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New stability results of generalized impulsive functional differential equations

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Dear editor,

An impulsive functional differential equation (IFDE) is a hybrid dynamical system including both continuous-time dynamic exhibited via a functional differential equation and discrete-time dynamics which are shown by impulses. To the best of our knowledge, impulsive phenomena, which are presented in the form of instantaneous jump, exist in the fields of engineering and nature. In recent years, a new class of impulsive equations called as generalized IFDEs have attracted much attention of researchers. In contrast to ordinary IFDEs, the typical characteristic of generalized IFDEs is that the state change at impulsive instants is dependent on the past state. Under Razumikhin method, some novel stability results are derived. For example, several global exponential stability and uniform stability criteria are derived in [1] for IFDEs with any time delays. The novel exponential stability criteria for delayed IFDEs with impulse time windows are obtained in [2]. The p-th moment exponential stability problem for stochastic IFDEs with Markovian switching has been coped with in [3]. In [4], some stability criteria for IFDEs with delayed impulses are provided. The exponential stability results for stochastic IFDEs with delayed impulses are presented in [5]. As a matter of fact, generalized IFDE can be viewed as the extended form of ordinary IFDE. Therefore, the stability results proposed in [1–5] are also suitable for ordinary IFDEs.

Although the innovative results for generalized IFDEs via Razumikhin method are convenient to judge the stability, there also exist some rigorous restrictions on impulses. Generally speaking, these restrictions could be classified into two categories. The first one is that all the impulses are required to be convergent (see [1, 2]). The second one is that the number of divergent impulses is restricted to be finite (see [3–5]). Obviously, these restrictions severely restrict the effectiveness of the results in [1–5]. For example, these stability results proposed in [1–5] could not be applied to investigating the stability of the generalized IFDEs with an infinite number of divergent impulses.

Stability results. Let N denote the set of natural numbers, $N_0 = N \cup \{0\}$, and $\mathrm{PC}([a,b],\mathbb{R}^n) = \{\varphi : [a,b] \to \mathbb{R}^n$ is continuous everywhere except at a finite number of points $t \in [a,b]$, at which both $\varphi(t^+)$ and $\varphi(t^-)$ exist$\}$.

We consider the following generalized IFDE:

$$\begin{cases} \dot{x}(t) = f(t, x, x_t), \ t \in [t_k, t_{k+1}), k \in N_0, \\ \Delta x(t_k) = R_k\left(x\left(t_k^-\right)\right) + S_k(x_{t_k^-}), \ k \in N, \\ x(t_0 + s) = \phi(s), \ s \in [-\tau, 0], \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $f : \mathbb{R} \times \mathbb{R}^n \times \mathrm{PC}([-\tau,0],\mathbb{R}^n) \to \mathbb{R}^n$, $R_k : \mathbb{R}^n \to \mathbb{R}^n$, $S_k : \mathrm{PC}([-\tau,0],\mathbb{R}^n) \to \mathbb{R}^n$, $\phi : \mathrm{PC}([-\tau,0],\mathbb{R}^n)$, $0 < \tau < \infty$, t_k, $k \in N$, is the impulsive time satisfying $t_0 < t_1 < \cdots < t_k < \cdots$ and $\lim_{k\to\infty} t_k = \infty$, $x_t = x(t+s)$ and $x_{t^-} = x(t^-+s)$, $s \in [-\tau, 0]$, respectively. Without loss of generality, we also assume that system (1) satisfies all the assumptions presented in [1, 4].

Definition 1. The origin of the system (1) is globally exponentially stable if there exist positive constants ε and M such that

$$\|x(t, t_0, \phi)\| \leqslant M\|\phi\|_\tau \mathrm{e}^{-\varepsilon(t-t_0)}, \quad t \geqslant t_0,$$

where $\|\phi\|_\tau = \sup_{s\in[-\tau,0]} \|\phi(s)\|$.

Let $\beta_0 = 1$, $c \in R$ and $\beta_i > 0$ for $i \geqslant 1$. Define

$$z(t) = \prod_{i=0}^{k} \beta_i \mathrm{e}^{c(t-t_0)}, \quad t \in [t_k, t_{k+1}), k \in N_0. \tag{2}$$

We say $z(t)$ is uniformly stable if there exist two constants $M \geqslant 1$ and $\varepsilon > 0$ such that

$$z(t^{**}) \leqslant Mz(t^*)\,\mathrm{e}^{-\varepsilon(t^{**}-t^*)}, \quad \forall t^{**} \geqslant t^* \geqslant t_0. \tag{3}$$

Lemma 1. Function $z(t)$ is uniformly stable if and only if, for any $\lambda \in (0, 1)$, there exists $\tilde{T}(\lambda) > 0$ such that for any

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finite $T \geqslant \tilde{T}(\lambda)$ and $t \geqslant t_0$,

$$\prod_{t<t_i\leqslant t+T} \beta_i \mathrm{e}^{cT} \leqslant \lambda, \quad \prod_{t<t_i\leqslant t+\theta} \beta_i \mathrm{e}^{c\theta} \leqslant d, \ \theta \in [0,T], \quad (4)$$

where $d \geqslant 1$ is some positive number.

Lemma 2. Let $y : \mathbb{R} \to \mathbb{R}_+$ be a piecewise continuous function such that $y(t) = y(t^+)$. Assume that $y(t)$ satisfies the following conditions:

(L1) $D^+y(t) \leqslant cy(t)$ for any $t \in [t_k, t_{k+1})$, $k \in N_0$, whenever $y(t+s) \leqslant qy(t)$, $s \in [-\tau, 0]$;

(L2) $y(t_k) \leqslant r_k y(t_k^-) + s_k \sup_{\theta\in[-\tau,0]}\{y(t_k^- + \theta)\}$, $k \in N$;

(L3) Function $z(t)$, which is defined by (2), is uniformly stable;

(L4) $\rho q \geqslant X_z(T)$.

In the above conditions, $q > 1$, $c \in \mathbb{R}$, $r_k, s_k > 0$, $\beta_k = r_k + s_k q$, $T \in \Theta_z$, and $\rho \in (0,1)$. Then, we get

$$y(t) \leqslant \sup_{s\in[-\bar{T},0]} \{y(\bar{t}_0 + s)\} \exp\left(\frac{\ln \varrho}{\bar{T}}(t - \bar{t}_0)\right), \quad (5)$$

where $\bar{T} = T + \tau$, $\varrho = \max\{\rho_T, \rho\}$ with $\rho_T = \sup_{t\geqslant t_0}\{\prod_{t<t_i\leqslant t+T} \beta_i\}\mathrm{e}^{cT}$, $\bar{t}_0 = t_0 + T$.

Theorem 1. Let Assumptions (H1)–(H5) in [1] hold. Assume that there exist a function $V(t, x(t)) \in \upsilon_0$, constants $q > 1$, $p > 0$, $c_1, c_2 > 0$, $c \in \mathbb{R}$, $r_k, s_k > 0$, satisfying the following conditions:

(T1) $c_1\|x\|^p \leqslant V(t,x) \leqslant c_2\|x\|^p$, for any $t \geqslant t_0 - \tau$ and $x \in \mathbb{R}^n$;

(T2) $D^+V(t,x(t)) \leqslant cV(t,x(t))$, whenever $V(t+s, x(t+s)) \leqslant qV(t,x(t))$ for any $t \in [t_k, t_{k+1})$, $k \in N_0$, and $s \in [-\tau, 0]$;

(T3) $V(t_k, x(t_k)) \leqslant r_k V(t_k^-, x(t_k^-)) + s_k \sup_{s\in[-\tau,0]} V(t_k^- + s, x(t_k^- + s))$ for any $k \in N$;

(T4) Function $z(t)$ is uniformly stable, where $z(t)$ is defined by (2) with $\beta_k = r_k + s_k q$, $k \in N$;

(T5) $q > X_z(T)$, where $T \in \Theta_z$.

Then, the origin of the system (1) is globally exponentially stable.

Proof. According to condition (T5), we know there exists some $\rho \in (0,1)$ such that

$$q\rho \geqslant X_z(T). \quad (6)$$

Let $y(t) = V(t, x(t))$. We have from Lemma 2, conditions (T2)–(T4) and (6) that, for any $t \in [\bar{t}_0, \infty)$,

$$y(t) \leqslant \sup_{s\in[-\bar{T},0]} \{y(\bar{t}_0 + s)\} \exp\left(\frac{\ln \varrho}{\bar{T}}(t - \bar{t}_0)\right), \quad (7)$$

where $\bar{T} = T + \tau$, $\varrho = \max\{\rho_T, \rho\}$ with $\rho_T = \sup_{t\geqslant t_0}\{\prod_{t<t_i\leqslant t+T} \beta_i\}\mathrm{e}^{cT}$. The above inequality and condition (T1) imply that

$$\|x(t)\| \leqslant \left(\frac{c_2}{c_1}\right)^{\frac{1}{p}} \sup_{s\in[-\bar{T},0]} \{\|x(\bar{t}_0 + s)\|\} \times \exp\left(\frac{\ln \varrho}{p\bar{T}}(t - \bar{t}_0)\right),$$

which yields the globally exponential stability of the system (1).

Remark 1. In Theorem 1, it is crucial to finding T and q such that

$$q > X_z(T) \quad (8)$$

and

$$\sup_{t\geqslant t_0} \left\{\prod_{t<t_i\leqslant t+T} (r_i + s_i q)\mathrm{e}^{cT}\right\} < 1. \quad (9)$$

Let δ be the positive number such that $\sup_{t\geqslant t_0}\{\prod_{t<t_i\leqslant t+\delta}(r_i + s_i)\} = 1$. It is obvious that $T > \delta$. By the definition of $X_z(T)$ and (8), we have

$$q > \sup_{t\geqslant t_0} \left\{\max_{s\in[0,T]} \left\{\prod_{t<t_i\leqslant t+s} (r_i + s_i)\right\}\right\} > \sup_{t\geqslant t_0} \left\{\max_{s\in[0,\delta]} \left\{\prod_{t<t_i\leqslant t+s} (r_i + s_i)\right\}\right\} \equiv X(\delta).$$

Moreover, it follows from (9) that

$$q < \max_i \left\{\frac{1 - r_i}{s_i}\right\} \equiv \bar{q}.$$

In the application, we could first find some $q \in (X(\delta), \bar{q})$ and then choose $T > \delta$ to satisfy both (8) and (9).

Remark 2. In Theorem 1, we restrict that

$$\sup_{t\geqslant t_0} \left\{\prod_{t<t_i\leqslant t+T} (r_i + s_i q)\right\} \mathrm{e}^{cT} < 1. \quad (10)$$

The above restriction admits that $r_k + s_k > 1$ for some $k \in N$. Namely, Theorem 1 permits that some impulses have destabilizing effect. Eq. (10) indicates that the destabilizing effect of divergent impulses can be compensated by stabilizing effect of other convergent impulses.

Remark 3. If for any $k \in N$, there exist some $l \in N$ and $\Gamma > 0$ such that $t_k = t_k + \Gamma$, $R_k = R_{k+l}$ and $S_k = S_{k+l}$, we say that the system (1) has periodical impulses. Obviously, we can choose $r_k = r_{k+l}$ and $s_k = s_{k+l}$. It is easy to show that $z(t)$ is uniformly stable if and only if

$$\chi_\Gamma = \prod_{t_0<t_i\leqslant t_0+\Gamma} (r_i + s_i q)\mathrm{e}^{c\Gamma} < 1.$$

Therefore, when the system (1) has periodical impulses, we can restrict that $T = \Gamma$ in Theorem 1.

Comparison with existing results. There exist some rigorous restrictions on impulses in the existing results, which can be summarized as the following cases.

Case A. Some results require that all the impulses must have stabilizing effect. For example, [1, Theorem 3.1] and [2, Theorem 1] restrict that

$$V(t_k, x(t_k)) \leqslant r_k V\left(t_k, x\left(t_k^-\right)\right) + s_k \sup_{s\in[-\tau,0]} V\left(t_k^- + s, x\left(t_k^- + s\right)\right),$$

and $r_k + s_k \leqslant \dfrac{1}{q}$. Obviously, $r_k + s_k < 1$ because of $q > 1$. This restriction implies that all the impulses are convergent.

Case B. Other results permit some impulses have destabilizing effect, but the number of divergent impulses must be finite. For example, [3, Thoerem 3.1] restricts that

$$\rho_1 + \rho_2 < 1,$$

and

$$V(t_k, x(t_k)) \leqslant \rho_1(1 + q_k)V\left(t_k^-, x\left(t_k^-\right)\right)$$

$$+\rho_2(1+e_k)\sup_{s\in[-\tau,0]} V\left(t_k^-+s, x\left(t_k^-+s\right)\right) \tag{11}$$

with $q_k, e_k \geqslant 0$ and

$$\sum_{k=1}^{\infty}\max\{q_k,e_k\}<\infty. \tag{12}$$

Clearly, there must exist some $K_2 \in N$ such that $(\rho_1+\rho_2)(1+\max\{q_k,e_k\}) < 1$ for $k > K_2$. Otherwise, there must exist some sequence of positive integers $\{k_i^2\}_{i=1}^{\infty}$ such that

$$\max\{q_{k_i^2}, e_{k_i^2}\} \geqslant \frac{1}{\rho_1+\rho_2}-1, \quad i\in N. \tag{13}$$

Eq. (13) implies that

$$\begin{aligned}\sum_{k=1}^{\infty}\max\{q_k,e_k\} &\geqslant \sum_{i=1}^{\infty}\max\{q_{k_i^2}, e_{k_i^2}\}\\ &\geqslant \lim_{i\to\infty}\left(\frac{1}{\rho_1+\rho_2}-1\right)i=\infty,\end{aligned}$$

which conflicts with (12). Consequently, the number of divergent impulses is not greater than K_2.

In Theorem 1, we only restrict that Eq. (10) holds. This restriction neither requires that all impulses have stabilizing effect nor claims that the number of divergent impulses is finite. Therefore, Theorem 1 is more effective.

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