm{E}\left\|\hat{\sigma}(s)\right\|_2^2$, which, together with (17), implies that $\mathrm{E}\left\|\psi(t)\right\|_2^2\leqslant\hbar_2/\hbar_1\left\|\psi(0)\right\|_2^2\mathrm{e}^{-\kappa/\hbar_2(t-t_0)}+(2\hbar_2^2d_2/\hbar_1\kappa)\sup_{t_0\leqslant s\leqslant t}\mathrm{E}\left\|\hat{\sigma}(s)\right\|_2^2$. Furthermore,

$$\mathrm{E}\left\|\boldsymbol{x}(t)-\boldsymbol{x}^*\right\|_2^2\leqslant\hbar_2/\hbar_1\left\|x(0)\right\|_2^2\mathrm{e}^{-\kappa/\hbar_2(t-t_0)}+(2\hbar_2^2d_2/\hbar_1\kappa)\sup_{t_0\leqslant s\leqslant t}\mathrm{E}\left\|\hat{\sigma}(s)\right\|_2^2. \tag{26}$$

Clearly, $\frac{\hbar_2}{\hbar_1}\left\|x(0)\right\|_2^2\mathrm{e}^{-\frac{\kappa}{\hbar_2}(t-t_0)}$ is a class-$\mathcal{KL}$ function and $\frac{2d_2\hbar_2^2}{\hbar_1\kappa}\sup_{t_0\leqslant s\leqslant t}\mathrm{E}\left\|\hat{\sigma}(s)\right\|_2^2$ is a class-$\mathcal{K}$ function. According to Definition 1 and Lemma 1, it is derived from (26) that system (14) is exponentially NOS stable in the second moment. So the solution of problem (3) is exponentially NOS stable in the second moment. It is further derived that the algorithm (4) can solve problem (3).

**Remark 5.** In Theorem 1, the sufficient condition (15) corresponding to $\mathrm{E}\left\|\boldsymbol{x}(t)-\boldsymbol{x}^*\right\|_2^2$ is obtained in the form of matrix inequalities. Despite the computational time being lost by solving matrix inequalities and performing the Jordan decomposition associated with the Laplacian matrix $\mathcal{L}$, conservatism of the established criterion is alleviated. Note that the determination for value range of parameters in system (11) must be prior to the grid search method employment, which will accelerate the acquisition of the best optimization parameters and decrease the estimation error of the solution at the same time.

**Remark 6.** Compared with the existing results [13–15,19,21], this paper has the following advantages.

(1) For the DOP, the dynamics of agents are integral-type systems and they do not consider the effect of external surroundings on system characteristics [13–15,21]. However, the dynamics of agents in this paper is extended to a class of nonlinear systems disturbed by a stationary process, which is very appropriate to model noisy disturbances in practice, and it is different from that by a white noise process in [19].

(2) Although a co-coercivity of vector field method is employed to select the optimal parameters of system based on stochastic differential equations in [19] to tight the minimum of the second moment of the estimation error of the solution, this method is very complicated, and the obtained parameters are some functions defined on a bounded interval. In addition, this method do not directly reflect the relationship between the system parameters and the magnitude of the estimation error of the DOP solutions. On the contrary, we use the grid search method, which is easily implemented in real applications, to determine the optimal parameters of the system for reducing the estimation error concerned. Moreover, the figures

depicted using this method can directly reflect the relationship between the selected parameters and the size of the estimation error of the solution.

(3) The obtained results can be applied to the average consensus problem for a class of random nonlinear multi-agent systems. Thus, our results provide an alternative solution to achieve average consensus in the presence of stationary processes.

## 4 Applications to the average consensus problem in the presence of a stationary process

For distributed collaborative control of multi-agent systems, the DOP is considered an extension of the consensus problem by adding a global cost function. The average consensus problem is a special issue of the consensus problem in which its solution equals an average value of all initial states of agents. In this section, we show that the results in Section 3 can be applied to the average consensus problem for random nonlinear systems (2) by modifying the corresponding algorithm. To be specific, we consider the problem given in the following definition.

**Definition 2** ([40])**.** If states of agents converge to the average of their initial states for the target system, i.e., $\lim_{t\to\infty} x_i(t) = x^* = \frac{1}{n}\sum_{j=1}^{n} x_j(0),\ i \in \mathcal{V},\ \forall t \geqslant 0$, then average consensus is said to be reached.

In order to solve the problem given in Definition 2, the proposed algorithm (4) can be modified as

$$\begin{aligned} u_i &= \dot{\varsigma}_i + \varsigma_i, \\ \dot{q}_i &= \chi\xi \sum_{j\in N_i} a_{ij}(x_i - x_j), \\ \dot{\varsigma}_i &= -\chi(x_i - x_i(0)) - \xi \sum_{j\in N_i} a_{ij}[(x_i - x_j) + (q_i - q_j)], \end{aligned} \tag{27}$$

where $x_i(0)$ is the initial state of $x_i$, and other variables and notations are the same as those in (4).

**Remark 7.** It is worth noting that the difference between the proposed algorithms (27) and (4) is the gradient term. The gradient term is $\nabla f_i(x_i)$ in (4), but it is replaced with $x_i - x_i(0)$ in (27). It is derived from the first-order optimality condition $\sum_{i=1}^{n} \nabla f_i(x^*) = 0$ that $x^* = \lim_{t\to\infty} x_i(t) = 1/n\sum_{j=1}^{n} x_j(0)$, which implies that average consensus is achieved.

Applying Theorem 1 to the distributed average consensus problem for random nonlinear multi-agent systems (2), we have the following result.

**Theorem 2.** Under Assumptions 1–7, there are positive constants $l_1$, $l_2$, $c_1$, $c_2$, $c_3$, $c_4$, $p_1$, $b_1$, $b_2$, $\xi$, $\chi$, and the diagonal positive-definite matrix $\hat{P}$ such that the solution of the random nonlinear system composed of (2), (27) and (7) is exponentially NOS stable in the second moment under the condition (15).

*Proof.* For each agent $i$, let its local cost function be

$$f_i(x_i) = 0.5(x_i - x_i(0))^2, \tag{28}$$

which satisfies Assumption 2. We transform the average consensus problem into the DOP with the form of (3):

$$\min_{\boldsymbol{x}\in R^n} F(\boldsymbol{x}) = \sum_{i=1}^{n} 0.5(x_i - x_i(0))^2 \quad \text{s.t.} \quad \lim_{t\to\infty} \mathrm{E}\,|x_i - x_j| = 0. \tag{29}$$

For problem (29), when the effect of the environment on system performance is ignored, its minimum point is $x^* = \frac{1}{n}\sum_{j=1}^{n} x_j(0)$. Moreover, suppose that Assumptions 1, 3–7 hold, using similar proof idea to that in Theorem 1, we prove that the system composed by (2), (27) and (7) is exponentially NOS stable in the second moment. It is derived from Lemma 1 that the solution of the problem (29) is exponentially NOS stable in the second moment. Therefore, it is obtained that the algorithm (27) solves the distributed average consensus problem given in Definition 2 for random nonlinear multi-agent systems (2).

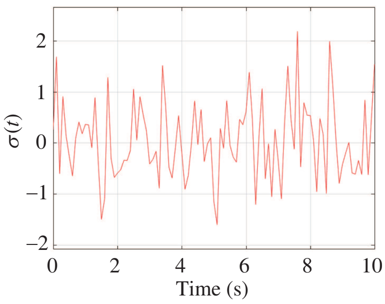


**Figure 1** (Color online) A curve of a second-order stationary process $\sigma(t)$.

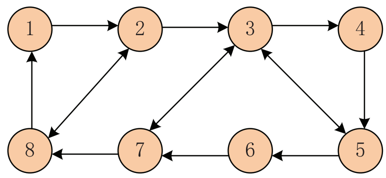


**Figure 2** (Color online) The communication topology among eight agents.

## 5 Simulation

In this section, an example is provided to illustrate the effectiveness of the proposed algorithm. Consider a multi-agent system consisting of 8 agents that operates in a random vibration environment, where the dynamics of each agent is modeled by a coupled FitzHugh-Nagumo type system (2) [26] with $g_{i1}(z_i, x_i, w_i) = \hat{\tau}_{i1}(x_i - \hat{\tau}_{i2} z_i)$ and $g_{i2}(z_i, x_i, w_i) = x_i(x_i - \hat{\tau}_{i3})(1 - x_i)$, where $\hat{\tau}_{ij} = \tau'_{ij} + w_i > 0$, $i = 1, \ldots, 8$, $j = 1, 2, 3$, are some real coefficients with the uncertainty $w_i$ influencing on the nominal value $\tau'_{ij}$, and $\hat{\tau}_i = (\hat{\tau}_{i1}, \hat{\tau}_{i2}, \hat{\tau}_{i3}) = (1, 2, 1)$. Random process $\sigma^y(t)$ $(y = 0, k)$ is a second-order stationary process. We use MATLAB to generate a desired stationary process. Particularly, the target random process is a bandlimited white noise instead of pure white noise [33]. Based on the fact, it is physically achievable and can be generated by a module of Band-Limited White Noise in Simulink. Set the noise power $A = 0.05$ and the correlation time $t_c = 0.1$, and then $\mathrm{E}\left\|\sigma^y(t)\right\|_2^2 = A/t_c = 0.5$. Thus, the stationary process that satisfies Assumptions 4 and 5 is depicted in Figure 1. The communication topology among agents is modeled by a strongly-connected and weight-balanced digraph as shown in Figure 2, which implies that Assumption 1 is satisfied. In addition, let the weight of each edge be $a_{ij} = 1$ if $(i, j) \in \mathcal{E}$. The objective of all agents is to cooperatively solve an optimization problem in the form of (3) based on a multi-agent framework, where the local cost functions associated with each agent are respectively

$$f_1(x_1) = 0.8(x_1 - 2)^2, \; f_2(x_2) = 2x_2^2 + 3x_2 + 1, \; f_3(x_3) = 3\mathrm{e}^{0.3x_3}, \; f_4(x_4) = 3x_4^2 + 0.5\mathrm{e}^{2x_4} + 6,$$
$$f_5(x_5) = \frac{x_5^2 + 3}{\sqrt{x_5^2 + 1}} + x_5^2, \; f_6(x_6) = 0.8\mathrm{e}^{2x_6} + 2x_6, \; f_7(x_7) = 4x_7^2 - 6x_7, \; f_8(x_8) = 2\mathrm{e}^{-3x_8} + 2x_8^2 + 2x_8.$$

Each $f_i(x_i)$ is strongly convex, which implies that Assumption 2 is satisfied. Assume that Assumptions 6 and 7 associated with nonlinear system (2) are also satisfied. The initial state of $\boldsymbol{x}$ is set to $\boldsymbol{x}(0) = [1, 2, 2, -2, 3, 1, 0, 0.5]^{\mathrm{T}}$, and the initial states of the remainder of variables can be arbitrarily chosen. After calculation, it is easy to know that the state $x_i$ converges to the optimal solution $x^* = 0.1065$ of problem (3) in the absence of a stationary process, and the minimum value of the global cost function is $F(\boldsymbol{x}^*) = 19.2643$.

Next, consider the optimization problem (3) in the presence of a stationary process. According to Theorem 1, the grid search method is employed to depict the relationship between the second moment $\mathrm{E}\|\boldsymbol{x}(t) - \boldsymbol{x}^*\|_2^2$ of the estimation error and parameters $\chi$, $\xi$ as shown in Figure 3. Select two pairs of parameters ($\chi = 2.45$, $\xi = 1.38$) and ($\chi = 2.45$, $\xi = 0.72$) from Figure 3 to perform the simulation analysis, and let the general-force matrices $\varphi^0(z, t) = \varphi^1(x, t) = \varphi^2(\varsigma, t) = \varphi^3(q, t) = \varphi^4(x, t) = \varphi^5(x, t) = \varphi^6(q, t)$. Here, we only present a detailed description of the components of $\varphi^1(x, t)$ for the sake of brevity, $\varphi^1_{ii}(x_i, t) = 0.05 + r^*\sin(t)$ if $i = j$, and $\varphi^1_{ij}(x_i, t) = 0.05 + 0.5r^*\sin(t)$ if $(i, j) \in \mathcal{E}$, where $r^*$ is randomly selected in $[0, 0.1]$. It is derived that Assumption 7 is satisfied with two constants $L_0 = 0.15$ and $L_1 = 0.05$. It is observed from Figure 4 that the trajectories of state $x_i$ of agent $i$ converge to a bounded neighborhood of the optimal solution of problem (3). The reason for this case is that the dynamics of agents are disturbed by a stationary process. In addition, it is also derived that when we focus on the minimum of the second moment $\mathrm{E}\|\boldsymbol{x}(t) - \boldsymbol{x}^*\|_2^2$ of the estimation error, the parameter $\xi$ is smaller. This further illustrates the effectiveness and correctness of the obtained results.

Finally, we show that the obtained results can be applied to average consensus for a multi-agent system

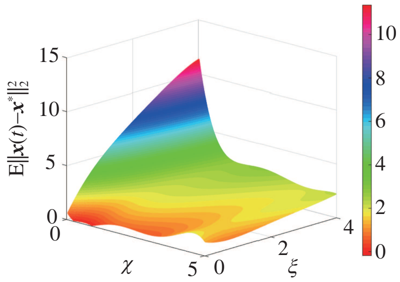


**Figure 3** (Color online) The relationship between E $\|\boldsymbol{x}(t) - \boldsymbol{x}^*\|_2^2$ and parameters $\chi$, $\xi$.

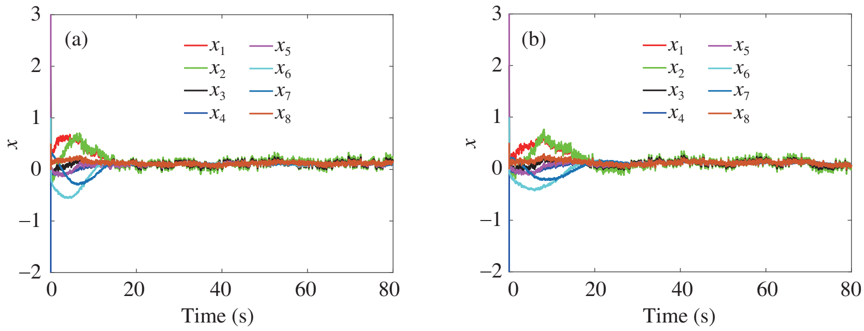


**Figure 4** (Color online) Trajectories of system states $\boldsymbol{x}$ under (4) with different parameters (a) $\chi = 2.45$, $\xi = 1.38$ and (b) $\chi = 2.45$, $\xi = 0.72$.

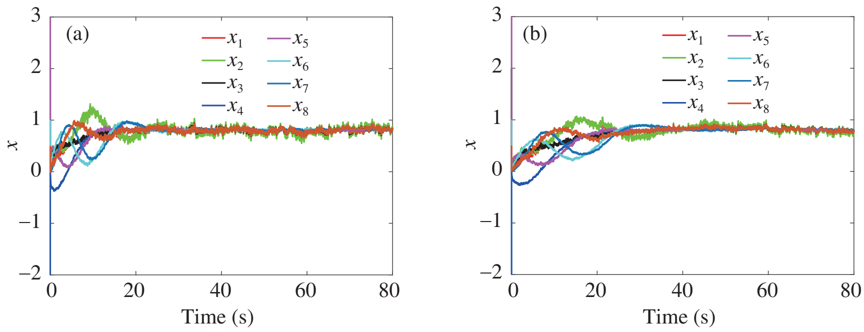


**Figure 5** (Color online) Trajectories of system states $\boldsymbol{x}$ under (27) with different parameters (a) $\chi = 2.45$, $\xi = 1.38$ and (b) $\chi = 2.45$, $\xi = 0.72$.

in the presence of a stationary process, where the communication topology among agents, the initial states of all variables, two pairs of parameters and random noise are chosen the same as those in the DOP (3). However, the local cost functions of agents are modified as $f_i(x_i) = \frac{1}{2}(x_i - x_i(0))^2$, $i = 1, 2, \ldots, 8$. The simulation results are presented in Figure 5. It is observed from Figure 5 that the trajectories of $x_i$ converge to a bounded domain of the average consensus point $x^* = 0.9375$. Furthermore, the size of the bounded neighborhood is excessively dependent on the parameter $\xi$. In other words, when we overemphasize consensus, the parameter $\xi$ is smaller.

## 6 Conclusion

In this paper, a continuous-time algorithm was proposed to solve the DOP for a class of nonlinear multi-agent systems disturbed by a stationary process over a strongly-connected and weight-balanced digraph. The NOS exponential stability criterion for the target system based on RDEs is established to evaluate the quality of solutions for the given problem. In addition, the obtained results can be applied to the distributed average consensus problem in the presence of a stationary process. In future, we plan to focus on developing distributed optimization algorithms under time-varying unbalanced directed graphs.

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