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• **Supplementary File** •

Active disturbance rejection control for fractional reaction-diffusion equations with spatially varying diffusivity and time delay

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Appendix A Proof of Lemma 1

We first prove that ${}_0^C D_t^\alpha \sin\beta t$ is bounded on $t \in [0,\infty)$. Indeed, from the definition of fractional derivative, we have

$$\begin{aligned}
{}_0^C D_t^\alpha \sin\beta t &= \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-s)^{-\alpha}\sin\beta s \mathrm{d}s \\
&= \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^t (t-s)^{-\alpha}\sin\beta s \mathrm{d}s - \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^0 (t-s)^{-\alpha}\sin\beta s \mathrm{d}s \\
&= \beta^\alpha \sin\left(\beta t + \frac{\alpha\pi}{2}\right) + \frac{\alpha}{\Gamma(1-\alpha)}\int_{-\infty}^0 (t-\varsigma)^{-\alpha-1}\sin\beta\varsigma \mathrm{d}\varsigma,
\end{aligned}$$

where the first item is according to [1, Formula 115]. Let $h(t) = \int_{-\infty}^0 (t-\varsigma)^{-\alpha-1}\sin\beta\varsigma \mathrm{d}\varsigma$. Its continuity leads to its boundedness on $[0,1]$. In addition, for $t \geqslant 1$, $|h(t)| \leqslant \int_{-\infty}^0 (t-\varsigma)^{-\alpha-1}\mathrm{d}\varsigma \leqslant \frac{1}{\alpha}$, which yields that $h(t)$ is bounded for $t \geqslant 0$. Hence, ${}_0^C D_t^\alpha \sin\beta t$ is bounded on $t \in [0,\infty)$.

Next, we show that ${}_0^C D_t^\alpha \cos\beta t$ is bounded on $t \in [0,\infty)$. By the definition of fractional derivative again, we get

$$\begin{aligned}
{}_0^C D_t^\alpha \cos\beta t &= \frac{-\beta}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\sin\beta s \mathrm{d}s = \frac{-\beta}{\Gamma(1-\alpha)}\int_0^t \frac{\sin\beta(t-s)}{s^\alpha}\mathrm{d}s \\
&= \frac{\beta}{\Gamma(1-\alpha)}\left[\cos\beta t\int_0^t \frac{\sin\beta s}{s^\alpha}\mathrm{d}s - \sin\beta t\int_0^t \frac{\cos\beta s}{s^\alpha}\mathrm{d}s\right] \\
&= \frac{\beta^\alpha}{\Gamma(1-\alpha)}\left[\cos\beta t\int_0^t \frac{\sin s}{s^\alpha}\mathrm{d}s - \sin\beta t\int_0^t \frac{\cos s}{s^\alpha}\mathrm{d}s\right].
\end{aligned}$$

Denote $\zeta(t) = \cos\beta t\int_0^t \frac{\sin s}{s^\alpha}\mathrm{d}s - \sin\beta t\int_0^t \frac{\cos s}{s^\alpha}\mathrm{d}s$. Clearly, $\zeta(t)$ is continuous on $[0,\infty)$. It follows from [2, Page 284, A.4.11] that for $\alpha \in (0,1)$,

$$\int_0^\infty \frac{\cos s}{s^\alpha}\mathrm{d}s = \Gamma(1-\alpha)\sin\frac{\alpha\pi}{2} \quad \textit{and} \quad \int_0^\infty \frac{\sin s}{s^\alpha}\mathrm{d}s = \Gamma(1-\alpha)\cos\frac{\alpha\pi}{2}$$

hold, which imply that $\int_0^t \frac{\sin s}{s^\alpha}\mathrm{d}s$ and $\int_0^t \frac{\cos s}{s^\alpha}\mathrm{d}s$ are bounded on $[0,\infty)$. By the boundedness of $\sin\beta t$, $\cos\beta t$, we know that $\zeta(t)$ is bounded, and thus, ${}_0^C D_t^\alpha \cos\beta t$ is bounded.

Appendix B Proof of Lemma 2

Let $\widetilde{\Psi}(\xi,t) = \widehat{\Psi}(\xi,t) + \xi^2 d(t)$. Then $\widetilde{\Psi}(x,t)$ satisfies

$$\begin{cases}
{}_0^C D_t^\alpha \widetilde{\Psi}(\xi,t) = a(\xi)\widetilde{\Psi}_{\xi\xi}(\xi,t) + c(\xi)\widetilde{\Psi}(\xi,t-\tau) - \kappa\widetilde{\Psi}(\xi,t) + f(\xi,t),\ t \geqslant 0,\ \xi \in (0,1), \\
\widetilde{\Psi}(0,t) = 0,\ \ \widetilde{\Psi}(1,t) = 0,\ t \geqslant 0, \\
\widetilde{\Psi}(\xi,\theta) = \widetilde{\Psi}_0(\xi,\theta) = \Psi_0(\xi,\theta) - \Phi_0(\xi,\theta) + \xi^2 d(0),\ \xi \in [0,1],\ \theta \in [-\tau,0],
\end{cases} \tag{B1}$$

where $f(\xi,t) = -2a(\xi)d(t) + \xi^2\left(\kappa d(t) - c(\xi)d(t-\tau) + {}_0^C D_t^\alpha d(t)\right)$ and $d(\theta) \triangleq d(0)$ for $\theta \in [-\tau,0]$.

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Define operator $\mathcal{A}: L^2(0,1) \to L^2(0,1)$, $\mathcal{A}\varphi(\xi) = a(\xi)\varphi''(\xi) - \kappa\varphi(\xi)$. $D(\mathcal{A}) = \{\varphi \in H^2(0,1) | \varphi(0) = \varphi(1) = 0\}$, and operator $\mathcal{A}_1 : L^2(0,1) \to L^2(0,1)$, $\mathcal{A}_1\varphi(\xi) = c(\xi)\varphi(\xi)$. We claim that $\mathcal{A}$ is the generator of a C_0-semigroup in $L^2(0,1)$. Indeed, for any $\varphi \in D(\mathcal{A})$,

$$\begin{aligned}\langle \mathcal{A}\varphi, \varphi\rangle_{L^2(0,1)} = \langle a(\xi)\varphi''(\xi) - \kappa\varphi(\xi), \varphi(\xi)\rangle_{L^2(0,1)} &= \int_0^1 a(\xi)\varphi''(\xi)\varphi(\xi) - \kappa\varphi^2(\xi)\mathrm{d}\xi \\ &= -\int_0^1 a(\xi)\left(\varphi'(\xi)\right)^2 - \kappa\varphi^2(\xi)\mathrm{d}\xi - \int_0^1 a'(\xi)\varphi'(\xi)\varphi(\xi)\mathrm{d}\xi.\end{aligned}$$

From $\int_0^1 a'(\xi)\varphi'(\xi)\varphi(\xi)\mathrm{d}\xi = -\int_0^1 a''(\xi)\varphi^2(\xi)\mathrm{d}\xi - \int_0^1 a'(\xi)\varphi'(\xi)\varphi(\xi)\mathrm{d}\xi$, we see

$$\int_0^1 a'(\xi)\varphi'(\xi)\varphi(\xi)\mathrm{d}\xi = -\frac{1}{2}\int_0^1 a''(\xi)\varphi^2(\xi)\mathrm{d}\xi, \tag{B2}$$

and thus $\langle \mathcal{A}\varphi, \varphi\rangle_{L^2(0,1)} \leqslant -\int_0^1 a(\xi)\left(\varphi'(\xi)\right)^2 - \left(\kappa - \frac{K}{2}\right)\varphi^2(\xi)\mathrm{d}\xi \leqslant 0$. Hence, $\mathcal{A}$ is a dissipative operator on $L^2(0,1)$. Since $\forall\lambda > 0$ and $Y(\xi) \in L^2(0,1)$,

$$\begin{cases} -a(\xi)\varphi''(\xi) + (\lambda + \kappa)\varphi(\xi) = Y, \\ \varphi(0) = \varphi(1) = 0 \end{cases}$$

exists a unique solution in $L^2(0,1)$. We know the range of $\lambda I - \mathcal{A}$ is $L^2(0,1)$, where I denotes the identity operator. According to Lumer-Phillips Theorem, we get that $\mathcal{A}$ generates a C_0-semigroup on $L^2(0,1)$. It's clear that $\mathcal{A}_1$ is bounded. Then, by the semigroup method, we have that (B1) has a unique solution on $[0,\tau]$. By repeating this procedure step-by-step on $[\tau, 2\tau]$, $[2\tau, 3\tau]$, $\cdots$, we obtain the solution of (B1) for $t \geqslant 0$ and so does for (3).

Choose Lyapunov function $L(t) = \frac{1}{2}\int_0^1 \widetilde{\Psi}^2(\xi,t)\mathrm{d}\xi$, by inequality ${}_0^C D_t^\alpha \rho^2(t) \leqslant 2\rho(t){}_0^C D_t^\alpha \rho(t)$, we have

$$\begin{aligned}{}_0^C D_t^\alpha L(t) &\leqslant \int_0^1 \widetilde{\Psi}(\xi,t){}_0^C D_t^\alpha \widetilde{\Psi}(\xi,t)\mathrm{d}\xi = \int_0^1 \widetilde{\Psi}(\xi,t)(a(\xi)\widetilde{\Psi}_{\xi\xi}(\xi,t) + c(\xi)\widetilde{\Psi}(\xi,t-\tau) - \kappa\widetilde{\Psi}(\xi,t) + f(\xi,t))\mathrm{d}\xi \\ &= -\int_0^1 a(\xi)\widetilde{\Psi}_\xi^2(\xi,t)\mathrm{d}\xi - \kappa\int_0^1 \widetilde{\Psi}^2(\xi,t)\mathrm{d}\xi + \int_0^1 \widetilde{\Psi}(\xi,t)f(\xi,t)\mathrm{d}\xi \\ &\quad - \int_0^1 a'(\xi)\widetilde{\Psi}(\xi,t)\widetilde{\Psi}_\xi(\xi,t)\mathrm{d}\xi + \int_0^1 c(\xi)\widetilde{\Psi}(\xi,t)\widetilde{\Psi}(\xi,t-\tau)\mathrm{d}\xi.\end{aligned}$$

Similar to (B2), we have $\int_0^1 a'(\xi)\widetilde{\Psi}(\xi,t)\widetilde{\Psi}_\xi(\xi,t)\mathrm{d}\xi = -\frac{1}{2}\int_0^1 a''(\xi)\widetilde{\Psi}^2(\xi,t)\mathrm{d}\xi$. According to Young's Inequality, we know $\int_0^1 \widetilde{\Psi}(\xi,t)f(\xi,t)\mathrm{d}\xi \leqslant \varepsilon\int_0^1 \widetilde{\Psi}^2(\xi,t)\mathrm{d}\xi + \frac{1}{4\varepsilon}\int_0^1 f^2(\xi,t)\mathrm{d}\xi$ and

$$\int_0^1 c(\xi)\widetilde{\Psi}(\xi,t)\widetilde{\Psi}(\xi,t-\tau)\mathrm{d}\xi \leqslant \varepsilon\int_0^1 \widetilde{\Psi}^2(\xi,t)\mathrm{d}\xi + \frac{\widetilde{M}^2}{4\varepsilon}\int_0^1 \widetilde{\Psi}^2(\xi,t-\tau)\mathrm{d}\xi,$$

where ε is chosen to satisfy $0 < 4\varepsilon < 2\kappa - K$ and $1 - \frac{\widetilde{M}^2}{2\varepsilon}(2\kappa - K - 4\varepsilon)^{-1} > 0$. Here, ε can be chosen provided by the restriction $\kappa > \frac{K}{2} + \sqrt{2}\widetilde{M}$. Therefore, we get

$${}_0^C D_t^\alpha L(t) \leqslant -(2\kappa - K - 4\varepsilon)L(t) + \frac{\widetilde{M}^2}{2\varepsilon}L(t-\tau) + \frac{1}{4\varepsilon}\int_0^1 f^2(\xi,t)\mathrm{d}\xi. \tag{B3}$$

This yields

$$\begin{aligned}L(t) \leqslant& E_\alpha\left(-(2\kappa - K - 4\varepsilon)t^\alpha\right)L(0) + \frac{\widetilde{M}^2 K_t}{2\varepsilon}\int_0^t (t-\varsigma)^{\alpha-1}E_{\alpha,\alpha}\left(-(2\kappa - K - 4\varepsilon)(t-\varsigma)^\alpha\right)\mathrm{d}\varsigma \\ &+ \frac{C_1}{16\varepsilon^2}\int_0^t (t-\varsigma)^{\alpha-1}E_{\alpha,\alpha}\left(-(2\kappa - K - 4\varepsilon)(t-\varsigma)^\alpha\right)\mathrm{d}\varsigma,\end{aligned} \tag{B4}$$

where $K_t = \sup\limits_{s\in[-\tau,t]} L(s)$ and $E_{\alpha,\beta}(z) = \sum\limits_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\beta)}$ is the two parameter Mittag-Leffer function. From $\frac{\mathrm{d}}{\mathrm{d}t}\left[t^\alpha E_{\alpha,\alpha+1}(-\lambda t^\alpha)\right]$ $= t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)$, one see that

$$\int_0^t (t-\varsigma)^{\alpha-1}E_{\alpha,\alpha}\left(-(2\kappa - K - 4\varepsilon)(t-\varsigma)^\alpha\right)\mathrm{d}\varsigma = t^\alpha E_{\alpha,\alpha+1}\left(-(2\kappa - K - 4\varepsilon)t^\alpha\right).$$

It is well-known that $E_{\alpha,\alpha+1}\left(-(2\kappa - K - 4\varepsilon)t^\alpha\right) = (2\kappa - K - 4\varepsilon)^{-1}t^{-\alpha} + \mathcal{O}\left(t^{-2\alpha}\right)$, which gives $\lim\limits_{t\to+\infty} t^\alpha E_{\alpha,\alpha+1}(-(2\kappa$ $-K - 4\varepsilon)t^\alpha) = (2\kappa - K - 4\varepsilon)^{-1}$. Hence,

$$\begin{aligned}&\int_0^t (t-\varsigma)^{\alpha-1}E_{\alpha,\alpha}\left(-(2\kappa - K - 4\varepsilon)(t-\varsigma)^\alpha\right)\mathrm{d}\varsigma \\ \leqslant& \int_0^\infty s^{\alpha-1}E_{\alpha,\alpha}\left(-(2\kappa - K - 4\varepsilon)(t-s)^\alpha\right)\mathrm{d}s = (2\kappa - K - 4\varepsilon)^{-1},\end{aligned}$$

which implies

$$L(t) \leqslant E_\alpha\left((2\kappa - K - 4\varepsilon)^{-1}t^\alpha\right)L(0) + \frac{\widetilde{M}^2 K_t}{2\varepsilon}(2\kappa - K - 4\varepsilon)^{-1} + \frac{C_1}{16\varepsilon^2}(2\kappa - K - 4\varepsilon)^{-1}. \tag{B5}$$

Taking the supremum on $[-\tau, t]$ on both sides of (B5), we have

$$K_t \leqslant \sup_{s\geqslant 0} E_\alpha\left((2\kappa - K - 4\varepsilon)^{-1} s^\alpha\right) L(0) + K_0 + \frac{\widetilde{M}^2 K_t}{2\varepsilon}(2\kappa - K - 4\varepsilon)^{-1} + \frac{C_1}{16\varepsilon^2}(2\kappa - K - 4\varepsilon)^{-1}.$$

This, together with $E_\alpha\left(-(2\kappa - K - 4\varepsilon)s^\alpha\right) \leqslant 1$ for all $s \geqslant 0$, yields

$$K_t \leqslant \left(L(0) + K_0 + \frac{C_1}{16\varepsilon^2}(2\kappa - K - 4\varepsilon)^{-1}\right)\left(1 - \frac{\widetilde{M}^2}{2\varepsilon}(2\kappa - K - 4\varepsilon)^{-1}\right)^{-1}.$$

Taking the supremum on both sides of (B4) on $t \geqslant 0$, we conclude $\sup_{t\geqslant 0} L(t) < +\infty$. This, jointly with the boundedness of $\xi^2 d(t)$, leads to $\sup_{t\geqslant 0}\left\|\widehat{\Psi}(\xi,t)\right\|_{L^2(0,1)} < +\infty$.

When $d(t) = 0$, we see $f(\xi,t) = 0$, and thus, by (B3) and the fractional Halanay inequality [3, Corollary 1], we know $\lim_{t\to\infty} L(t) = 0$, that is, $\lim_{t\to\infty}\left\|\widetilde{\Psi}(\xi,t)\right\| = 0$. Since $\widetilde{\Psi}(\xi,t) = \xi^2 d(t) + \widehat{\Psi}(\xi,t)$, we conclude $\lim_{t\to\infty}\left\|\widehat{\Psi}(\xi,t)\right\| = 0$. Moveover, when $\tau = 0$ and $d(t) = 0$, we get ${}_0^C D_t^\alpha L(t) \leqslant -\left(2\kappa - K - 2\widetilde{M}\right)L(t)$, which implies the Mittag-Leffler stability of $\widehat{\Psi}(\xi,t)$.

Appendix C Proof of Lemma 3

Define $\widetilde{L}(t) = \frac{1}{2}\int_0^1 \Theta^2(\xi,t)\mathrm{d}\xi$. Similar to Lemma 1, we can derive the uniqueness and existence for the solutions of (5) We can calculate

$$\begin{aligned}
{}_0^C D_t^\alpha \widetilde{L}(t) \leqslant& -\int_0^1 a(\xi)\Theta_\xi^2(\xi,t)\mathrm{d}\xi + \int_0^1 c(\xi)\Theta(\xi,t-\tau)\Theta(\xi,t)\mathrm{d}\xi \\
&- \kappa\int_0^1 \Theta^2(\xi,t)\mathrm{d}\xi - \frac{1}{2}a'(1)\Theta^2(1,t) + \frac{1}{2}\int_0^1 a''(\xi)\Theta^2(\xi,t)\mathrm{d}\xi \\
\leqslant& -\left(\kappa - \frac{K}{2} - 2\varepsilon\right)\int_0^1 \Theta^2(\xi,t)\mathrm{d}\xi + \frac{\widetilde{M}^2}{8\varepsilon}\int_0^1 \Theta^2(\xi,t-\tau)\mathrm{d}\xi \\
=& -(2\kappa - K - 4\varepsilon)\widetilde{L}(t) + \frac{\widetilde{M}^2}{4\varepsilon}\widetilde{L}(t-\tau),
\end{aligned}$$

provided that $a'(1) \geqslant 0$. Since $2\kappa - K - 4\varepsilon > \widetilde{M}^2/(2\varepsilon)$, we see $2\kappa - K - 4\varepsilon > \widetilde{M}^2/(4\varepsilon)$ and thus, by the fractional Halanay inequality [3, Corollary 1], we get $\lim_{t\to\infty}\widetilde{L}(t) = 0$, which leads to $\lim_{t\to\infty}\|\Theta(\xi,t)\|_{L^2(0,1)} = 0$. Moreover, when $\tau = 0$, we obtain ${}_0^C D_t^\alpha \widetilde{L}(t) \leqslant -(2\kappa - K - 4\varepsilon - \widetilde{M}^2/(4\varepsilon))\widetilde{L}(t)$. Hence, $\widetilde{L}(t) \leqslant E_\alpha(-(2\kappa - K - 4\varepsilon - \widetilde{M}^2/(4\varepsilon)))\widetilde{L}(0)$, which gives the desired result and ends the proof.

Appendix D Proof of Lemma 4

Clearly, system (5) can be written as ${}_0^C D_t^\alpha \Theta(\xi,t) = \mathcal{A}_0\Theta(\xi,t) - \kappa\Theta(\xi,t) + c\Theta(\xi,t)$ with $\Theta(\xi,\theta) = \Theta_0(\xi,\theta)$. By the assumption, it is easy to see that ${}_0^C D_t^\alpha \mathcal{A}_0\Theta(\xi,t) = \mathcal{A}_0\mathcal{A}_0\Theta(\xi,t) - \kappa\mathcal{A}_0\Theta(\xi,t) + c\mathcal{A}_0\Theta(\xi,t)$ with $\mathcal{A}_0\Theta(\xi,\theta) = \mathcal{A}_0\Theta_0(\xi,\theta) \in L^2(0,1)$. By Lemma 3, system (5) with initial value $\Theta_0(\xi,\theta)$ has a solution $\Theta(\xi,t)$ and system (5) with initial value $\mathcal{A}_0\Theta_0(\xi,\theta)$ has a solution $\mathcal{A}_0\Theta(\xi,t)$. This leads to $\left\|\Theta_{\xi\xi}(\xi,t)\right\|_{L^2(0,1)} \leqslant \|\mathcal{A}_0\Theta(\xi,t)\|_{L^2(0,1)} \to 0$. By Sobolev Interpolation Inequality, there is a $C_1 > 0$, such that

$$\left\|\Theta_\xi(\xi,t)\right\|_{L^2(0,1)} \leqslant C_1\left(\|\Theta(\xi,t)\|_{L^2(0,1)} + \left\|\Theta_{\xi\xi}(\xi,t)\right\|_{L^2(0,1)}\right) \to 0.$$

It follows from the trace theorem that there exists $C_2 > 0$, such that

$$|\Theta(1,t)| \leqslant C_2\left(\|\Theta(\xi,t)\|_{L^2(0,1)} + \left\|\Theta_\xi(\xi,t)\right\|_{L^2(0,1)}\right) \to 0.$$

When $\tau = 0$, similarly, we see that for some $M_1, \mu_1 > 0$,

$$\left\|\Theta_{\xi\xi}(\xi,t)\right\|_{L^2(0,1)} \leqslant \|\mathcal{A}_0\Theta(\xi,t)\|_{L^2(0,1)} \leqslant M_1 E_\alpha\left(-\mu_1 t^\alpha\right)\|\mathcal{A}_0\Theta_0(\xi,0)\|_{L^2(0,1)},$$

and thus $\left\|\Theta_\xi(\xi,t)\right\|_{L^2(0,1)} \leqslant (C_1+1)M_1 E_\alpha\left(-\mu_1 t^\alpha\right)\|\mathcal{A}_0\Theta_0(\xi,0)\|_{L^2(0,1)}$. Hence,

$$|\Theta(1,t)| \leqslant (C_1+1)(C_2+1)M_1 E_\alpha\left(-\mu_1 t^\alpha\right)\|\mathcal{A}_0\Theta_0(\xi,0)\|_{L^2(0,1)} \leqslant M E_\alpha\left(-\mu t^\alpha\right),$$

where $\mu = \mu_1$ and $M = (C_1+1)(C_2+1)M_1\|\mathcal{A}_0\Theta_0(\xi,0)\|_{L^2(0,1)}$.

Appendix E Proof of Lemma 5

Proof. Let $\widetilde{\Phi}(\xi,t) = \widehat{\Phi}(\xi,t) - \Theta(\xi,t)$. From (5) and (11) we know $\widetilde{\Phi}(\xi,t)$ satisfies

$$\begin{cases}
{}_0^C D_t^\alpha \widetilde{\Phi}(\xi,t) = a(\xi)\widetilde{\Phi}_{\xi\xi}(\xi,t) + c(\xi)\widetilde{\Phi}(\xi,t-\tau) - \kappa\widetilde{\Phi}(\xi,t), \\
\widetilde{\Phi}(0,t) = \widetilde{\Phi}(1,t) = 0, \\
\widetilde{\Phi}(\xi,\theta) = \widetilde{\Phi}_0(\xi,\theta).
\end{cases} \tag{E1}$$

Similar to Lemma 3, (E1) has a unique solution. Hence, the solution of (11) is unique provided that the solution of (5) is unique. According to Lemma 2 and Lemma 3, both $\Theta(\xi,t)$ and $\widetilde{\Phi}(\xi,t)$ are asymptotically stable, and $\widehat{\Phi}(\xi,t)$ is thus asymptotically stable. Moreover, when $\tau = 0$, we see $\widetilde{\Phi}(\xi,t)$ and $\Theta(\xi,t)$ are Mittag-Leffler stable. Therefore, $\widehat{\Phi}(\xi,t)$ is Mittag-Leffler stable. □

Appendix F Proof of Theorem 1

Let $\widehat{\Psi}(\xi,t)=-\Phi(\xi,t)+\Psi(\xi,t)$, $\Theta(\xi,t)=-\widehat{\Psi}(\xi,t)-\Upsilon(\xi,t)$ and introduce backstepping transform (7). (12) is converted into

$$\begin{cases} {}_0^CD_t^\alpha\widehat{\Psi}(\xi,t)=a(\xi)\widehat{\Psi}_{\xi\xi}(\xi,t)+c(\xi)\widehat{\Psi}(\xi,t-\tau)-\kappa\widehat{\Psi}(\xi,t),\\ \widehat{\Psi}(0,t)=0,\ \widehat{\Psi}(1,t)=-d(t),\\ {}_0^CD_t^\alpha\Theta(\xi,t)=a(\xi)\Theta_{\xi\xi}(\xi,t)+c(\xi)\Theta(\xi,t-\tau)-\kappa\Theta(\xi,t),\\ \Theta(0,t)=\Theta_\xi(1,t)=0,\\ {}_0^CD_t^\alpha\widehat{\Phi}(\xi,t)=a(\xi)\widehat{\Phi}_{\xi\xi}(\xi,t)+c(\xi)\widehat{\Phi}(\xi,t-\tau)-\kappa\widehat{\Phi}(\xi,t),\\ \widehat{\Phi}(0,t)=0,\ \widehat{\Phi}(1,t)=\Theta(1,t),\\ \widehat{\Psi}(\xi,\theta)=\widehat{\Psi}_0(\xi,\theta),\ \Theta(\xi,\theta)=\Theta_0(\xi,\theta),\ \widehat{\Phi}(\xi,\theta)=\widehat{\Phi}_0(\xi,\theta), \end{cases} \tag{F1}$$

where $\widehat{\Psi}_0(\xi,\theta)=\Psi_0(\xi,\theta)-\Phi_0(\xi,\theta)$, $\Theta_0(\xi,\theta)=-\Upsilon_0(\xi,\theta)-\Psi_0(\xi,\theta)+\Phi_0(\xi,\theta)$ and $\widehat{\Phi}_0(\xi,\theta)=\Phi_0(\xi,\theta)-\int_0^\xi k(\xi,\eta)\Phi_0(\eta,\theta)\mathrm{d}\eta$. Since $(\Phi_0(\xi,\theta),\ \Psi_0(\xi,\theta),\ \Upsilon_0(\xi,\theta))^\top\in\left(C(-\tau,0;L^2(0,1))\right)^3$, we see $(\widehat{\Psi}_0(\xi,\theta)$, $\Theta_0(\xi,\theta),\ \widehat{\Phi}_0(\xi,\theta))^\top\in\left(C(-\tau,0;L^2(0,1))\right)^3$. According to Lemmas 2, 3 and 4, (F1) has a unique solution. In addition, $\|(\Theta(\xi,t),\ \widehat{\Phi}(\xi,t))\|_{L^2(0,1)}\to0$, as $t\to\infty$ and $\sup_{t\geqslant0}\left\|\widehat{\Psi}(\xi,t)\right\|_{L^2(0,1)}<+\infty$. Since

$$\begin{bmatrix}\Phi(\xi,t)\\ \Psi(\xi,t)\\ \Upsilon(\xi,t)\end{bmatrix}=\begin{bmatrix}0 & 0 & (I+Q)^{-1}\\ I & 0 & (I+Q)^{-1}\\ -I & -I & 0\end{bmatrix}\begin{bmatrix}\widehat{\Psi}(\xi,t)\\ \Theta(\xi,t)\\ \widehat{\Phi}(\xi,t)\end{bmatrix},$$

we can easily conclude $\lim_{t\to\infty}\|\Phi(\xi,t)\|_{L^2(0,1)}=0$ and $\sup_{t\geqslant0}\|(\Psi(\xi,t),\ \Upsilon(\xi,t))\|_{L^2(0,1)}<+\infty$.

When $\tau=0$, we get $\|\Theta(\xi,t),\ \widehat{\Phi}(\xi,t)\|_{L^2(0,1)}\leqslant M_1E_\alpha\left(-\mu_1t^\alpha\right)$ holds for some $M_1,\ \mu_1>0$ and $\sup_{t\geqslant0}\|\widehat{\Psi}(\xi,t)\|_{L^2(0,1)}<+\infty$. Hence, $\Phi(\xi,t)$ is Mittag-Leffler stable. Moreover, when $d(t)=0$, according to Lemma 2, we know the Mittag-Leffler stability of $\widehat{\Psi}(\xi,t)$, which leads to the result that $\Psi(\xi,t)$ and $\Upsilon(\xi,t)$ are also Mittag-Leffler stable.

Appendix G Stabilization for system with Neumann boundary condition

Now we consider the Neumann boundary system described by

$$\begin{cases} {}_0^CD_t^\alpha\Phi(\xi,t)=a(\xi)\Phi_{\xi\xi}(\xi,t)+b(\xi)\Phi(\xi,t)+c(\xi)\Phi(\xi,t-\tau),\\ \Phi(0,t)=0,\quad \Phi_\xi(1,t)=u(t)+d(t),\\ \Phi(\xi,\theta)=\Phi_0(\xi,\theta). \end{cases} \tag{G1}$$

We introduce one auxiliary system as

$$\begin{cases} {}_0^CD_t^\alpha\Psi(\xi,t)=a(\xi)\Psi_{\xi\xi}(\xi,t)+b(\xi)\Phi(\xi,t)+c(\xi)\Phi(\xi,t-\tau)+\kappa\left(\Phi(\xi,t)-\Psi(\xi,t)\right),\\ \Psi(0,t)=0,\quad \Psi_\xi(1,t)=u(t),\\ \Psi(\xi,\theta)=\Psi_0(\xi,\theta), \end{cases} \tag{G2}$$

where κ satisfies $\kappa>\frac{K}{2}$. Let $\widehat{\Psi}(\xi,t)=\Psi(\xi,t)-\Phi(\xi,t)$, we see $\widehat{\Psi}(\xi,t)$ satisfies

$$\begin{cases} {}_0^CD_t^\alpha\widehat{\Psi}(\xi,t)=a(\xi)\widehat{\Psi}_{\xi\xi}(\xi,t)-\kappa\widehat{\Psi}(\xi,t),\\ \widehat{\Psi}(0,t)=0,\quad \widehat{\Psi}_\xi(1,t)=-d(t),\\ \widehat{\Psi}(\xi,0)=\widehat{\Psi}_0(\xi). \end{cases} \tag{G3}$$

Lemma G.1. If $d(t)$ and $\widehat{\Psi}_0(\xi,0)\in L^2(0,1)$, the solution of (G3) is unique with $\sup_{t\geqslant0}\left\|\widehat{\Psi}(\xi,t)\right\|_{L^2(0,1)}<+\infty$. Further, when $d(t)=0$, (G3) is Mittag-Leffler stable.

Proof. Let $\widetilde{\Psi}(\xi,t)=\widehat{\Psi}(\xi,t)+\frac{\xi^2}{2}d(t)$, then $\widetilde{\Psi}(\xi,t)$ satisfies

$$\begin{cases} {}_0^CD_t^\alpha\widetilde{\Psi}(\xi,t)=a(\xi)\widetilde{\Psi}_{\xi\xi}(\xi,t)-\kappa\widetilde{\Psi}(\xi,t)+f(\xi,t),\\ \widetilde{\Psi}(0,t)=0,\quad \widetilde{\Psi}_\xi(1,t)=0,\\ \widetilde{\Psi}(\xi,0)=\widetilde{\Psi}_0(\xi)=\Psi_0(\xi,0)-\Phi_0(\xi,0)+\dfrac{\xi^2}{2}d(0). \end{cases}$$

Here $f(\xi,t)=-a(\xi)d(t)+\frac{\xi^2}{2}\left(\kappa d(t)+{}_0^CD_t^\alpha d(t)\right)$. The rest of proof is analogous to Lemma 2. □

Then we design the second auxiliary system as

$$\begin{cases} {}_0^CD_t^\alpha\Upsilon(\xi,t)=a(\xi)\Upsilon_{\xi\xi}(\xi,t)-\kappa\Upsilon(\xi,t),\\ \Upsilon(0,t)=0,\quad \Upsilon(1,t)=\Phi(1,t)-\Psi(1,t),\\ \Upsilon(\xi,0)=\Upsilon_0(\xi). \end{cases}$$

Define $\Theta(\xi,t)=-\Upsilon(\xi,t)-\widehat{\Psi}(\xi,t)$, which satisfies

$$\begin{cases} {}_0^C D_t^\alpha \Theta(\xi,t)=a(\xi)\Theta_{\xi\xi}(\xi,t)-\kappa\Theta(\xi,t),\\ \Theta(0,t)=0,\ \ \Theta(1,t)=0,\\ \Theta(\xi,0)=\Theta_0(\xi)=-\Upsilon_0(\xi)-\Psi_0(\xi,0)+\Phi_0(\xi,0).\end{cases} \tag{G4}$$

And we draw two lemmas as follows.

Lemma G.2. Suppose $a'(1)\geqslant 0$. If $\Theta_0(\xi)\in L^2(0,1)$, then the solution of (G4) is unique and is Mittag-Leffler stable.

Lemma G.3. If $\Theta_0(\xi)\in D(\mathcal{A})$, $\Theta_\xi(1,t)$ is Mittag-Leffler stable.

Proof. From the definition of $\mathcal{A}$, (G4) is converted to

$$\begin{cases} {}_0^C D_t^\alpha \Theta(\xi,t)=\mathcal{A}\Theta(\xi,t),\\ \Theta(\xi,0)=\Theta_0(\xi),\end{cases}$$

whose solution is $\Theta(\xi,t)=S_\alpha(t)\Theta_0(\xi)$. Then we get $\|\Theta(\xi,t)\|_{L^2(0,1)}\leqslant M_1E_\alpha(-\mu_1t^\alpha)\|\Theta_0(\xi)\|_{L^2(0,1)}$and $\|S_\alpha(t)\mathcal{A}\Theta_0(\xi)\|_{L^2(0,1)}$ $\leqslant M_1E_\alpha(-\mu_1t^\alpha)\|\mathcal{A}\Theta_0(\xi)\|_{L^2(0,1)}$, for some M_1, $\mu_1>0$. Hence,

$$\begin{aligned}\left\|\Theta_{\xi\xi}(\xi,t)\right\|_{L^2(0,1)}&\leqslant\|\mathcal{A}\Theta(\xi,t)\|_{L^2(0,1)}+\kappa\|\Theta(\xi,t)\|_{L^2(0,1)}\\ &\leqslant M_1E_\alpha(-\mu_1t^\alpha)\left(\|\mathcal{A}\Theta_0(\xi)\|_{L^2(0,1)}+\kappa\|\Theta_0(\xi)\|_{L^2(0,1)}\right).\end{aligned}$$

Sobolev Interpolation Inequality implies $\left\|\Theta_\xi(\xi,t)\right\|_{L^2(0,1)}\leqslant C_1\left(\|\Theta(\xi,t)\|_{L^2(0,1)}+\|\Theta_{\xi\xi}(\xi,t)\|_{L^2(0,1)}\right)$. It follows from Sobolev trace theorem that

$$\begin{aligned}\left|\Theta_\xi(1,t)\right|&\leqslant C_2\left(\left\|\Theta_\xi(\xi,t)\right\|_{L^2(0,1)}+\left\|\Theta_{\xi\xi}(\xi,t)\right\|_{L^2(0,1)}\right)\\ &\leqslant(C_1+1)\,C_2(\|\Theta(\xi,t)\|_{L^2(0,1)}+\left\|\Theta_{\xi\xi}(\xi,t)\right\|_{L^2(0,1)})\leqslant ME_\alpha(-\mu t^\alpha),\end{aligned}$$

for $C_2>0$, where $\mu=\mu_1$ and $M=(C_1+1)\,C_2M_1\left(\|\mathcal{A}\Theta_0(\xi)\|_{L^2(0,1)}+(\kappa+1)\|\Theta_0(\xi)\|_{L^2(0,1)}\right)$. □

Based on Lemma G.3 estimator $\Upsilon_\xi(1,t)$ can compensate $d(t)$. Then, when $c(\xi)=c$, using the backstepping transform, (G1) is transformed into

$$\begin{cases} {}_0^C D_t^\alpha \widehat{\Phi}(\xi,t)=a(\xi)\widehat{\Phi}_{\xi\xi}(\xi,t)+c\widehat{\Phi}(\xi,t-\tau)-\kappa\widehat{\Phi}(\xi,t),\\ \widehat{\Phi}(0,t)=0,\ \widehat{\Phi}_\xi(1,t)=u(t)+d(t)-k(1,1)\Phi(1,t)-\displaystyle\int_0^1 k_\xi(1,\eta)\Phi(\eta,t)\mathrm{d}\eta,\end{cases} \tag{G5}$$

where $\widehat{\Phi}(\xi,\theta)=\widehat{\Phi}_0(\xi,\theta)$ and the kernel function $k(\xi,\eta)$ satisfies (8).

Since $\Upsilon_\xi(1,t)$ compensates the disturbance, we design $u(t)$ as

$$u(t)=-\Upsilon_\xi(1,t)+k(1,1)\Phi(1,t)+\int_0^1 k_\xi(1,\eta)\Phi(\eta,t)\mathrm{d}\eta. \tag{G6}$$

And (G5) is converted to

$$\begin{cases} {}_0^C D_t^\alpha \widehat{\Phi}(\xi,t)=a(\xi)\widehat{\Phi}_{\xi\xi}(\xi,t)+c\widehat{\Phi}(\xi,t-\tau)-\kappa\widehat{\Phi}(\xi,t),\\ \widehat{\Phi}(0,t)=0,\ \widehat{\Phi}_\xi(1,t)=\Theta_\xi(1,t),\\ \widehat{\Phi}(\xi,\theta)=\widehat{\Phi}_0(\xi,\theta).\end{cases} \tag{G7}$$

Similar to Lemma 5, we immediately obtain the following lemma.

Lemma G.4. If $\widehat{\Phi}_0(\xi,\theta)\in C(-\tau,0;L^2(0,1))$, then (G7) has a unique solution with $\lim\limits_{t\to\infty}\|\widehat{\Phi}(\xi,t)\|_{L^2(0,1)}=0$. Moreover, if $\tau=0$, then $\|\widehat{\Phi}(\xi,t)\|_{L^2(0,1)}\leqslant ME_\alpha(-\mu t^\alpha)$ holds for some M, $\mu>0$.

With control (G6), we focus on the closed-loop system of (G1):

$$\begin{cases} {}_0^C D_t^\alpha \Phi(\xi,t)=a(\xi)\Phi_{\xi\xi}(\xi,t)+b(\xi)\Phi(\xi,t)+c(\xi)\Phi(\xi,t-\tau),\\ \Phi(0,t)=0,\ \Phi_\xi(1,t)=-\Upsilon_\xi(1,t)+k(1,1)\Phi(1,t)+\displaystyle\int_0^1 k_\xi(1,\eta)\Phi(\eta,t)\mathrm{d}\eta+d(t),\\ {}_0^C D_t^\alpha \Psi(\xi,t)=a(\xi)\Psi_{\xi\xi}(\xi,t)+b(\xi)\Phi(\xi,t)+c(\xi)\Phi(\xi,t-\tau)-\kappa\left(\Psi(\xi,t)-\Phi(\xi,t)\right),\\ \Psi(0,t)=0,\ \Psi_\xi(1,t)=-\Upsilon_\xi(1,t)+k(1,1)\Phi(1,t)+\displaystyle\int_0^1 k_\xi(1,\eta)\Phi(\eta,t)\mathrm{d}\eta,\\ {}_0^C D_t^\alpha \Upsilon(\xi,t)=a(\xi)\Upsilon_{\xi\xi}(\xi,t)-\kappa\Upsilon(\xi,t),\\ \Upsilon(0,t)=0,\ \Upsilon(1,t)=\Phi(1,t)-\Psi(1,t),\\ \Phi(\xi,\theta)=\Phi_0(\xi,\theta),\ \Psi(\xi,0)=\Psi_0(\xi),\ \Upsilon(\xi,0)=\Upsilon_0(\xi).\end{cases} \tag{G8}$$

Let $\mathcal{H}=[L^2(0,1)]^3$ and we conclude:

Theorem G.5. Assume $d(t)$ and $a'(1)\geqslant 0$. For any $(\Phi_0(\xi,\theta),\ \Psi_0(\xi),\ \Upsilon_0(\xi))^\top\in C(-\tau,0;L^2(0,1))\times[L^2(0,1)]^2$, the unique solution of (G8) in $C(0,\infty;\mathcal{H})$ satisfying

$$\lim_{t\to\infty}\|\Phi(\xi,t)\|_{L^2(0,1)}=0\quad and\quad \sup_{t\geqslant 0}\|[\Psi(\xi,t),\ \Upsilon(\xi,t)]\|_{L^2(0,1)}<+\infty.$$

Moreover, if $\tau=0$, then $\|\Phi(\xi,t)\|_{L^2(0,1)}\leqslant ME_\alpha(-\mu t^\alpha)$ holds for M, $\mu>0$. Further, if $d(t)=0$, $\Psi(\xi,t)$, $\Upsilon(\xi,t)$ are also Mittag-Leffler stable.

The proof of this theorem is analogous to Theorem 1, we omit the details.

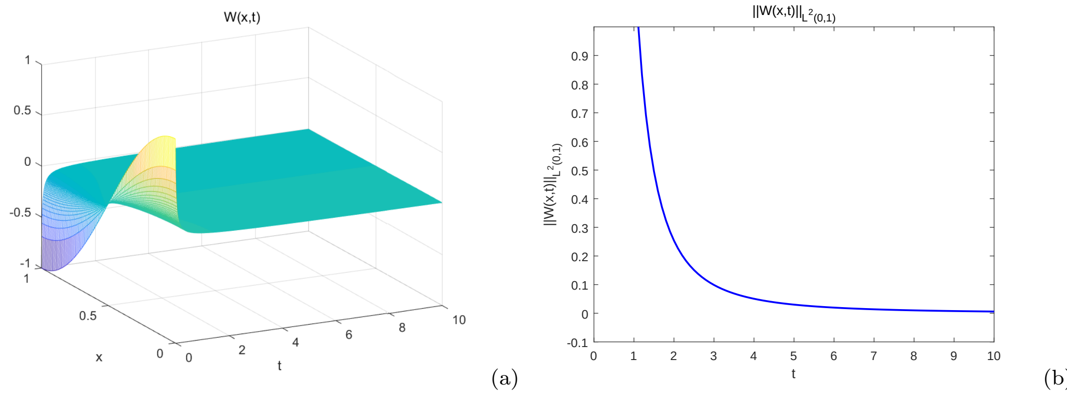

Fig.1: (a) Simulation result of $\Phi(\xi,t)$ in (G8) with controller (H2); (b) the $L^2(0,1)$-norm of $\Phi(\xi,t)$ with controller (H2).

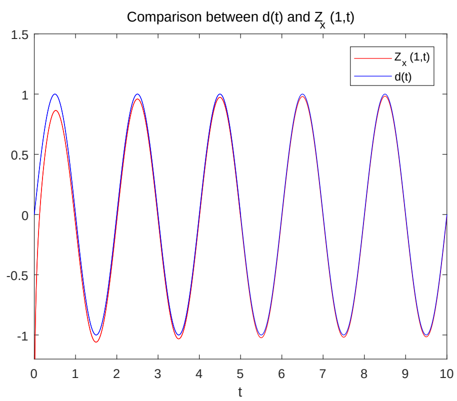

Fig.2: Comparison between $d(t)$ and $\Upsilon_\xi(1,t)$.

Appendix H An illustrative Example

In (G8), choose $\alpha=0.9$, $\tau=0.1$, $a(\xi)=\left(1+\xi^2\right)^2$, $b(\xi)\equiv 2$, $c(\xi)\equiv 1$, $d(t)=\sin\pi t$ and $\kappa=12$. The initial conditions are taken as $\Phi_0(\xi,\theta)=(\theta+1)\cos\pi\xi$, $\Psi_0(\xi,\theta)=\xi^2\theta^2$ and $\Upsilon_0(\xi)=\cos 2\xi$, $\theta\in[-0.1,0]$. It's easy to check that $a'(1)=8\geqslant 0$, $\kappa>\frac{K}{2}+\sqrt{2}\widetilde{M}=8+\sqrt{2}$, $d(t)$ is bounded and $\left(\Phi_0(\xi,\theta),\,\Psi_0(\xi,\theta),\,\Upsilon_0(\xi)\right)^\top\in\widetilde{H}$. According to Lemma 1, ${}_0^C D_t^\alpha d(t)$ is also bounded, and thus, all the conditions in Theorem G.5 hold. With these parameters, the kernel function $k(\xi,\eta)$ in (8) can be provided by

$$k(\xi,\eta)=-14\left(1+\xi^2\right)^{\frac{1}{2}}\left(1+\eta^2\right)^{-\frac{3}{2}}\arctan\eta\,\frac{I_1\left(\sqrt{12\left(\arctan^2\xi-\arctan^2\eta\right)}\right)}{\sqrt{12\left(\arctan^2\xi-\arctan^2\eta\right)}}. \tag{H1}$$

Hence, $k(1,1)=-\frac{7\pi}{8}$, $k_\xi(1,\eta)=-7\sqrt{2}(1+\eta^2)^{-\frac{3}{2}}\arctan\eta\,\frac{I_1(\sqrt{12(\frac{\pi^2}{16}-\arctan^2\eta)})}{\sqrt{12(\frac{\pi^2}{16}-\arctan^2\eta)}}$ and the feedback controller can be provided by

$$u(t)=-\Upsilon_\xi(1,t)-\frac{7\pi}{8}\Phi(1,t)-7\sqrt{2}\int_0^1\left(1+\eta^2\right)^{-\frac{3}{2}}\arctan\eta\,\frac{I_1\left(\sqrt{12\left(\frac{\pi^2}{16}-\arctan^2\eta\right)}\right)}{\sqrt{12\left(\frac{\pi^2}{16}-\arctan^2\eta\right)}}\Phi(\eta,t)\mathrm{d}\eta. \tag{H2}$$

Fig.1(a) shows the state $\Phi(\xi,t)$ of (G8) with feedback controller (H2). It's seen in Fig.1(b) that the $L^2(0,1)$-norm of $\Phi(\xi,t)$, i.e., $\|\Phi(\xi,t)\|_{L^2(0,1)}$ tends to zero as $t\to+\infty$, which implies that with the effect of controller (H2), the closed-loop system of (G8) is asymptotically stable. Moreover, Fig.2 compares $d(t)$ with $\Upsilon_\xi(1,t)$, which indicates that $\Upsilon_\xi(1,t)$ can behave as an estimator of the disturbance $d(t)$. Note that the initial time of Υ is $t=0$ and the stability analysis focuses on the system behavior when t is large enough, thus here the initial points of time in Fig.1 and Fig.2 are both 0.

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