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• **Supplementary File** •

Quasi-synchronization of bounded confidence opinion dynamics with stochastic asynchronous rule

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Appendix A Proof of Theorem 1

To prove Theorem 1, some lemmas are needed. The following lemma provides a basic tool for analyzing the noise-induced properties of bounded confidence opinion dynamics, which roughly states that when a random walk has a uniform positive probability entering a region within a finite time, it will almost surely enter that region in finite time.

Lemma 1. Let $\{w_t, t \geqslant 1\}$ be a random walk on R^n, $\{T_i : \Omega \to \mathbb{N}^+, i \geqslant 1\}$ be a sequence of increasing random variables. For $D \subset R^n$, denote $T = \inf_{t\geqslant 1}\{t : w_t \in D\}$ and $\bar{D} = R^n - D$. If for any $T_i, i \geqslant 1$, there is a constant $0 < p \leqslant 1$ such that $\mathbb{P}\Big\{w_{T_{i+1}} \in D\Big| \bigcap_{k\leqslant i}\{w_{T_k} \in \bar{D}\}\Big\} \geqslant p$, then $\mathbb{P}\{T < \infty\} = 1$.

Proof: The proof of Lemma 1 was given in the last part of Proposition 3.1 in [1]. □

Lemma 2. Given any initial state $x(0) \in [0,1]^n$, $\epsilon \in (0,1]$ of the system (1)-(3), and any $\lambda \in (0,1]$, define $T = \inf_{t\geqslant 0}\{t : d_{\mathcal{V}}(t) \leqslant \lambda\epsilon\}$, then $\mathbb{P}\{T < \infty\} = 1$ for all $0 < \delta \leqslant \frac{\lambda\epsilon}{2}$.

Proof: Notice (C2) and consider the following noise protocol: for all $i \in \mathcal{V}$, $t \geqslant 0$

$$\begin{cases} \xi_i(t+1) \in [a, \delta], \quad \text{if } \min\limits_{j\in\mathcal{V}} x_j(t) \leqslant \widetilde{x}_i(t) \leqslant \min\limits_{j\in\mathcal{V}} x_j(t) + \frac{d_{\mathcal{V}}(t)}{2}; \\ \xi_i(t+1) \in [-\delta, -a], \quad \text{if } \min\limits_{j\in\mathcal{V}} x_j(t) + \frac{d_{\mathcal{V}}(t)}{2} < \widetilde{x}_i(t) \leqslant \max\limits_{j\in\mathcal{V}} x_j(t), \end{cases} \tag{A1}$$

where

$$\widetilde{x}_i(t) = \begin{cases} \alpha_i(t)x_i(t) + (1-\alpha_i(t))\frac{\sum_{j\in\mathcal{N}_i(t)} x_j(t)}{|\mathcal{N}_i(t)|}, \quad \text{if } i \in \mathcal{U}(t) \text{ and } \mathcal{N}_i(t) \neq \emptyset; \\ x_i(t), \quad \text{otherwise.} \end{cases} \tag{A2}$$

Since $\delta \leqslant \frac{\lambda\epsilon}{2}$, following a similar argument of the proof of Theorem 5 in [2], there exists $L > 0$ such that

$$\mathbb{P}\Big\{d_{\mathcal{V}}(mL) \leqslant \lambda\epsilon \Big| \bigcap_{j<m} d_{\mathcal{V}}(jL) > \lambda\epsilon\Big\} \geqslant \bar{p}^L \tag{A3}$$

for $m \geqslant 1$. Let $D = \{x \in R^n : \max_{i,j}|x_i - x_j| \leqslant \lambda\epsilon\}, T_k = kL, p = \bar{p}^L$, then by (A3) and Lemma 1, $\mathbb{P}\{T < \infty\} = 1$. □

In the following, we intend to analyze the system properties of the asynchronous model (1)-(3). Specially, the analysis methodology of quasi-synchronization i.m. of the proposed asynchronous model is quite different compared to the previous studies of quasi-synchronization a.s. of synchronous models. For $t \geqslant 0$, we take

$$m_t \in \Big\{i \in \mathcal{V} : x_i(t) = \min_{j\in\mathcal{V}} x_j(t)\Big\},$$
$$M_t \in \Big\{i \in \mathcal{V} : x_i(t) = \max_{j\in\mathcal{V}} x_j(t)\Big\},$$

and define the event

$$A(t) = \{m_t \in \mathcal{U}(t), M_t \in \mathcal{U}(t)\}. \tag{A4}$$

$A(t)$ is the set of two agents with maximum and minimum opinion values who are also communicating agents at t. Then we have the following lemma.

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Lemma 3. Consider the system (1)-(3) and denote $L_0 = \frac{n(n-1)}{2}$. Given $\lambda \in (0,1]$, if there exists $T < \infty$ a.s. such that $d_{\mathcal{V}}(T) \leqslant \frac{\lambda\epsilon}{2}$, then

$$\mathbb{P}\Big\{ d_{\mathcal{V}}(T+t) \leqslant d_{\mathcal{V}}(T) + 2t\delta \leqslant \lambda\epsilon \Big\} = 1 \tag{A5}$$

for all $1 \leqslant t \leqslant L_0, \delta \in \Big(0, \frac{\alpha\lambda\epsilon}{2n(n-1)^2}\Big]$, and

$$\mathbb{P}\Big\{ d_{\mathcal{V}}\Big(T+L_0\Big) \leqslant \frac{\lambda\epsilon}{2} - \frac{\alpha\lambda\epsilon}{2(n-1)} \Big| \bigcap_{r=T}^{T+L_0-1} A(r) \Big\} = 1. \tag{A6}$$

Proof: For convenience of notation, suppose $T = 0$ a.s. To prove (A5), we only need to prove

$$d_{\mathcal{V}}(t) \leqslant d_{\mathcal{V}}(0) + 2t\delta \leqslant \lambda\epsilon, \quad a.s. \tag{A7}$$

for $1 \leqslant t \leqslant L_0$.

Since $|\xi_i(t)| \leqslant \delta$ a.s., from (1), $d_{\mathcal{V}}(t) \leqslant d_{\mathcal{V}}(t-1) + 2\delta \leqslant \ldots \leqslant d_{\mathcal{V}}(0) + 2t\delta$, implying the first part of (A7). The second part of (A7) can be directly obtained by $1 \leqslant t \leqslant \frac{n(n-1)}{2}$, $\alpha \leqslant \frac{1}{n}$ and $\delta \leqslant \frac{\alpha\lambda\epsilon}{2n(n-1)^2}$.

Now we proceed to prove (A6). By (3)(a) and (4), for $t \geqslant 0$

$$\begin{aligned}\mathbb{P}\{A(t)\} &= \sum_{k=2}^{n} \frac{1}{C_n^k} p_k C_{n-2}^{k-2} = \sum_{k=2}^{n} \frac{k(k-1)}{n(n-1)} p_k \\ &\geqslant \frac{2}{n(n-1)}(p_2 + \ldots + p_n) \\ &= \frac{2}{n(n-1)}(1 - p_0 - p_1) > 0.\end{aligned} \tag{A8}$$

By (3)(c), $\{A(t), t \geqslant 0\}$ are independent, then we can gain

$$\mathbb{P}\Big\{ \bigcap_{r=0}^{L_0-1} A(r) \Big\} = \prod_{r=0}^{L_0-1} \mathbb{P}\{A(r)\} \geqslant \Big(\frac{2(1-p_0-p_1)}{n(n-1)} \Big)^{L_0} > 0. \tag{A9}$$

To prove (A6), we assume $\mathbb{P}\Big\{ \bigcap_{r=0}^{L_0-1} A(r) \Big\} = 1$ without loss of generality, and we then need to prove

$$d_{\mathcal{V}}(L_0) \leqslant \frac{\lambda\epsilon}{2} - \frac{\alpha\lambda\epsilon}{2(n-1)}, \quad a.s. \tag{A10}$$

By the above assumption, we know $\mathbb{P}\{A(0)\} = 1$, which implies that m_0 and M_0 are communicating agents at $t = 0$. And, $d_{\mathcal{V}}(0) \leqslant \frac{\lambda\epsilon}{2} \leqslant \epsilon$ implies that m_0 and M_0 are neighbors to each other. From (1), we know

$$x_{m_0}(1) = \alpha_{m_0}(0)x_{m_0}(0) + (1 - \alpha_{m_0}(0)) \frac{\sum_{j \in \mathcal{N}_{m_0}(0)} x_j(0)}{|\mathcal{N}_{m_0}(0)|} + \xi_{m_0}(1) \tag{A11}$$

then, by $\alpha \leqslant \alpha_{m_0}(0) \leqslant 1 - \alpha$ and $|\xi_{m_0}(1)| \leqslant \delta$ a.s., it follows a.s.

$$\begin{aligned}x_{m_0}(1) - x_{m_0}(0) =& (1 - \alpha_{m_0}(0) \frac{\sum_{j \in \mathcal{N}_{m_0}(0)} (x_j(0) - x_{m_0}(0))}{|\mathcal{N}_{m_0}(0)|} + \xi_{m_0}(1) \\ \geqslant& (1 - \alpha_{m_0}(0)) \frac{x_{M_0}(0) - x_{m_0}(0)}{n-1} + \xi_{m_0}(1) \\ \geqslant& \frac{\alpha}{n-1} d_{\mathcal{V}}(0) - \delta.\end{aligned} \tag{A12}$$

Similarly, we can get

$$x_{M_0}(1) = \alpha_{M_0}(0)x_{M_0}(0) + (1 - \alpha_{M_0}(0)) \frac{\sum_{j \in \mathcal{N}_{M_0}(0)} x_j(0)}{|\mathcal{N}_{M_0}(0)|} + \xi_{M_0}(1) \tag{A13}$$

and a.s.

$$\begin{aligned}x_{M_0}(1) - x_{M_0}(0) =& (1 - \alpha_{M_0}(0) \frac{\sum_{j \in \mathcal{N}_{M_0}(0)} (x_j(0) - x_{M_0}(0))}{|\mathcal{N}_{M_0}(0)|} + \xi_{M_0}(1) \\ \leqslant& (1 - \alpha_{M_0}(0)) \frac{x_{m_0}(0) - x_{M_0}(0)}{n-1} + \xi_{M_0}(1) \\ \leqslant& -\frac{\alpha}{n-1} d_{\mathcal{V}}(0) + \delta.\end{aligned} \tag{A14}$$

Equations (A12) and (A14) yield a.s.

$$\begin{aligned}|x_{M_0}(1) - x_{m_0}(1)| \leqslant& d_{\mathcal{V}}(0) - \frac{2\alpha}{n-1} d_{\mathcal{V}}(0) + 2\delta \\ \leqslant& \frac{\lambda\epsilon}{2} - \Big(\frac{\alpha\lambda\epsilon}{n-1} - 2\delta \Big)\end{aligned} \tag{A15}$$

For any $i \in U(0)$, we know $x_{m_0}(0) \leqslant x_i(0) \leqslant x_{M_0}(0)$. Hence, following a similar argument as illustrated above, (A15) implies a.s.

$$|x_i(1) - x_j(1)| \leqslant \frac{\lambda\epsilon}{2} - \Big(\frac{\alpha\lambda\epsilon}{n-1} - 2\delta\Big) \tag{A16}$$

for any $i, j \in U(0)$.

Equation (A16) yields that once two agents are communicating at $t = 0$, their distance at $t = 1$ has an upper bound which is represented by the right side of (A16). During the following movement, their distance can exceed the upper bound only when they are not communicating agents simultaneously. In this case, their distance may increase by no more than 2δ after each time step. Since $\delta \leqslant \frac{\alpha\lambda\epsilon}{2n(n-1)^2}$, then a.s.

$$|x_i(t) - x_j(t)| \leqslant \frac{\lambda\epsilon}{2} - \Big(\frac{\alpha\lambda\epsilon}{n-1} - 2t\delta\Big) \leqslant \frac{\lambda\epsilon}{2} - \frac{\alpha\lambda\epsilon}{2(n-1)} \tag{A17}$$

for all $i, j \in U(0)$ and $1 \leqslant t \leqslant \frac{n(n-1)}{2}$.

Given any $2 \leqslant t_0 \leqslant L_0 - 1$, by (A7), we can get $d_{\mathcal{V}}(t_0) \leqslant d_{\mathcal{V}}(0) + 2t_0\delta$. Similar to the process of obtaining (A15), we have a.s.

$$\begin{aligned} |x_{M_{t_0}}(t_0+1) - x_{m_{t_0}}(t_0+1)| \leqslant & d_{\mathcal{V}}(t_0) - \frac{2\alpha}{n-1} d_{\mathcal{V}}(t_0) + 2\delta \\ \leqslant & \frac{\lambda\epsilon}{2} - \Big(\frac{\alpha\lambda\epsilon}{n-1} - 2t_0\delta\Big) + 2\delta \end{aligned} \tag{A18}$$

Then just as the process of obtaining the equation (A17),

$$\begin{aligned} |x_i(t) - x_j(t)| \leqslant & \frac{\lambda\epsilon}{2} - \Big(\frac{\alpha\lambda\epsilon}{n-1} - 2t_0\delta\Big) + \Big(L_0 - t_0\Big)(2\delta) \\ \leqslant & \frac{\lambda\epsilon}{2} - \frac{\alpha\lambda\epsilon}{2(n-1)}, \quad a.s. \end{aligned} \tag{A19}$$

for all $i, j \in U(t_0)$ and $t_0 + 1 \leqslant t \leqslant L_0$.

For any $i \in \mathcal{V}$, since $\mathbb{P}\Big\{\bigcap_{r=0}^{L_0-1} A(r)\Big\} = 1$ by assumption, it is certain that $i \in A(t)$ for some $0 \leqslant t \leqslant L_0$, or $x_{m_t}(t) \leqslant x_i(t) \leqslant x_{M_t}(t)$ for all $0 \leqslant t \leqslant L_0$. For both cases, by (A17) and (A19), we have

$$\Big|x_i(L_0) - x_j(L_0)\Big| \leqslant \frac{\lambda\epsilon}{2} - \frac{\alpha\lambda\epsilon}{2(n-1)}, \; a.s. \tag{A20}$$

for all $i, j \in \mathcal{V}$. Hence (A10) can be obtained. This complete the proof. □

Lemma 3 indicates that, once the system enters a region which is narrow enough, it will not get away from the region, provided some special agents with extreme opinion values are always communicating.

Furthermore, in order to achieve the final result of quasi-synchronization i.m. of the asynchronous model, we would like to introduce the following lemma. In the proof of the lemma, we introduce a stopping time as a bridge to obtain the moment property of $d_{\mathcal{V}}(t)$.

Lemma 4. Suppose the noises $\{\xi_i(t)\}_{i\in\mathcal{V}, t\geqslant 1}$ are given in Theorem 1. Let $x(0) \in [0,1]^n$, $\epsilon \in (0,1]$ be arbitrarily given, then for any $\mu \in (0,1]$, there is $\bar{\delta} = \bar{\delta}(\mu, n, \epsilon, \alpha, p_0, p_1) > 0$, such that $\lim_{t\to\infty} \mathbf{E}\, d_{\mathcal{V}}(t) \leqslant \mu\epsilon$ for all $\delta \in (0, \bar{\delta}]$.

Proof: Denote $\tilde{p} = \frac{2}{n(n-1)}(1 - p_0 - p_1)$, $L_0 = \frac{n(n-1)}{2}$, $L = \min\Big\{l > 0 : (1 - \tilde{p}^{L_0})^l \leqslant \frac{\mu\epsilon}{2}\Big\}$ and $T = \inf_{t\geqslant 0}\Big\{t : d_{\mathcal{V}}(t) \leqslant \frac{\mu\epsilon}{4(1+L_0L)}\Big\}$, then

$$d_{\mathcal{V}}(T) \leqslant \frac{\mu\epsilon}{4(1+L_0L)}, a.s. \tag{A21}$$

Denote

$$\bar{\delta} = \min\Big\{\frac{\alpha\mu\epsilon}{2n(n-1)^2}, \frac{\mu\epsilon}{8(1+L_0L)}\Big\}, \tag{A22}$$

and we next prove that

$$\mathbf{E}\, d_{\mathcal{V}}(T+k) \leqslant \mu\epsilon \tag{A23}$$

for all $k \geqslant 1$ and $\delta \in (0, \bar{\delta}]$.

In order to prove (A23), we first consider $\mathbb{P}\Big\{d_{\mathcal{V}}(T+k) > \frac{\mu\epsilon}{2}\Big\}$ for any given $k > 0$. Take $\lambda = \frac{\mu}{2}$ in Lemma 3, and notice that $d_{\mathcal{V}}(T) \leqslant \frac{\mu\epsilon}{4(1+L_0L)} \leqslant \frac{\mu\epsilon}{4}$ a.s., then by (A5) in Lemma 3 and (A21), we can get

$$d_{\mathcal{V}}(T+k) \leqslant d_{\mathcal{V}}(T) + 2k\delta \leqslant \frac{\mu\epsilon}{4(1+L_0L)} + 2L_0L\bar{\delta} \leqslant \frac{\mu\epsilon}{4}, a.s. \tag{A24}$$

for $1 \leqslant k \leqslant L_0L$, implying

$$\mathbb{P}\Big\{d_{\mathcal{V}}(T+k) > \frac{\mu\epsilon}{2}\Big\} = 0, \qquad 1 \leqslant k \leqslant L_0L. \tag{A25}$$

Next we consider $\mathbb{P}\Big\{d_{\mathcal{V}}(T+k) > \frac{\mu\epsilon}{2}\Big\}$ for $k > L_0L$. Since $d_{\mathcal{V}}(T) \leqslant \frac{\mu\epsilon}{4(1+L_0L)} \leqslant \frac{\mu\epsilon}{4}$ a.s., by Lemma 3, we know that once $A(t)$ occurs for $T \leqslant t \leqslant T + L_0 - 1$, then $d_{\mathcal{V}}(T+L_0) \leqslant \frac{\mu\epsilon}{4} - \frac{\alpha\mu\epsilon}{4(n-1)}$ a.s. By $\delta \leqslant \frac{\alpha\mu\epsilon}{8(n-1)L_0L}$, we can get

$$d_{\mathcal{V}}(T+L_0+t) \leqslant d_{\mathcal{V}}(T+L_0) + 2\delta t \leqslant \frac{\mu\epsilon}{4}, a.s. \tag{A26}$$

for $1 \leqslant t \leqslant L_0L$.

Denote $B(s,r)=\bigcap_{t=T+s+(r-1)L_0}^{T+s+rL_0-1} A(t), s\geqslant 0, r\geqslant 1$. (A26) implies that when there is a moment T such that $d_{\mathcal{V}}(T)\leqslant\frac{\mu\epsilon}{4}$, and $A(t)$ occurs in the following L_0 times, then $d_{\mathcal{V}}(T+t)\leqslant\frac{\mu\epsilon}{4}$ for all $L_0\leqslant t\leqslant L_0+L_0L$. In other words, once $B(0,1)$ occurs, $d_{\mathcal{V}}(t)$ can not exceed $\frac{\mu\epsilon}{4}$ during the next L_0L steps. By (A24), $d_{\mathcal{V}}(T+t)\leqslant\frac{\mu\epsilon}{4}$ a.s. for all $1\leqslant t\leqslant L_0L$. Hence, if there is $k>L_0L$ such that $d_{\mathcal{V}}(T+k)>\frac{\mu\epsilon}{4}$ a.s., there must exist a period of length L_0L and some integer $s\geqslant 0$ such that anyone of $\{B(s,1),\dots,B(s,L)\}$ cannot happen, i.e.,

$$\Big\{d_{\mathcal{V}}(T+k)>\frac{\mu\epsilon}{4}\Big\}\subset\Big\{\bigcap_{r=1}^{L}\{\Omega-B(s,r)\}\Big\}. \tag{A27}$$

By (3) and (A8), $\{A(t),t\geqslant 0\}$ are i.i.d., and so are $\{B(s,r),s\geqslant 0,r\geqslant 1\}$ by strong Markov property. As a result, for any given $k>L_0L$, we can get by (A27)

$$\begin{aligned}\mathbb{P}\Big\{d_{\mathcal{V}}(T+k)>\frac{\mu\epsilon}{4}\Big\}\leqslant&\mathbb{P}\Big\{\bigcap_{r=1}^{L}\{\Omega-B(s,r)\}\Big\}\\=&(1-\mathbb{P}\{B(0,1)\})^L\end{aligned} \tag{A28}$$

where Ω is the sample space.

By (A9) and (A28), we have

$$\begin{aligned}\mathbb{P}\Big\{d_{\mathcal{V}}(T+k)>\frac{\mu\epsilon}{4}\Big\}\leqslant&(1-\mathbb{P}\{B(0,1)\})^L\\=&\Big(1-\prod_{r=0}^{L_0-1}\mathbb{P}\{A(T+r)\}\Big)^L\\\leqslant&\Big(1-\tilde{p}^{L_0}\Big)^L\end{aligned} \tag{A29}$$

for any given $k>L_0L$.

Since $d_{\mathcal{V}}(t)\leqslant 1$ a.s. for all $t\geqslant 0$, by (A25), (A29) and the definition of L, it follows

$$\begin{aligned}\mathbf{E}\,d_{\mathcal{V}}(T+k)=&\mathbf{E}\Big(d_{\mathcal{V}}(T+k)I_{\{d_{\mathcal{V}}(T+k)\leqslant\frac{\mu\epsilon}{2}\}}+d_{\mathcal{V}}(T+k)I_{\{d_{\mathcal{V}}(T+k)>\frac{\mu\epsilon}{2}\}}\Big)\\\leqslant&\frac{\mu\epsilon}{2}+\mathbb{P}\Big\{d_{\mathcal{V}}(T+k)>\frac{\mu\epsilon}{2}\Big\}\\\leqslant&\frac{\mu\epsilon}{2}+(1-\tilde{p}^{L_0}\Big)^L\leqslant\mu\epsilon\end{aligned} \tag{A30}$$

for all $k\geqslant 0$.

Subsequently, given any $t\geqslant 0$, we gain by (A30)

$$\begin{aligned}\mathbf{E}\,d_{\mathcal{V}}(t)=&\mathbf{E}\Big(d_{\mathcal{V}}(t)I_{\{T\leqslant t\}}+d_{\mathcal{V}}(t)I_{\{T>t\}}\Big)\\=&\mathbf{E}\Big(\sum_{k=0}^{t}d_{\mathcal{V}}(T+k)I_{\{T=t-k\}}\Big)+\mathbf{E}\,d_{\mathcal{V}}(t)I_{\{T>t\}}\\=&\sum_{k=0}^{t}\mathbf{E}\,d_{\mathcal{V}}(T+k)I_{\{T=t-k\}}+\mathbf{E}\,d_{\mathcal{V}}(t)I_{\{T>t\}}\\\leqslant&\mu\epsilon\sum_{k=0}^{t}\mathbf{E}\,I_{\{T=t-k\}}+\mathbf{E}\,d_{\mathcal{V}}(t)I_{\{T>t\}}\\\leqslant&\mu\epsilon\mathbb{P}\{T\leqslant t\}+\mathbb{P}\{T>t\}.\end{aligned} \tag{A31}$$

By Lemma 2, $\mathbb{P}\{T<\infty\}=1$, we can thus obtain

$$\begin{aligned}\limsup_{t\to\infty}\mathbf{E}\,d_{\mathcal{V}}(t)\leqslant&\limsup_{t\to\infty}\Big(\mu\epsilon\mathbb{P}\{T\leqslant t\}+\mathbb{P}\{T>t\}\Big)\\\leqslant&\mu\epsilon.\end{aligned} \tag{A32}$$

This completes the proof. □

Proof of Theorem 1: Take $\mu=1$ in Lemma 4, and we obtain the conclusion. □

Appendix B Quasi-synchronization i.m. of DW model

Let $\mathcal{U}(t)=\{i_1,i_2\}$ in the system (1)-(3), where $\{i_1,i_2\}\subset\mathcal{V}$ is an arbitrary choice of two agents and $\beta\in(0,1]$. The standard DW model has a form as

$$\begin{aligned}x_{i_r}(t+1)=&\begin{cases}\beta x_{i_r}(t)+(1-\beta)x_{i_{3-r}}(t), \text{ if } |x_{i_1}(t)-x_{i_2}(t)|\leqslant\epsilon;\\ x_{i_r}(t), \text{ otherwise.}\end{cases}\\x_k(t+1)=&x_k(t),\quad k\notin\mathcal{U}(t),\end{aligned} \tag{B1}$$

where $r=1,2$.

Let us consider the following noisy DW model

$$x_i(t+1)=(\tilde{x}_i(t)+\xi_i(t+1))_{[0,1]}, \tag{B2}$$

where $\tilde{x}_i(t)$ is the right side of (B1).

Corollary 1. (Quasi-synchronization i.m. of DW model) Given any $x(0)\in[0,1]^n,\epsilon\in(0,1]$ for the system (B2), there exists $\bar{\delta}=\bar{\delta}(n,\epsilon,\beta)>0$, such that $\limsup\limits_{t\to\infty}\mathbf{E}\,d_{\mathcal{V}}(t)\leqslant\epsilon$ for all $\delta\in(0,\bar{\delta}]$.

Appendix C Proof of Theorem 2

Proof: We only need to prove that for any $x(0) \in [0,1]^n, \epsilon \in (0,1)$ and $\delta > 0$, $\limsup_{t\to\infty} d_{\mathcal{V}}(t) = 1$ a.s., i.e.

$$\begin{aligned}\mathbb{P}\Big\{\bigcup_{g=0}^{\infty}\{d_{\mathcal{V}}(t)<1, t\geqslant g\}\Big\} =&1-\mathbb{P}\Big\{\bigcap_{g=0}^{\infty}\bigcup_{t=g}^{\infty}\{d_{\mathcal{V}}(t)=1\}\Big\}\\ =&1-\mathbb{P}\Big\{\limsup_{t\to\infty} d_{\mathcal{V}}(t)=1\Big\}=0.\end{aligned} \tag{C1}$$

Given any $g \geqslant 0$, denote $T = \inf_{t\geqslant g}\{t : d_{\mathcal{V}}(t) = 1\}$, then by Lemma 1, we need to prove that, for any initial state $x(0) \in [0,1]^n$, there are $t_L > g, 0 < p < 1$ such that $\mathbb{P}\{d_{\mathcal{V}}(t_L) = 1\} \geqslant p$.

Since $\mathbf{E}\,\xi_1(1) = 0, \mathbf{E}\,\xi_1^2(1) > 0$ and $|\xi_i(1)| \leqslant \delta$ a.s., there exist constants $0 < a \leqslant \delta, 0 < \bar{p} < 1$ such that

$$\mathbb{P}\{a<\xi_1(1)\leqslant\delta\}\geqslant\bar{p},\ \mathbb{P}\{-\delta\leqslant\xi_1(1)<-a\}\geqslant\bar{p}. \tag{C2}$$

For $t \geqslant g$, consider the following noise protocol

$$\begin{cases}\xi_i(t+1)\in[-\delta,-a], & \text{if } \min_{j\in\mathcal{V}} x_j(t)\leqslant x_i(t)\leqslant\min_{j\in\mathcal{V}} x_j(t)+\frac{d_{\mathcal{V}}(t)}{2};\\ \xi_i(t+1)\in[a,\delta], & \text{if } \min_{j\in\mathcal{V}} x_j(t)+\frac{d_{\mathcal{V}}(t)}{2}<x_i(t)\leqslant\max_{j\in\mathcal{V}} x_j(t).\end{cases} \tag{C3}$$

Denote $A(t)^C = \Omega - A(t), t \geqslant 0$, where $A(t)$ is defined in (A4), then by $p_n < 1$ we have

$$\begin{aligned}\mathbb{P}\{\{A(t)^C\} =&1-\mathbb{P}\{A(t)\}=1-\sum_{k=2}^{n}\frac{1}{C_n^k}p_kC_{n-2}^{k-2}\\ =&1-\sum_{k=2}^{n-1}\frac{k(k-1)}{n(n-1)}p_k-p_n\\ \geqslant&1-p_n-\frac{n-2}{n}(p_2+\ldots+p_{n-1})\\ \geqslant&1-p_n-\frac{n-2}{n}(1-p_n)\\ =&\frac{2(1-p_n)}{n}>0.\end{aligned} \tag{C4}$$

Hence by the independence of $\{\xi_i(t), \mathcal{U}(t), i \in \mathcal{V}, t \geqslant 1\}$ and (C2), we can get

$$\begin{aligned}\mathbb{P}\{d_{\mathcal{V}}(t+1)\geqslant d_{\mathcal{V}}(t)+a\}\geqslant&\mathbb{P}\{A(t)^C,\ \text{protocol }(C3)\text{ occures at }t\}\\ =&\mathbb{P}\{A(t)^C\}\cdot\mathbb{P}\{\text{protocol }(C3)\text{ occures at }t\}\\ \geqslant&\tilde{p}\bar{p}^n>0,\end{aligned} \tag{C5}$$

where $\tilde{p} = \frac{2(1-p_n)}{n}$.

Denote $t_L = \lceil \frac{1}{a} \rceil$, then under the protocol (C3)

$$\max_{i\in\mathcal{V}} x_i(g+t_L)=1,\quad \min_{i\in\mathcal{V}} x_i(g+t_L)=0,$$

yielding $d_{\mathcal{V}}(g + t_L) = 1$. By (C5) and the independence of $\{\xi_i(t), \mathcal{U}(t), i \in \mathcal{V}, t \geqslant 0\}$, we gain

$$\begin{aligned}\mathbb{P}\{d_{\mathcal{V}}(g+t_L)=1\}\geqslant&\prod_{t=g+1}^{g+t_L}\mathbb{P}\{A(t)^C,\ \ \text{protocol (C3) occurs at }t\}\\ \geqslant&\tilde{p}^{t_L}\bar{p}^{nt_L}>0.\end{aligned} \tag{C6}$$

Let $p = \tilde{p}^{t_L}\bar{p}^{nt_L}$, and this completes the proof. □

Theorem 2 shows that quite different from the synchronous model, the asynchronous model cannot achieve quasi-synchronization a.s., no matter how small the non-zero noise is.

Appendix D Proof of Theorem 3

Proof: Since $\mathbb{P}\Big\{\limsup_{t\to\infty} d_{\mathcal{V}}(t) > \epsilon\Big\} = 0$,

$$\mathbb{P}\Big\{\lim_{s\to\infty}\bigcup_{t\geqslant s}\{d_{\mathcal{V}}(t)>\epsilon\}\Big\}=\lim_{s\to\infty}\mathbb{P}\Big\{\bigcup_{t\geqslant s}\{d_{\mathcal{V}}(t)>\epsilon\}\Big\}=0,$$

where the first equation follows from the exchange theorem of limit operation and probability measure (refer to Corollary 1.5.2 [3]). By $d_{\mathcal{V}}(t) \leqslant 1$ a.s., we obtain

$$\begin{aligned}\mathbf{E}\,d_{\mathcal{V}}(t)=&\,\mathbf{E}\Big(d_{\mathcal{V}}(t)I_{\{d_{\mathcal{V}}(t)\leqslant\epsilon\}}+d_{\mathcal{V}}(t)I_{\{d_{\mathcal{V}}(t)>\epsilon\}}\Big)\\ \leqslant&\,\epsilon\mathbb{P}\{d_{\mathcal{V}}(t)\leqslant\epsilon\}+\mathbb{P}\{d_{\mathcal{V}}(t)>\epsilon\}.\end{aligned} \tag{D1}$$

Consequently, we can get

$$\begin{aligned}\limsup_{t\to\infty}\mathbf{E}\,d_{\mathcal{V}}(t)\leqslant&\limsup_{t\to\infty}\Big(\epsilon\mathbb{P}\{d_{\mathcal{V}}(t)\leqslant\epsilon\}+\mathbb{P}\{d_{\mathcal{V}}(t)>\epsilon\}\Big)\\ \leqslant&\epsilon.\end{aligned}$$

This completes the proof. □

Appendix E Simulations

In this section, we present a simulation result to help understand the meaning of quasi-synchronization i.m. Let $n = 40, \epsilon = 0.1, \alpha_i(t) = \frac{1}{|\mathcal{N}_i(t)|+1}$, and $|\mathcal{U}(t)|$ is randomly selected from $0, 1, \ldots, n$ with equal probability at each time. The initial opinion values are randomly generated on $[0, 1]$, then we add independent noises with uniform distribution on $[-\delta, \delta]$ to the system (1)-(3). Take $\delta = 0.01$, then Fig. E1 shows that the system achieves a synchronization at about $t = 10000$. However, it is not the almost sure quasi-synchronization (refer to [2] for the simulation study of almost sure quasi-synchronization), since it can be calculated that $\max_t d_{\mathcal{V}}(t) = 0.1105 > \epsilon = 0.1$ from $t = 15000$ to 40000. Such an approximate synchronization can be measured by quasi-synchronization i.m.

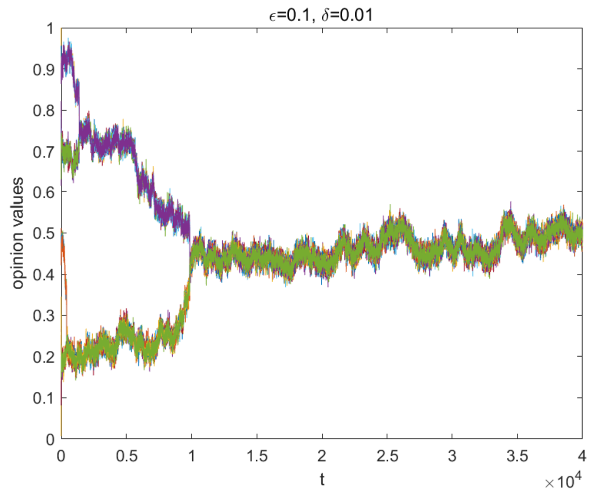

Figure E1 Opinion evolution of system (1)-(3) of 40 agents. The initial system states are randomly generated on $[0, 1]$, confidence threshold $\epsilon = 0.1$, noise strength $\delta = 0.01$.

References

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