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Active disturbance rejection control for fractional reaction-diffusion equations with spatially varying diffusivity and time delay

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Dear editor,

Fractional systems have been identified as an excellent tool for describing sub-diffusion phenomena owing to the memory property of the fractional derivatives. Among all the research fields in the control theory of fractional systems, stabilization occupies an important place. Ref. [1] discussed the boundary stabilization for fractional PDEs (partial differential equations) for the first time, where a feedback controller was designed to stabilize fractional wave equations using numerical methods, but without rigorous proof owing to the lack of mathematical tools. Recently, the introduction of the fractional Lyapunov method [2] made the stability study more promising. However, in practical engineering, unknown external disturbance always follows the controllers as by-products. In this case, control for systems without uncertainty may no longer work.

To overcome the disturbance, Han [3] firstly introduced the active disturbance rejection control (ADRC), which provided a new effective way to cope with external disturbance. The main idea of ADRC is to design an observer for the unknown disturbance, which is utilized to compensate the effects of disturbance in closed-loop systems. Nowadays, ADRC is widely applied to stabilize PDEs [4, 5], while it is also gradually used in fractional systems [6] with abundant applications in real world such as the nylon tube extrusion line and the motion control chip.

However, there are no reports on the stabilization of fractional PDEs via ADRC except a new published research study [7], where the authors developed a fractional diffusion equation with constant diffusivity. Specifically, compared to the constant diffusivity considered in [7], observers for systems with non-constant diffusivity will contain the diffusivity term, which can more effectively reveal the characteristics of the considered systems. Moreover, it should be noted that the non-constant diffusivity indicates that the spatial distribution in the considered region could be non-uniform. In addition, it is known that time delay frequently occurs in various dynamic systems, which results in an unstable effect and may lead to poor performance of the control systems.

Hence, in this study, the state feedback stabilization problem will be discussed based on the ADRC approach involving the unknown disturbance for fractional reaction-diffusion equations with variable diffusion coefficient and time delay:

$$\begin{cases} {}_0^C D_t^\alpha \Phi(\xi,t) = a(\xi)\Phi_{\xi\xi}(\xi,t) + b(\xi)\Phi(\xi,t) \\ \qquad\qquad\qquad + c(\xi)\Phi(\xi,t-\tau),\ t \geqslant 0,\ \xi \in (0,1), \\ \Phi(0,t) = 0,\ \Phi(1,t) = u(t) + d(t),\ \ t \geqslant 0, \\ \Phi(\xi,\theta) = \Phi_0(\xi,\theta),\ \xi \in [0,1],\ \theta \in [-\tau,0], \end{cases} \tag{1}$$

where ${}_0^C D_t^\alpha \Phi(\xi,t) \triangleq \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial}{\partial s}\Phi(\xi,s)\mathrm{d}s$ is the Caputo derivative of Φ for $\alpha \in (0,1)$. $a(\xi)$ is the diffusivity satisfying $a(\xi) > 0$, twice continuously differentiable, and $a'(1) \geqslant 0$. $b(\xi)$ and $c(\xi)$ are continuous. τ represents the time delay, while u is the control input. $\Phi_0(\xi,\theta) \in C(-\tau,0;L^2(0,1))$ and the disturbance $d(t) \in H^{\alpha,\infty}(0,\infty) \triangleq \{\phi \mid \phi \in L^\infty(0,\infty),\ {}_0^C D_t^\alpha \phi(t) \in L^\infty(0,\infty)\}$. The open-loop system of (1) may become unstable even if $d \equiv 0$. This can be confirmed by the following example. It is assumed that $a(\xi) = 1$, $b(\xi) = b$ and $c(\xi) = c$, while $l(t)$ is the solution of ${}_0^C D_t^\alpha l(t) = (-\pi^2 + b)l(t) + cl(t-\tau)$ with $l(\theta) = 1$ for $\theta \in [-\tau,0]$. Clearly, $\Phi(x,t) = l(t)\sin\pi\xi$ is the solution of (1) and $\Phi_0(\xi,\theta) = \sin\pi\xi$. It has been earlier reported that $l(t) \to \infty$ as $t \to \infty$, where b or τ is large enough, leading to unstable (1).

The main specialities of this study were the three following aspects: (1) Design disturbance estimators for Dirichlet/Neumann boundary control to avoid the high gains presented in existing studies. (2) Obtain the asymptotical stability for the original systems by rejecting the external disturbance and achieve the Mittag-Leffler stabilization for the system without delay. (3) Verify the boundedness of the fractional derivative of the harmonic signals.

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Definition 1 ([2]). Eq. (1) is called Mittag-Leffler stable if $\|\Phi(\xi,t)\|_{L^2(0,1)} \leqslant mE_\alpha(-\mu t^\alpha)$, where $0<\alpha<1$, $m,\mu>0$ and $E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+1)}$.

Lemma 1. For $0<\alpha<1$, $\beta\in\mathbb{R}$, the α-th order Caputo fractional derivatives of $\sin\beta t$, $\cos\beta t$ are both bounded for $t\geqslant 0$.

Our aim is to seek a control input $u(t)$ that stabilizes (1) by rejecting the disturbance $d(t)$. To that end, we separate the controller and the disturbance into two systems, and then introduce an auxiliary system to estimate $d(t)$. By assumption on a,c, two constants, $K,\widetilde{M}>0$, s.t. $|a''(\xi)|\leqslant K$ and $|c(\xi)|\leqslant\widetilde{M}$, could be extracted. The first auxiliary system is then designed:

$$\begin{cases} {}_0^C D_t^\alpha \Psi(\xi,t)=a(\xi)\Psi_{\xi\xi}(\xi,t)+b(\xi)\Psi(\xi,t) \\ \qquad +c(\xi)\Psi(\xi,t-\tau)-\kappa\left(\Psi(\xi,t)-\Phi(\xi,t)\right), \\ \Psi(0,t)=0,\ \ \Psi(1,t)=u(t), \\ \Psi(\xi,\theta)=\Psi_0(\xi,\theta), \end{cases} \tag{2}$$

where κ satisfies $\kappa>\frac{K}{2}+\sqrt{2}\widetilde{M}$. When $\widehat{\Psi}(\xi,t)=\Psi(\xi,t)-\Phi(\xi,t)$, $\widehat{\Psi}(\xi,t)$ satisfies

$$\begin{cases} {}_0^C D_t^\alpha \widehat{\Psi}(\xi,t)=a(\xi)\widehat{\Psi}_{\xi\xi}(\xi,t) \\ \qquad +c(\xi)\widehat{\Psi}(\xi,t-\tau)-\kappa\widehat{\Psi}(\xi,t), \\ \widehat{\Psi}(0,t)=0,\ \ \widehat{\Psi}(1,t)=-d(t), \\ \widehat{\Psi}(\xi,\theta)=\widehat{\Psi}_0(\xi,\theta)=\Psi_0(\xi,\theta)-\Phi_0(\xi,\theta). \end{cases} \tag{3}$$

Lemma 2. If $d(t)\in H^{\alpha,\infty}(0,\infty)$ and $\widehat{\Psi}_0(\xi,\theta)\in C(-\tau,0;L^2(0,1))$, the solution of (3) is unique with $\sup_{t\geqslant 0}\|\widehat{\Psi}(\xi,t)\|_{L^2(0,1)}<+\infty$. Moreover, when $d(t)=0$, we have $\lim_{t\to\infty}\|\widehat{\Psi}(\xi,t)\|=0$. Further, if $\tau=0$, then two positive numbers M, μ, exist such that $\|\widehat{\Psi}(\xi,t)\|_{L^2(0,1)}\leqslant ME_\alpha(-\mu t^\alpha)$.

Here, the second auxiliary system is presented:

$$\begin{cases} {}_0^C D_t^\alpha \Upsilon(\xi,t)=a(\xi)\Upsilon_{\xi\xi}(\xi,t) \\ \qquad +c(\xi)\Upsilon(\xi,t-\tau)-\kappa\Upsilon(\xi,t), \\ \Upsilon(0,t)=0,\ \ \Upsilon_\xi(1,t)=\Phi_\xi(1,t)-\Psi_\xi(1,t), \\ \Upsilon(\xi,\theta)=\Upsilon_0(\xi,\theta). \end{cases} \tag{4}$$

Define $\Theta(\xi,t)=-\Upsilon(\xi,t)-\widehat{\Psi}(\xi,t)$. It is directly verified that $\Theta(\xi,t)$ satisfies

$$\begin{cases} {}_0^C D_t^\alpha \Theta(\xi,t)=a(\xi)\Theta_{\xi\xi}(\xi,t) \\ \qquad +c(\xi)\Theta(\xi,t-\tau)-\kappa\Theta(\xi,t), \\ \Theta(0,t)=\Theta_\xi(1,t)=0, \\ \Theta(\xi,\theta)=\Theta_0(\xi,\theta). \end{cases} \tag{5}$$

Lemma 3. If $\Theta_0(\xi,\theta)\in C(-\tau,0;L^2(0,1))$, Eq. (5) has a unique solution with $\lim_{t\to\infty}\|\Theta(\xi,t)\|_{L^2(0,1)}=0$. Moreover, if $\tau=0$, then for some $M,\mu>0$, $\|\Theta(\xi,t)\|_{L^2(0,1)}\leqslant ME_\alpha\left(-\mu t^\alpha\right)$ holds.

Lemma 4. If the operator $\mathcal{A}_0$ is calculated by $\mathcal{A}_0\phi(\xi)=a(\xi)\phi''(\xi)$ with $D(\mathcal{A}_0)=\{\phi\in H^2(0,1):\phi(0)=\phi'(1)=0\}$ and $\Theta_0(\xi,\theta)\in C(-\tau,0;D(\mathcal{A}_0))$, then $\lim_{t\to\infty}|\Theta(1,t)|=0$. Moreover, if $\tau=0$, $\Theta(1,t)$ is Mittag-Leffler stable.

Given that $\Theta(1,t)=d(t)-\Upsilon(1,t)$, it is known that $\Upsilon(1,t)\to d(t)$, which provides an estimate for the disturbance $d(t)$. Although this estimate is under a smooth initial state, we will show that it is enough to asymptotically stabilize the system (1) for the general initial state. Based on the arguments above, a disturbance estimator for (1) is

$$\begin{cases} {}_0^C D_t^\alpha \Psi(\xi,t)=a(\xi)\Psi_{\xi\xi}(\xi,t)+b(\xi)\Phi(\xi,t) \\ \qquad +c(\xi)\Phi(\xi,t-\tau)-\kappa(\Psi(\xi,t)-\Phi(\xi,t)), \\ \Psi(0,t)=0,\ \ \Psi(1,t)=u(t), \\ {}_0^C D_t^\alpha \Upsilon(\xi,t)=a(\xi)\Upsilon_{\xi\xi}(\xi,t) \\ \qquad +c(\xi)\Upsilon(\xi,t-\tau)-\kappa\Upsilon(\xi,t), \\ \Upsilon(0,t)=0,\ \ \Upsilon_\xi(1,t)=\Phi_\xi(1,t)-\Psi_\xi(1,t), \\ \Psi(\xi,\theta)=\Psi_0(\xi,\theta),\ \ \Upsilon(\xi,\theta)=\Upsilon_0(\xi,\theta). \end{cases} \tag{6}$$

By employing $\Upsilon(1,t)$ as a compensation of disturbance $d(t)$, we are looking for a disturbance-based controller for (1). To this end, $c(\xi)=c$ is set and a backstepping transformation is introduced:

$$\widehat{\Phi}(\xi,t)\triangleq\Phi(\xi,t)-\int_0^\xi k(\xi,\eta)\Phi(\eta,t)\mathrm{d}\eta, \tag{7}$$

where $k(\xi,\eta)$ is the solution of

$$\begin{cases} a(\xi)k_{\xi\xi}(\xi,\eta)-a(\eta)k_{\eta\eta}(\xi,\eta)-2a'(\eta) \\ \qquad \cdot k_\eta(\xi,\eta)=(a''(\eta)+b(\eta)+\kappa)k(\xi,\eta), \\ k(\xi,\xi)=-\dfrac{1}{2\sqrt{a(\xi)}}\displaystyle\int_0^\xi\dfrac{b(\omega)+\kappa}{\sqrt{a(\omega)}}\mathrm{d}\omega. \end{cases} \tag{8}$$

Hence, the transformation (7) converts (1) to

$$\begin{cases} {}_0^C D_t^\alpha \widehat{\Phi}(\xi,t)=a(\xi)\widehat{\Phi}_{\xi\xi}(\xi,t) \\ \qquad +c(\xi)\widehat{\Phi}(\xi,t-\tau)-\kappa\widehat{\Phi}(\xi,t), \\ \widehat{\Phi}(0,t)=0,\ \ \widehat{\Phi}(1,t)=u(t)+d(t) \\ \qquad -\displaystyle\int_0^1 k(1,\eta)\Phi(\eta,t)\mathrm{d}\eta, \\ \widehat{\Phi}(\xi,\theta)=\widehat{\Phi}_0(\xi,\theta). \end{cases} \tag{9}$$

Based on (9) and given that $\Upsilon(1,t)$ compensates $d(t)$, the controller $u(t)$ is designed as

$$u(t)=-\Upsilon(1,t)+\int_0^1 k(1,\eta)\Phi(\eta,t)\mathrm{d}\eta. \tag{10}$$

In this case, the closed-loop system of (9) is

$$\begin{cases} {}_0^C D_t^\alpha \widehat{\Phi}(\xi,t)=a(\xi)\widehat{\Phi}_{\xi\xi}(\xi,t) \\ \qquad +c(\xi)\widehat{\Phi}(\xi,t-\tau)-\kappa\widehat{\Phi}(\xi,t), \\ \widehat{\Phi}(0,t)=0,\ \ \widehat{\Phi}(1,t)=\Theta(1,t), \\ \widehat{\Phi}(\xi,\theta)=\widehat{\Phi}_0(\xi,\theta). \end{cases} \tag{11}$$

Lemma 5. If $\widehat{\Phi}_0\in C(-\tau,0;L^2(0,1))$, then Eq. (11) has a unique solution with $\lim_{t\to\infty}\|\widehat{\Phi}(\xi,t)\|_{L^2(0,1)}=0$. Moreover, if $\tau=0$, it is Mittag-Leffler stable.

Finally, we turn to the closed-loop system of (1) under control (10):

$$\begin{cases} {}_0^C D_t^\alpha \Phi(\xi,t)=a(\xi)\Phi_{\xi\xi}(\xi,t)+b(\xi)\Phi(\xi,t) \\ \qquad +c(\xi)\Phi(\xi,t-\tau), \\ {}_0^C D_t^\alpha \Psi(\xi,t)=a(\xi)\Psi_{\xi\xi}(\xi,t)+b(\xi)\Phi(\xi,t) \\ \qquad +c(\xi)\Psi(\xi,t-\tau)-\kappa\left(\Psi(\xi,t)-\Phi(\xi,t)\right), \\ {}_0^C D_t^\alpha \Upsilon(\xi,t)=a(\xi)\Upsilon_{\xi\xi}(\xi,t) \\ \qquad +c(\xi)\Upsilon(\xi,t-\tau)-\kappa\Upsilon(\xi,t), \\ \Phi(0,t)=\Psi(0,t)=\Upsilon(0,t)=0, \\ \Phi(1,t)=\Psi(1,t)+d(t), \\ \Psi(1,t)=-\Upsilon(1,t)+\displaystyle\int_0^1 k(1,\eta)\Phi(\eta,t)\mathrm{d}\eta, \\ \Upsilon_\xi(1,t)=\Phi_\xi(1,t)-\Psi_\xi(1,t), \\ (\Phi,\Psi,\Upsilon)|_{(\xi,\theta)}=(\Phi_0,\Psi_0,\Upsilon_0)|_{(\xi,\theta)}. \end{cases} \tag{12}$$

Considering (12) in $\mathcal{H}=[L^2(0,1)]^3$, the following conclusion is drawn.

Theorem 1. It is assumed that $d(t)\in H^{\alpha,\infty}(0,\infty)$ and $a'(1)\geqslant 0$. For any $\left((\Phi_0,\Psi_0,\Upsilon_0)|_{(\xi,\theta)}\right)^{\rm T}\in(C(-\tau,0;L^2(0,1)))^3$, Eq. (12) has a unique solution in $C(0,\infty;\mathcal{H})$ with $\lim_{t\to\infty}\|\Phi(\xi,t)\|_{L^2(0,1)}=0$ and

$$\sup_{t\geqslant 0}\|(\Psi(\xi,t),\Upsilon(\xi,t))\|_{L^2(0,1)}<+\infty.$$

Moreover, if $\tau=0$, then for some M, $\mu>0$, $\|\Phi(\xi,t)\|_{L^2(0,1)}\leqslant ME_\alpha(-\mu t^\alpha)$ holds, while if $d\equiv 0$, the Mittag-Leffler stabilization of $\Psi(\xi,t)$ and $\Upsilon(\xi,t)$ is achieved.

Remark 1. The design of the Neumann boundary controller, i.e., considering $\Phi_\xi(1,t)=u(t)+d(t)$ in (1), is based on the Dirichlet boundary controller with different design procedure and proofs. More information is included about this part in Appendix G.

Conclusion. We analyzed the stabilization problem of a reaction-diffusion system with spatially varying diffusivity and a time delay subject to boundary disturbance by ADRC. Disturbance estimators are designed consisting of two auxiliary systems, where no high gain was involved. Based on these, a feedback controller is also proposed by compensating the disturbance. Thus, it is demonstrated that the original fractional system is asymptotically stable, while the state of two auxiliary systems is bounded. Particularly, the Mittag-Leffler stability for the original system is guaranteed when the time delay vanishes.

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Supporting information Appendixes A–H. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

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