SCIENCE CHINA
Information Sciences

• LETTER •

January 2022, Vol. 65 119203:1–119203:2
https://doi.org/10.1007/s11432-020-2950-x

Quasi-synchronization of bounded confidence opinion dynamics with a stochastic asynchronous rule

Wei SU[1,2], Xue-Qiao WANG[3], Ge CHEN[4] & Kai SHEN[5]*

[1]*School of Automation and Electrical Engineering, University of Science and Technology Beijing, Beijing* 100083*, China;*
[2]*Key Laboratory of Knowledge Automation for Industrial Processes, Ministry of Education, Beijing* 100083*, China;*
[3]*Beijing Key Laboratory of Information Service Engineering, Beijing Union University, Beijing* 100101*, China;*
[4]*Key Laboratory of Systems and Control, Chinese Academy of Sciences, Beijing* 100190*, China;*
[5]*School of Automation, Beijing Institute of Technology, Beijing* 100081*, China*

Received 25 March 2020/Accepted 30 April 2020/Published online 12 October 2021

Citation Su W, Wang X-Q, Chen G, et al. Quasi-synchronization of bounded confidence opinion dynamics with a stochastic asynchronous rule. Sci China Inf Sci, 2022, 65(1): 119203, https://doi.org/10.1007/s11432-020-2950-x

Dear editor,

One of the interesting topics in opinion dynamics is the noise-induced synchronization of opinion systems. Su et al. [1–3] conducted a rigorous theoretical analysis of this phenomenon based on the Hegselmann-Krause (HK) model.

The HK model is a typical bounded confidence opinion model with a synchronous opinion updating rule, i.e., all agents continuously check and decide whether or not they will update their opinions at each time [4]. However, a more practical case is that people are more likely to communicate with each other in an asynchronous manner, with only a random fraction of agents communicating with each other at each time. A popular model with an asynchronous rule is the well-known Deffuant-Weisbuch (DW) model [5].

Some studies in this field have focused on the noise-driven properties of asynchronous opinion models. Similar to the HK model, the DW model too was shown to result in noise-driven synchronization in simulation studies [6]. Baccelli et al. [7] studied a noisy DW model over a graph and established some sufficient conditions for the system to converge. The sufficient conditions reported are essentially related to the state connectivity of the system, which cannot be predetermined easily in bounded confidence models. Zhang et al. [8] analyzed the consensus property for a noisy DW model, and introduced the definition of T-robust consensus. However, a complete study of noise-induced synchronization of asynchronous models is lacking.

We propose a generalized stochastic asynchronous model, such that the HK and DW models are its special examples. In this study, we proved that unlike the HK model, the stochastic asynchronous model cannot achieve the noise-driven synchronization almost surely (a.s.). Finally, by introducing the definition of quasi-synchronization in mean (i.m.), we proved that the stochastic asynchronous model can achieve quasi-synchronization i.m. under driving by noise. This result naturally implies the noise-induced synchronization of the DW model, i.e., for any initial state, under the driving influence of noise, the DW model will achieve quasi-synchronization i.m. Furthermore, the quasi-synchronization i.m. is weaker than the quasi-synchronization a.s.

Asynchronous model. Let $\mathcal{V}=\{1,2,\ldots,n\}, n\geqslant 2$ denote a set of n agents, $x_i(t)\in[0,1]$, $i\in\mathcal{V}$, $t\geqslant 0$ be the opinion state of agent i at time t, $\mathcal{U}(t)\subset\mathcal{V}$ be the set of communicating agents at time t, and $\xi_i(t), i\in\mathcal{V}, t\geqslant 1$ be the noise.

To proceed, we first propose our general bounded confidence model with the stochastic asynchronous rule. Let $y_{[0,1]}=0,y,1$ for $y<0$, $0\leqslant y\leqslant 1$, $y>1$, respectively. Then, the update rule of the stochastic opinion dynamics yields the following:

$$x_i(t+1)=\begin{cases}(\tilde{x}_i(t)+\xi_i(t+1))_{[0,1]}, & \text{if } i\in\mathcal{U}(t),\ \mathcal{N}_i(t)\neq\emptyset;\\ (x_i(t)+\xi_i(t+1))_{[0,1]}, & \text{otherwise},\end{cases}\tag{1}$$

where $\tilde{x}_i(t)=\alpha_i(t)x_i(t)+(1-\alpha_i(t))\frac{\sum_{j\in\mathcal{N}_i(t)}x_j(t)}{|\mathcal{N}_i(t)|}$, $\alpha_i(t)\in[\alpha,1-\alpha]$ with $\alpha\in(0,\frac{1}{n}]$,

$$\mathcal{N}_i(t)=\{j\in\mathcal{U}(t)-\{i\}\mid |x_j(t)-x_i(t)|\leqslant\epsilon\}\tag{2}$$

is the neighbor set of i at t, and $\epsilon\in(0,1]$ represents the confidence threshold of the agents. Here, $|\cdot|$ is the cardinal number of a set or the absolute value of a real number.

From (1), we can see that when an agent $i\in\mathcal{U}(t)$, i.e., i is a communicating agent at t, i will communicate with other communicating agents at t, and then update its opinion according to the bounded confidence mechanism. When i is not a communicating agent at t, it will not communicate with other agents, while its opinion can only be affected by environmental noise.

Now, we discuss the communicating set $\mathcal{U}(t)$. In reality, people seldom update opinion values simultaneously at each time. Hence, the elements of $\mathcal{U}(t)$ are randomly selected from $\mathcal{V}$. Formally, we suppose that for $t\geqslant 0$, $\mathcal{U}(t)$ satisfies

* Corresponding author (email: kshen@bit.edu.cn)

the following:

$$\begin{aligned}
&\text{(a) } \mathbb{P}\{|\mathcal{U}(t)|=k\}=p_k,\ \text{where } 0\leqslant k\leqslant n,\ 0\leqslant p_k\leqslant 1,\\
&\quad \text{and } \sum_k p_k=1,\ 0\leqslant p_0+p_1<1;\\
&\text{(b) for any } i_1,i_2\in\mathcal{V},\ i_1\neq i_2,\\
&\quad \mathbb{P}\{i_1,i_2\in\mathcal{U}(t)\}=\mathbb{P}\{i_1\in\mathcal{U}(t)\}\mathbb{P}\{i_2\in\mathcal{U}(t)\},\\
&\quad \mathbb{P}\{i_1\in\mathcal{U}(t)\}=\mathbb{P}\{i_2\in\mathcal{U}(t)\};\\
&\text{(c) } \mathcal{U}(t),\xi_i(t),i\in\mathcal{V},t\geqslant 0 \text{ are independent.}
\end{aligned} \tag{3}$$

Eq. (3) provides the assumptions of the communicating agents. The assumptions are natural, and we will further provide a methodological explanation for them. Eq. (3)(a) assumes that the number of communicating agents is randomly selected at each time and that the number of communicating agents is not always 0 or 1. This is because we suppose that the communication occurs among agents with a positive probability. Eq. (3)(b) assumes that each agent is independently selected to be the communicating agent with an equal probability at each time. With Eq. (3)(b), we have

$$\mathbb{P}\{\mathcal{U}(t)=\{i_1,\ldots,i_k\}\}=\frac{1}{C_n^k}p_k, \tag{4}$$

where $\{i_1,\ldots,i_k\}\subset\mathcal{V}$ is an arbitrary choice of k agents, and C_n^k is the combinatorial symbol that abbreviates $\frac{n!}{k!(n-k)!}$ for $0\leqslant k\leqslant n$. Eq. (4) indicates that given k communicating agents, any choice of a combination of k agents has an equal probability. Eq. (3)(c) assumes that the event in which agents are communicating and the noises are mutually independent at each time, and also the updating process of the communicating agents and noises are mutually independent for different time points.

To study the noise-induced synchronization of (1)–(3), we introduce the definition of quasi-synchronization a.s. and quasi-synchronization i.m.

Definition 1. Let $d_{\mathcal{V}}(t)=\max_{i,j\in\mathcal{V}}|x_i(t)-x_j(t)|$.

(1) If $\mathbb{P}\{\limsup_{t\to\infty}d_{\mathcal{V}}(t)\leqslant\epsilon\}=1$, we say that the system (1)–(3) achieves quasi-synchronization a.s.;

(2) If $\limsup_{t\to\infty}\mathrm{E}\,d_{\mathcal{V}}(t)\leqslant\epsilon$, we say that the system (1)–(3) achieves quasi-synchronization i.m.

Quasi-synchronization i.m. of asynchronous systems. When there is no noise in the system (1)–(3) (i.e., $\xi_i(t)\equiv 0$, $i\in\mathcal{V}$, $t\geqslant 1$), the system cannot reach synchronization ($\lim_{t\to\infty}d_{\mathcal{V}}(t)=0$) under some initial state $x(0)\in[0,1]^n$ and $\epsilon\in(0,1)$. However, when there is noise, the following theorem is applicable.

Theorem 1. Suppose that the noises $\{\xi_i(t)\}_{i\in\mathcal{V},t\geqslant 1}$ are zero-mean random variables, each with an independent and identical distribution (i.i.d.), and $\mathrm{E}\,\xi_1^2(1)>0$, $|\xi_1(1)|\leqslant\delta$ a.s. for $\delta>0$. Then given any $x(0)\in[0,1]^n$ and $\epsilon\in(0,1]$, there exists $\bar{\delta}=\bar{\delta}(n,\epsilon,\alpha,p_0,p_1)>0$ such that the system (1)–(3) achieves quasi-synchronization i.m. for all $\delta\in(0,\bar{\delta}]$.

Using Theorem 1, the quasi-synchronization i.m. of the DW model can be obtained directly. In fact, let $\mathcal{U}(t)=\{i_1,i_2\}$ in the system (1)–(3), where $\{i_1,i_2\}\subset\mathcal{V}$ is an arbitrary choice of two agents, and $\alpha_i(t)\equiv\beta\in(0,1]$. The system (1)–(3) becomes a standard noisy DW model. A more detailed conclusion is provided in Appendix B.

Quasi-synchronization a.s. of asynchronous systems. When the system (1)–(3) is synchronous (i.e., $|\mathcal{U}(t)|\equiv n,t\geqslant 1$ or $p_n=1$), the system can achieve quasi-synchronization a.s. for all $\delta\in(0,\frac{\alpha\epsilon}{n(n-1)^2})$ (this is actually the case of the HK model [1]). However, when the model is asynchronous, i.e., $p_n<1$, we reach the following conclusion.

Theorem 2. Suppose that the noises $\{\xi_i(t)\}_{i\in\mathcal{V},t\geqslant 1}$ are zero-mean i.i.d. random variables, and $\mathrm{E}\,\xi_1^2(1)>0$, $|\xi_i(t)|\leqslant\delta$ a.s. for $\delta>0$. If $p_n<1$, then the system (1)–(3) cannot achieve quasi-synchronization a.s. for any $x(0)\in[0,1]^n$, $\epsilon\in(0,1)$ and $\delta>0$.

Theorem 2 shows that unlike the synchronous model, the asynchronous model cannot achieve quasi-synchronization a.s., regardless of how small the non-zero noise is.

Quasi-synchronization a.s. and quasi-synchronization i.m. Theorems 1 and 2 indicate that for the system (1)–(3), quasi-synchronization i.m. does not necessarily imply quasi-synchronization a.s. The following theorem reveals that the converse conclusion is true, i.e., quasi-synchronization a.s. leads to quasi-synchronization i.m.

Theorem 3. If $\mathbb{P}\{\limsup_{t\to\infty}d_{\mathcal{V}}(t)\leqslant\epsilon\}=1$, then $\limsup_{t\to\infty}\mathrm{E}\,d_{\mathcal{V}}(t)\leqslant\epsilon$.

Discussions. In this study, the noise-induced synchronization of a general asynchronous opinion dynamics of bounded confidence with stochastic rules was examined. The results lay a theoretical foundation for developing a noise-based control strategy of more general complex social opinion systems. The results also highlight the noise-driven properties of loosely coupled self-organizing particle robotic systems proposed by Li et al. [9].

Acknowledgements This work was supported by the National Key Research and Development Program of Ministry of Science and Technology of China (Grant No. 2018AAA0101002), National Natural Science Foundation of China (Grant Nos. 61803024, 11688101, 61906016), Young Elite Scientists Sponsorship Program by CAST (Grant No. 2018QNRC001), and Beijing Institute of Technology Research Fund Program for Young Scholars.

Supporting information Appendixes A–E. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

References

1 Su W, Chen G, Hong Y G. Noise leads to quasi-consensus of Hegselmann-Krause opinion dynamics. Automatica, 2017, 85: 448–454
2 Su W, Guo J, Chen X, et al. Noise-induced synchronization of Hegselmann-Krause dynamics in full space. IEEE Trans Autom Control, 2019, 64: 3804–3808
3 Chen G, Su W, Ding S Y, et al. Heterogeneous Hegselmann-Krause dynamics with environment and communication noise. IEEE Trans Autom Control, 2020, 65: 3409–3424
4 Hegselmann R, Krause U. Opinion dynamics and bounded confidence models, analysis, and simulation. J Artif Soc Soc Simul, 2002, 5: 1–33
5 Deffuant G, Neau D, Amblard F, et al. Mixing beliefs among interacting agents. Advs Complex Syst, 2000, 3: 87–98
6 Carro A, Toral R, Miguel M S. The role of noise and initial conditions in the asymptotic solution of a bounded confidence, continuous-opinion model. J Stat Phys, 2013, 151: 131–149
7 Baccelli F, Chatterjee A, Vishwanath S. Pairwise stochastic bounded confidence opinion dynamics: heavy tails and stability. IEEE Trans Autom Control, 2017, 62: 5678–5693
8 Zhang J B, Zhao Y Y. The robust consensus of a noisy Deffuant-Weisbuch model. Math Probl Eng, 2018, 2018: 1065451
9 Li S G, Batra R, Brown D, et al. Particle robotics based on statistical mechanics of loosely coupled components. Nature, 2019, 567: 361–365