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• **Supplementary File** •

Social Conformity Creates Consensus and Strong Diversity of Hegselmann-Krause Opinion Dynamics

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Appendix A Proof of Theorem 1, Theorem 2, Propositon 1, Proposition 2

Theorem 1. In Model I, suppose Assumption 1 holds, opinion consensus can be reached in finite time, and the upper bound of convergence time is $T^* = \left[\frac{n(1-\varepsilon)}{2p\varepsilon}\right] + 1$.

Theorem 2. In Model II, suppose that Assumption 1 holds, consider the condition that

$$N_i(t) \neq \varnothing, \forall i \in V, t \in \mathbb{N} \tag{A1}$$

then the consensus of private opinion $y(t)$ and expressed opinion $\hat{y}(t)$ both can be reached in finite time T_1^*, and we have $y(t) = \hat{y}(t)$, for $t > T_1^*$.

Proposition 1. Suppose that the time when Model II converges to stable state is T^*, if $\delta_{ij}^t \to 0$, $i, j \in V$, for $t > T^*$, then we have $\Delta_{ij}^t \to 0$, thus strong diversity of expressed opinions between agent i and j exists.

Proposition 2. In Model II, suppose that Assumption 1 holds, for all $t \in \mathbb{N}$, we have

$$|\hat{y}_{max}(t) - \hat{y}_{min}(t)| \leqslant |y_{max}(t) - y_{min}(t)|$$

The following is the proof of Theorem 1.

Denote $y_{max}(t)$ and $y_{min}(t)$ as the upper and lower bound of value set $\{y_i(t), i \in V\}$. If $y_{max}(0) - y_{min}(0) \leqslant \varepsilon$, then consensus of group will be achieved directly. So we just consider common conditions: $y_{max}(t) - y_{min}(t) > \varepsilon$. In the set $\{y_i(t), i \in V\}$, we have $y_{min}(t) < y_{max}(t) - \varepsilon$, and $y_{max}(t) > y_{min}(t) + \varepsilon$. So we have $y_{avg}(t) = \frac{1}{n}\sum_{i\in V} y_i(t) < \frac{ny_{max}-\varepsilon}{n}$, for $\forall i \in V$.

Then, we can rewrite the formula of model I as

$$\begin{aligned} y_i(t+1) &= \frac{1-p_i}{1+|N_i(t)|}\left[y_i + \sum_{j\in N_i(t)} y_j(t)\right] + p_i y_{avg}(t) \\ &< (1-p_i)y_{max}(t) + p_i\frac{ny_{max}(t)-\varepsilon}{n} = y_{max}(t) - \frac{p_i\varepsilon}{n} \end{aligned} \tag{A2}$$

Because $i \in V$ is an arbitrary element, for $y_{max}(t)$ and $y_{min}(t)$, we have $y_{max}(t+1) < y_{max}(t) - \frac{p\varepsilon}{n}$ and $y_{min}(t+1) > y_{min}(t) + \frac{p\varepsilon}{n}$.

Set $\Delta y(t) = y_{max}(t) - y_{min}(t)$ as gap of group opinion, then we have

$$\Delta y(t+1) < \Delta y(t) - \frac{2p\varepsilon}{n}$$

For $\Delta y(0) \leqslant 1$, and after iterative calculations we have

$$\Delta y(t) < 1 - \frac{2p\varepsilon}{n}t$$

The $\Delta y(t)$ is monotone decreasing over t, until $\Delta y(t) \leqslant \varepsilon$. After that, group consensus will be reached with finitely steps.

$$T^* = \left[\frac{n(1-\varepsilon)}{2p\varepsilon}\right] + 1 \tag{A3}$$

It has been proved that if confidence interval $\varepsilon > 0$ and group pressure $p > 0$ in Model I, the group can always converge to consensus in finite time.

The following is the proof of Theorem 2.

Let's start with several related definitions and propositions which will be used later.

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Definition 1. Coefficient of Ergodicity [1]: For a stochastic matrix $M = \{m_{ij}\}$, define its coefficient of ergodicity as $\rho(M) = 1 - \min_{i,j} \sum \min\{m_{ik}, m_{jk}\}$.

Proposition 3. (see [1]) For any stochastic matrices M_1 and M_2, it follows that $\rho(M_1 M_2) \leqslant \rho(M_1)\rho(M_2)$.

Definition 2. Scrambling matrix [2]: For any non-negative n order matrix M, if there exists $k \in \{1, 2, \cdots, n\}$ such that for arbitrary i and j with $i, j \in \{1, 2, \cdots, n\}$, $i \neq j$, m_{ik} and m_{jk} are both positive, then we call M a scrambling matrix.

Furthermore, the scrambling matrix has the following property.

Proposition 4. (see [2]) Let $\mathcal{M} = \{M_i\}, i = 1, 2, \cdots$ be a compact set of scrambling stochastic. Then for each infinite sequence $M_{i_1}, M_{i_1}, \cdots$ there exists a row vector $\boldsymbol{c}$ such that

$$\lim_{j\to\infty} M_{i_j} M_{i_{j-1}} \cdots M_{i_1} = \mathbf{1c} \tag{A4}$$

We can rebuild the interaction of agents associated with Model II by using a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The system state could be described as $y = [y_1, y_2, \cdots, y_n]^\top$, $\hat{y} = [\hat{y}_1, \hat{y}_2, \cdots, \hat{y}_n]^\top$. We define W as the influence matrix to describe the model formula in spatial form, the ij-element of the nonnegative matrix W is w_{ij}, connoting the impact of weights from i to j. So the opinion formation process described by Eq.(2) and Eq.(3), is captured, for $t = 0, 1, \cdots, \infty$, by the discrete-time system.

$$\begin{bmatrix} y(t+1) \\ \hat{y}(t+1) \end{bmatrix} = \begin{bmatrix} W_{11}^t & W_{12}^t \\ W_{21}^t & W_{22}^t \end{bmatrix} \begin{bmatrix} y(t) \\ \hat{y}(t) \end{bmatrix} \tag{A5}$$

Here, we build an new interactive network $\mathcal{G}[W^t]$ contains $2n$ nodes $\mathcal{V}^* = \{1, 2, \cdots 2n\}$. Nodes $\mathcal{V}_p^* = \{1, 2, \cdots n\}$ represent all agents' private opinions $y(t)$, and $\mathcal{V}_e^* = \{n+1, \cdots 2n\}$ represent expressed opinions $\hat{y}(t)$. Set $\alpha_i = 1/(1 + |N_i(t)|)$, the submatrix $W_{11}^t = diag\{\alpha_i\}$ and $W_{21}^t = diag\{(1 - p_i)\alpha_i\}$ are both diagonal matrices. W_{22}^t contains the link from $\hat{y}_{avg}(t)$ to $\hat{y}(t+1)$, having a specific structure, holding all elements to be positive.

$$W^t = \left[\begin{array}{ccc|ccc} + & & 0 & 0 & & \\ & \ddots & & & \ddots & \\ 0 & & + & & & 0 \\ \hline + & & 0 & + & \cdots & + \\ & \ddots & & \vdots & + & \vdots \\ 0 & & + & + & \cdots & + \end{array}\right]$$

The submatrix W_{12}^t describes influences between nodes in $\mathcal{V}_p^*$ and $\mathcal{V}_e^*$, in particular, all the diagonal elements of W_{12}^t are equal to zero. The non-zero elements in the ith row of W_{12}^t represent the neighbors of agent i, which are determined by $N_i(t) = \{1 \leqslant j \leqslant n, j \neq i | \, |y_i(t) - \hat{y}_j(t)| \leqslant \varepsilon_i\}$.

Lemma 1. Suppose that Assumption 1 and condition (A1) hold. In graph $\mathcal{G}[W^t]$, for $\forall i \in V_p^*$, node i is a root of node j, $j \in V^*$.

Proof. (1). If $j \in V_e^*$. We can always find a path from i to j in the directed graph $\mathcal{G}[W^t]$, as $i \to i+n \to j$, for $w_{ii}, w_{i+n,i}, w_{j,i+n} > 0$; (2). If $j \in V_p^*$. We have $N_i(t) \neq \varnothing$ for any $i \in \{1, 2, \cdots n\}$, so every row of W_{12}^t has at least one positive element, assume the jth row with positive element $w_{js} > 0$, then we have a path from i to j, as $i \to i+n \to s \to j$, for $w_{ii}, w_{i+n,i}, w_{s,i+n}, w_{js} > 0$.

Thus, for $\forall i \in V_p^*$, node i is a root of every other $2n - 1$ nodes.

Lemma 2. In graph $\mathcal{G}[W^t]$, there exists a positive integer k, for any time-intervals $[t_{j_k}, t_{j_{k+1}})$, such that any node i from set V_p^* is a strong root of every other $2n - 1$ nodes at time $t_{j_{k+1}}$ in the composition of $\mathcal{G}_t$ meeted through $[t_{j_k}, t_{j_{k+1}})$.

The following proposition will be used to prove Lemma 2.

Lemma 3. (Proposition 3 in [3]) suppose $n > 1$, and denote $\mathcal{G}_{p1}, \mathcal{G}_{p2}, \cdots, \mathcal{G}_{pk}$ be a finite sequence of rooted graphs in $\mathcal{G}$. If $\mathcal{G}_{p1}, \mathcal{G}_{p2}, \cdots, \mathcal{G}_{pk}$ are all rooted at v and $k \geqslant n - 1$, then $\mathcal{G}_{pk} \circ \mathcal{G}_{pk-1} \circ \cdots \circ \mathcal{G}_{p1}$ is strongly rooted at v.

Proof. From the lemma 3, for any integer $k \geqslant n - 1$, the composition of graphs as $\mathcal{G}_t \circ \mathcal{G}_{t+1} \circ \cdots \circ \mathcal{G}_{t+k}$ is strongly rooted at any node i, for all $i \in V_p^*$, because i is a root of each $\mathcal{G}_t$, there exists a directed edge $i \to j$, for all $j \in V^*$ in the composition of graphs $\mathcal{G}_t$. So node i is a neighbor of every other $2n - 1$ nodes at time $t_{j_{k+1}}$ in the composition of $\mathcal{G}_t$ meeted through $[t_{j_k}, t_{j_{k+1}})$.

Now we can present the consensus result of Theorem 1.

Proof. According to Lemma 2, we can find a positive $k \geqslant n-1$, such that node $i \in V_p^*$ is a neighbor of every other nodes in the composition of $\mathcal{G}_t$ meeted through $[t_{j_k}, t_{j_{k+1}})$. So the multiplication of matrix $\hat{W}^{\tau_j} = W^{t_{j_k}} W^{t_{j_k}+1} \cdots W^{t_{j_{k+1}}-1}$ has at least one positive column, that means $\hat{W}^{\tau_j}$ is a scrambling matrix.

Denote opinion set $Y = [y_1, \cdots, y_n, \hat{y}_1, \cdots, \hat{y}_n]^\top$, Eq. (A5) can be written as $Y(t+1) = W^t Y(t)$. so we have

$$\begin{aligned} Y(t_{j_k}) &= W^{t_{j_k}-1} \cdots W^{t_0} Y(0) \\ &= (W^{t_{j_k}-1} \cdots W^{t_{j_{k-1}}}) \cdots (W^{t_{j_1}-1} \cdots W^{t_0}) Y(0) \\ &= \hat{W}^{\tau_{j-1}} \cdots \hat{W}^{\tau_0} Y(0) \end{aligned} \tag{A6}$$

By the Proposition 4, There is always going to be a row vector $\boldsymbol{c}$ make

$$\lim_{j\to\infty} \hat{W}^{\tau_j}\hat{W}^{\tau_{j-1}}\cdots\hat{W}^{\tau_0} = \mathbf{1}\boldsymbol{c}$$

set $Y_{ss} = \boldsymbol{c}Y(0)$, then we get $\lim_{j\to\infty} Y(t_{j_k}) = Y_{ss}\mathbf{1}$.

Since a consensus will be reached for opinion set $Y(t) = [y_1,\cdots,y_n,\hat{y}_1,\cdots,\hat{y}_n]^\top$ contains both private opinions $y(t)$ and expressed opinions $\hat{y}(t)$. There exists a smallest T_0^* such that $\left|Y_{max}(T_0^*) - Y_{min}(T_0^*)\right| \leqslant \varepsilon$, so from Eq.(2) and (3), the consensus of expressed opinion $\hat{y}(t)$ and private opinion $y(t)$ are both reached after $T_1^* = T_0^* + 1$, and we have $y(t) = \hat{y}(t)$, for $t > T_1^*$.

Here, we can establish a new relationship between the scrambling matrices and rooted graphs, by consider the situation where each agent of V_p^* is strong rooted across some finite-length intervals. Hence, the consensus can be reached in compliance model under such condition (A1).

The following is the proof of Propusition 1.

We analyze the reasons in model II for two agents' private opinions which are very close $\delta_{ij}^t = |y_i(t) - y_j(t)| \to 0$ (less than ε), but do not communicate with each other:

- At time t, agent i updates his/her private opinion based on bounded confidence, so agent i can only seek 'neighbors' within his/her the confidence interval ε_i.
- Conformity behavior of compliance creates difference between private opinion y_i and expressed opinion $\hat{y}_i$ of agent i, $i \in V$.
- Due to the information asymmetry, agent i can only observe the expressed opinion $\hat{y}_j$ of other agent j, $j \in V, j \neq i$, but cannot obtain their private opinion y_j.

Thus, for two adjacent agents i and j, a higher level of conformity pressure p can make agent j's expressed opinion $\hat{y}_j$ beyond agent i's confidence interval ε_i, as $|y_i - \hat{y}_j| > \varepsilon_i$, and similarly, $|y_j - \hat{y}_i| > \varepsilon_j$. Thus, we have $j \notin N_i(t)$ and $i \notin N_j(t)$, and there is no communication between i and j.

Proof of Proposition 1. For agents i and j with $\delta_{ij}^t = |y_i(t) - y_j(t)| \to 0$, $i,j \in V$, $t > T^*$, that means $j \notin N_i(t)$ and $i \notin N_j(t)$. From Eq. (2) and (3), we have $y_i(t+1) = y_i(t)$, $y_j(t+1) = y_j(t)$. Then $\hat{y}_i(t+1) = (1-p)y_i(t+1) + p\hat{y}_{avg}(t)$, $\hat{y}_i(t+1) = (1-p)y_i(t+1) + p\hat{y}_{avg}(t)$, so

$$\Delta_{ij}^t = |\hat{y}_i(t) - \hat{y}_j(t)| = (1-p)\,|y_i(t) - y_j(t)| = (1-p)\delta_{ij}^t \tag{A7}$$

Therefore, the gap between expressed opinions Δ_{ij}^t will be limited by private opinions δ_{ij}^t. Because of the unobservability of private opinions, when $\delta_{ij}^t = |y_i(t) - y_j(t)| \to 0$, we have $\Delta_{ij}^t = |\hat{y}_i(t) - \hat{y}_j(t)| \to 0$. Hence, we prove that in the stable state, there exists a sufficient condition when two agents' private opinions are very close $\delta_{ij}^t \to 0$, strong diversity of expressed opinions $\Delta_{ij}^t \to 0$ can be observed.

The following is the proof of Propusition 2.

Proof. Set $y_{max}(t)$ and $y_{min}(t)$ as the upper and lower bound of value set $\{y_i(t), i \in V\}$; in the same way, we have $\hat{y}_{max}(t)$ and $\hat{y}_{min}(t)$ of $\{\hat{y}_i(t), i \in V\}$. For $t = 0$, we have $y_{min}(0) \leqslant \hat{y}_{min}(0) \leqslant \hat{y}_{avg}(0) \leqslant \hat{y}_{max}(0) \leqslant y_{max}(0)$, suppose $y_{max}(0)$ is corresponding to agent k, $k \in V$, then agent k's private opinon at time t is

$$y_k(1) = \frac{1}{1 + |N_k(0)|}\left[y_{max}(0) + \sum_{j\in N_k(0)} \hat{y}_j(0)\right] \tag{A8}$$

Because $N_k(0) = \{1 \leqslant j \leqslant n \cap j \neq k|\,|y_{max}(0) - \hat{y}_j(0)| \leqslant \varepsilon\}$, and $y_{max}(0) \geqslant \hat{y}_{max}(0) \geqslant \hat{y}_{avg}(0)$, so we have

$$\sum_{j\in N_k(0)} \hat{y}_j(0) \geqslant |N_k(0)|\,\hat{y}_{avg}(0)$$

Then from Eq.(A8), we have $y_k(1) \geqslant \hat{y}_{avg}(0)$, more generally, $y_{max}(1) \geqslant \hat{y}_{avg}(0)$, suppose $y_{max}(1)$ is corresponding to agent s, $s \in V$, and Assumption 1 holds with $p_i = p \in (0,1]$, $\forall i \in V$, from Eq. (3), we known $\hat{y}_{max}(1)$ is corresponding to agent s too. Next, the expressed opinion $\hat{y}_{max}(1)$ will be

$$\hat{y}_{max}(1) = (1-p)y_{max}(1) + p\hat{y}_{avg}(0) \leqslant (1-p)y_{max}(1) + py_{max}(1) = y_{max}(1)$$

Meanwhile, we can prove that $\hat{y}_{min}(1) \geqslant y_{min}(1)$. So the inequation

$$y_{min}(t) \leqslant \hat{y}_{min}(t) \leqslant \hat{y}_{avg}(t) \leqslant \hat{y}_{max}(t) \leqslant y_{max}(t)$$

holds at $t = 0, 1, 2, \cdots$, and we have $|\hat{y}_{max}(t) - \hat{y}_{min}(t)| \leqslant |y_{max}(t) - y_{min}(t)|$, for all $t \in \mathbb{N}$.

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